Markov Chains (Michaelmas 2020)

Primary references:
J. Norris, Markov Chains, Cambridge University Press
G. Grimmett and D. Welsh, Probability, An Introduction, Oxford University Press

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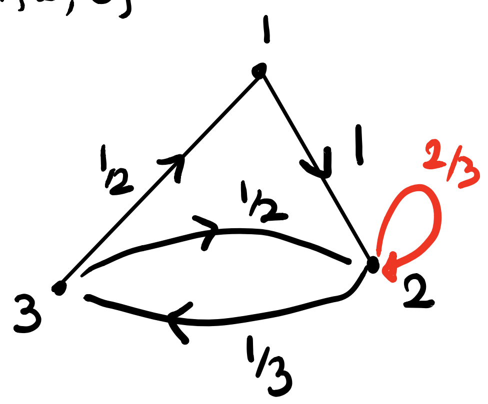
November 17, 2020

0. Examples.

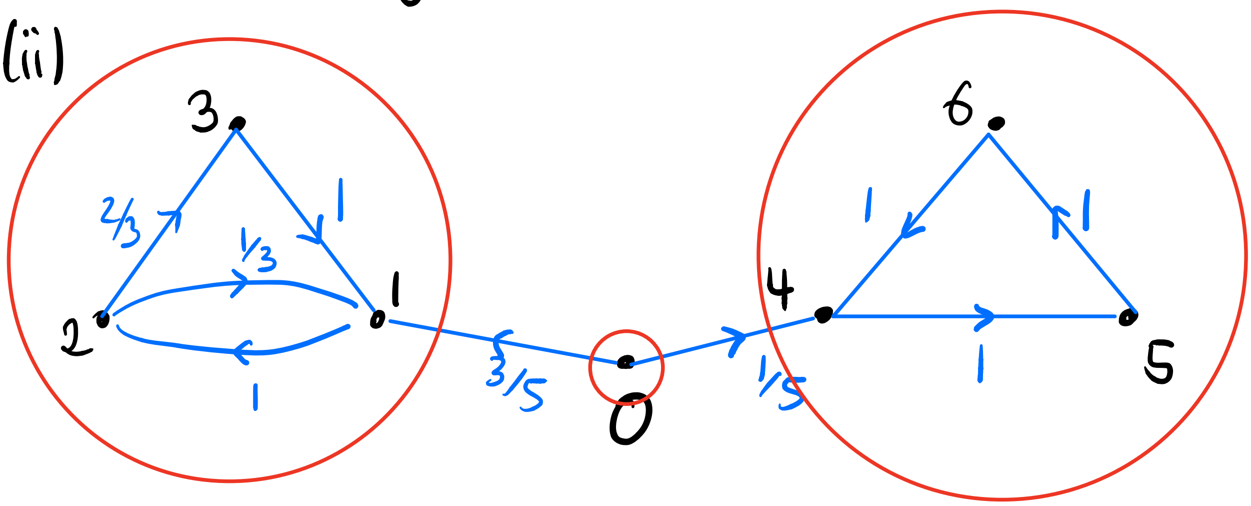
(i) $I = \{1, 2, 3\}$



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

diagram transition matrix

(ii)



Starting from 0, the probability to hit 6 is $\underline{1/4}$.

Starting from 1, the probability to hit 3 is $\underline{1}$.

Starting from 1, it takes on average $\underline{3}$ steps to hit 3.

Starting from 1, the long-run proportion of time spent in 2 is $\underline{3/8}$.

...

1. Definitions and basic properties

We will make the following standing assumptions:

- I is a countable set, the state space; $I = \{1, 2, \dots\}$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which all relevant random variables are defined.

Defn. A sequence of random variables $(X_n)_{n=0,1,\dots}$ is a Markov Chain if, for $n \geq 0$ and $i_0, \dots, i_{n+1} \in I$,

$$\mathbb{P}[X_{n+1} = i_{n+1} \mid \underbrace{X_0 = i_0, \dots, X_n = i_n}_{\text{has positive prob.}}] = \mathbb{P}[X_{n+1} = i_{n+1} \mid X_n = i_n]$$

It is homogeneous if, for all $i, j \in I$,

$$\mathbb{P}[X_{n+1} = j \mid X_n = i] = \mathbb{P}[X_1 = j \mid X_0 = i].$$

From now on, all Markov Chains are assumed homogeneous. Then a Markov Chain is characterised by:

(a) the initial distribution $\lambda = (\lambda_i)_{i \in I}$ given by
$$\lambda_i = \mathbb{P}[X_0 = i].$$

(b) the transition matrix $P = (p_{ij})_{i,j \in I}$ given by
$$p_{ij} = \mathbb{P}[X_1 = j \mid X_0 = i].$$

Facts. • λ is a distribution, i.e., $\lambda_i \geq 0$ for all $i \in I$, $\sum_{i \in I} \lambda_i = 1$.

• P is a stochastic matrix, i.e., $(p_{ij})_j$ is a distribution for every $i \in I$.

Defn. (X_n) is a Markov Chain with initial distrib. λ and transition matrix P, or (X_n) is Markov(λ, P), if (a) and (b) hold.

Thm. (X_n) is Markov(λ, P) iff for all $n \geq 0$, $i_0, \ldots, i_n \in I$,

$$\mathbb{P}[X_0 = i_0, \ldots, X_n = i_n] = \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}. \qquad (*)$$

Proof. Suppose (X_n) is Markov(λ, P). Then

$$\mathbb{P}[X_0 = i_0, \ldots, X_n = i_n] = \underbrace{\mathbb{P}[X_n = i_n \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}]}_{\substack{\mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}] \text{ by the} \\ = p_{i_{n-1} i_n} \quad \text{Markov prop.}}} \times \mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}].$$

$$= p_{i_{n-1} i_n} \mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}]$$

$$\overset{\text{(induct.)}}{=} p_{i_{n-1} i_n} p_{i_{n-2} i_{n-1}} \cdots p_{i_0 i_1} \underbrace{\mathbb{P}[X_0 = i_0]}_{\lambda_{i_0}}.$$

Conversely, assume (*) holds for all n and $i_0, \dots, i_n$.
For $n=0$, $\mathbb{P}[X_0 = i_0] = \lambda_{i_0}$.
Also, by (*) given

$$\mathbb{P}[X_n = i_n \mid X_0 = i_0, \dots, X_{n-1} = i_{n-1}] = \frac{\mathbb{P}[X_0 = i_0, \dots, X_n = i_n]}{\mathbb{P}[X_0 = i_0, \dots, X_{n-1} = i_{n-1}]}$$

$$= p_{i_{n-1} i_n}$$

Thus (a) & (b) hold, i.e., (X_n) is Markov(λ, P).

Let $\delta_i = (\delta_{ij} : j \in I)$ be the unit mass at $i \in I$:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

<u>Thm.</u> Let (X_n) is Markov(λ, P). Then conditional on $X_m = i$, $(X_{m+n})_{n \geq 0}$ is Markov(δ_i, P) and is independent of $X_0, \dots, X_m$.

<u>Proof.</u> It suffices to show

(i) $\mathbb{P}[X_m = i_m, \dots, X_{m+n} = i_{n+m} \mid X_m = i] = \delta_{i i_m} p_{i_m i_{m+1}} \cdots p_{i_{n+m-1} i_{n+m}}$
(Markov prop.)

and

(ii) for every event A determined by $X_1, \dots, X_m$ and every event B determined by $X_m, X_{m+1}, \dots$

$$\mathbb{P}[A \cap B \mid X_m = i] = \mathbb{P}[A \mid X_m = i]\, \mathbb{P}[B \mid X_m = i].$$

(indep.)

The previous theorem implies both for the elem. events

$$A = \{X_0 = i_0, \dots, X_m = i_m\}$$
$$B = \{X_m = i_m, \dots, X_{n+m} = i_{n+m}\}.$$

Indeed, after multiplying by $\mathbb{P}[X_m = i]$ the claim is

(i') $\mathbb{P}[X_m = i_m, \dots X_{m+n} = i_{m+n}] = \delta_{i i_m} p_{i_m i_{m+1}} \cdots p_{i_{n+m-1} i_{n+m}} \mathbb{P}[X_m = i]$

(which holds by prev. thm.)

(ii') $\mathbb{P}[A \cap B] = \mathbb{P}[A]\, \mathbb{P}[B \mid X_m = i]$

(which hold by prev. thm. & (i))

Now, any A and B in (ii) can be written as a countable union of elementary A and B, and the general claim follows by summing over the identities for elementary A and B.

<u>Notation</u>: We regard distributions and measures $(\lambda_i)_{i \in I}$ as row vectors.

$\sum \lambda_i = 1$. $\lambda_i \geq 0$ for $i \in I$.

Matrix multiplication:

$$(\lambda P)_j = \sum_{i\in I} \lambda_i p_{ij}, \quad (P^2)_{ij} = \sum_{k\in I} p_{ik} p_{kj} = p^{(2)}_{ij}, \quad (P^3)_{ij} = \cdots$$

with $P^0 = 1$ the $I\times I$ identity matrix $1_{ij} = \delta_{ij}$.

When $\lambda_i > 0$, write $\mathbb{P}_i[A] = \mathbb{P}[A \mid X_0 = i]$.

<u>Fact</u>. By the Markov property, $(X_n)_{n\geq 0}$ is Markov(δ_i, P) under $\mathbb{P}_i$. (So the behaviour of (X_n) under $\mathbb{P}_i$ does not depend on λ.)

<u>Thm</u>. Let $(X_n)_{n\geq 0}$ be Markov(λ, P). Then for all $n, m \geq 0$,

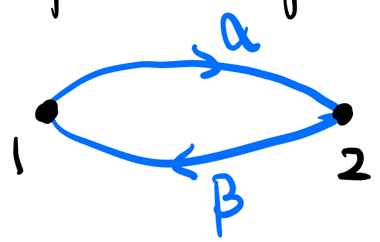
(a) $\mathbb{P}[X_n = j] = (\lambda P^n)_j$

(b) $\mathbb{P}_i[X_n = j] = p^{(n)}_{ij}$

<u>Proof</u>. (a)
$$\begin{aligned}\mathbb{P}[X_n = j] &= \sum_{i_0,\dots,i_{n-1}\in I} \mathbb{P}[X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = j] \\ &= \sum_{i_0,\dots,i_{n-1}} \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{n-2} i_{n-1}} p_{i_{n-1} j} \\ &= (\lambda P^n)_j .\end{aligned}$$

(b) Use the Markov property and $\lambda = \delta_i$ and (a).

Example The general two state Markov Chain is



$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \quad (\text{some } \alpha, \beta \in [0,1]).$$

$$P^{n+1} = P^n \cdot P \Rightarrow p_{11}^{(n+1)} = p_{12}^{(n)} \beta + p_{11}^{(n)} (1-\alpha)$$

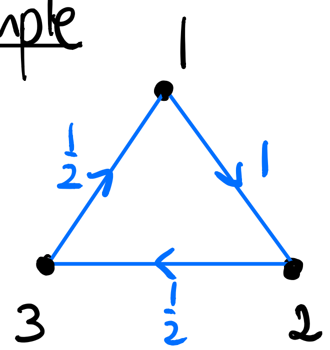
$$p_{12}^{(n)} + p_{11}^{(n)} = 1 \Rightarrow p_{11}^{(n+1)} = (1 - p_{11}^{(n)}) \beta + p_{11}^{(n)} (1-\alpha)$$

$$= p_{11}^{(n)} (1-\alpha-\beta) + \beta$$

Since $p_{11}^{(0)} = 1$, this recursion relation has unique soln:

$$p_{11}^{(n)} = \begin{cases} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \text{if } \alpha+\beta > 0 \\ 1 & \text{if } \alpha+\beta = 0. \end{cases}$$

Example



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

What is $p_{11}^{(n)}$?

<u>General method</u> to find $p_{ij}^{(n)}$ for an N state Markov Chain:

- Find the eigenvalues $\lambda_1, \dots, \lambda_N$ of P, i.e., roots of
  $\det(\lambda - P) = 0$.

- If all eigenvalues are distinct then $p_{ij}^{(n)}$ has the form
  $p_{ij}^{(n)} = a_1 \lambda_1^n + \cdots + a_N \lambda_N^n$ where the a_i are constants.

  If an eigenvalue λ is repeated once then the general form includes a term $(a+bn)\lambda^n$. Similar formulas hold for eigenvalues with higher multiplicity.

- As roots of a polynomial with real coefficients, any complex eigenvalues come in conjugate pairs. These are often best written in terms of sin and cos.

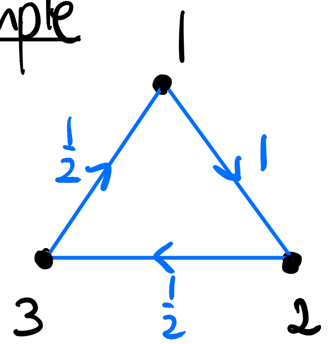
<u>Justification.</u> If P has distinct eigenvalues, then it can be diagonalised as

$$P = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U^{-1} \Rightarrow P^n = U \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_N^n \end{pmatrix} U^{-1}$$

$\Rightarrow p_{ij}^{(n)}$ is of the desired form.

If P has repeated eigenvalues, the more general claim can be seen from the Jordan normal form of P.

Example



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

What is $p_{11}^{(n)}$?

Eigenvalues: $0 = \det(\lambda - P) = \lambda(\lambda - \frac{1}{2})^2 - \frac{1}{4} = \frac{1}{4}(\lambda - 1)(4\lambda^2 + 1)$

$\Rightarrow \lambda = 1, \frac{i}{2}, -\frac{i}{2}$.

$\Rightarrow p_{11}^{(n)} = a + b\left(\frac{i}{2}\right)^n + c\left(\frac{-i}{2}\right)^n$ for some const. a, b, c.

$\left(\pm\frac{i}{2}\right)^n = \left(\frac{1}{2}\right)^n e^{\pm i\pi n/2} = \left(\frac{1}{2}\right)^n\left(\cos\left(\frac{1}{2}\pi n\right) \pm i \sin\left(\frac{1}{2}\pi n\right)\right)$

$\Rightarrow p_{11}^{(n)} = \alpha + \left(\frac{1}{2}\right)^n\left[\beta\cos\left(\frac{1}{2}\pi n\right) + \gamma\sin\left(\frac{1}{2}\pi n\right)\right]$

for some const. α, β, γ.

Note:

$$\left.\begin{aligned} 1 &= p_{11}^{(0)} = \alpha + \beta \\ 0 &= p_{11}^{(1)} = \alpha + \tfrac{1}{2}\gamma \\ 0 &= p_{11}^{(2)} = \alpha - \tfrac{1}{4}\beta \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= \tfrac{1}{5} \\ \beta &= \tfrac{4}{5} \\ \gamma &= -\tfrac{2}{5} \end{aligned}$$

$\Rightarrow p_{11}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n\left[\left(\frac{4}{5}\right)\cos\left(\frac{1}{2}\pi n\right) - \frac{2}{5}\sin\left(\frac{1}{2}\pi n\right)\right]$.

2. Class structure

<u>Defn.</u> For $i,j \in I$,

- i leads to j ($i \to j$) if $\mathbb{P}_i[X_n = j \text{ for some } n] > 0$,
- i communicates with j ($i \leftrightarrow j$) if $i \to j$ and $j \to i$.

<u>Thm.</u> For $i \neq j$ the following are equivalent:

(a) $i \to j$

(b) $p_{i_1 i_2} \cdots p_{i_{n-1} i_n} > 0$ for some $i_1, \dots, i_n$ with $i_1 = i$, $i_n = j$

(c) $p_{ij}^{(n)} > 0$ for some n.

<u>Proof.</u> Equivalence of (a) and (c) follows from

$$p_{ij}^{(n)} = \mathbb{P}_i[X_n = j] \leq \mathbb{P}_i[X_k = j \text{ for some } k] \leq \sum_{k=0}^{\infty} p_{ij}^{(k)}$$

Equivalence of (b) and (c) follows from

$$p_{ij}^{(n)} = \sum_{i_2 \cdots i_{n-1}} p_{i i_2} \cdots p_{i_{n-1} j}$$

<u>Prop.</u> The relation $i \leftrightarrow j$ is an equivalence relation.

<u>Proof.</u> We must show that $i \leftrightarrow j$ is reflexive, symmetric, and transitive. That $\leftrightarrow$ is is reflexive ($i \leftrightarrow i$) and symmetric ($i \leftrightarrow j$ implies $j \leftrightarrow i$) are clear from the definition. That $\leftrightarrow$ is transitive ($i \leftrightarrow j$ and $j \leftrightarrow k$ implies $i \leftrightarrow k$) follows from (b) of the thm.

Defn. The equivalence classes of $\leftrightarrow$ are called communicating classes. The chain is irreducible if there is a single communicating class, i.e., $i \leftrightarrow j$ for all $i, j \in I$.

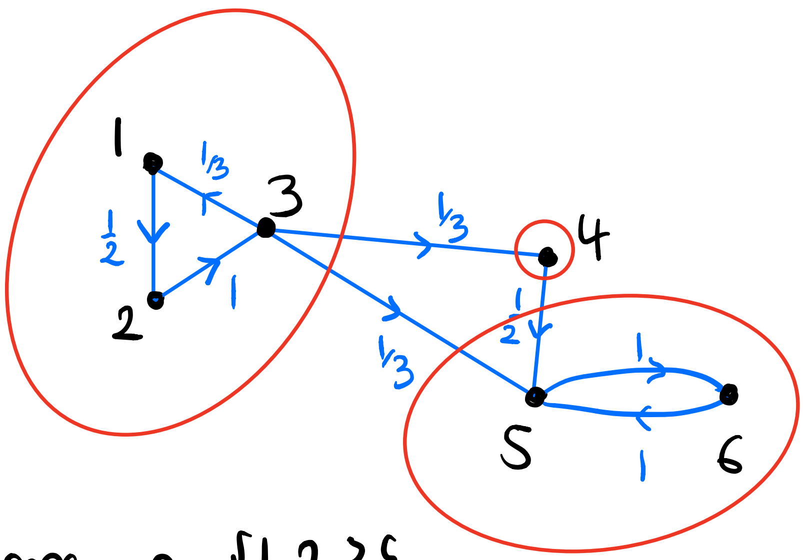
Defn. A subset $C \subset I$ is closed if

$$i \in C,\ i \to j \Rightarrow j \in C.$$

A state $i \in I$ is absorbing if $\{i\}$ is closed.

Example

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



The communicating classes are $\{1,2,3\}$, $\{4\}$, $\{5,6\}$.

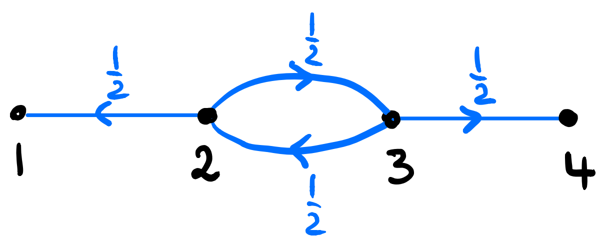
Only $\{5,6\}$ is closed.

3. Hitting and absorption probabilities

Defn Let (X_n) be a Markov Chain.

- The hitting time of a set $A \subset I$ is the random variable $H^A : \Omega \to \{0,1,2,\dots\} \cup \{+\infty\}$ given by
$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}, \quad \inf \emptyset = +\infty.$$
- The hitting probability of A is
$$h_i^A = \mathbb{P}_i[H^A < \infty] = \mathbb{P}_i[\text{hit } A].$$
If A is a closed class, h_i^A is called the absorption prob.
- The mean hitting time is the expected time to reach A
$$k_i^A = \mathbb{E}_i[H^A] = \mathbb{E}_i[\text{time to hit } A].$$

Example.



Starting from 2, what is the probability of absorption in 4? And how does it take until the chain is absorbed in 1 or 4?

Let $h_i = \mathbb{P}_i[\text{hit } 4]$ and $k_i = \mathbb{E}_i[\text{time to hit } 1 \text{ or } 4]$.

Note that

$$\begin{aligned} h_1 &= 0 & k_1 &= 0 \\ h_4 &= 1 & k_4 &= 0 \\ h_2 &= \tfrac{1}{2} h_1 + \tfrac{1}{2} h_3 & k_2 &= 1 + \tfrac{1}{2} k_1 + \tfrac{1}{2} k_3 \\ h_3 &= \tfrac{1}{2} h_2 + \tfrac{1}{2} h_4 & k_3 &= 1 + \tfrac{1}{2} k_2 + \tfrac{1}{2} k_4 \end{aligned}$$

$$\Rightarrow h_2 = \tfrac{1}{2}\left(\tfrac{1}{2} h_2 + \tfrac{1}{2}\right) = \tfrac{1}{4} h_2 + \tfrac{1}{4} = \tfrac{1}{3}$$

$$k_2 = 1 + \tfrac{1}{2}\left(1 + \tfrac{1}{2} k_2\right) = \tfrac{3}{2} + \tfrac{1}{4} k_2 = 2$$

__Thm.__ The vector of hitting probabilities $h^A = (h_i^A)_{i \in I}$ is the minimal nonnegative solution to

$$(*) \begin{cases} h_i^A = 1 & (i \in A) \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & (i \notin A). \end{cases}$$

typos in lecture!

Minimal means that if $x = (x_i)_{i \in I}$ is another soln. with $x_i \geq 0$ for all $i \in I$ then $h_i^A \leq x_i$ for all $i \in I$.

__Proof.__ Step 1: h^A is a solution to $(*)$.

If $X_0 = i \in A$ then clearly $H^A = 0$, so $h_i^A = 1$.

If $X_0 = i \in A$, then by the Markov property,

$$\mathbb{P}_i[H^A < \infty \mid X_1 = j] = \mathbb{P}_j[H^A < \infty] = h_j^A$$

$$\Rightarrow h_i^A = \mathbb{P}_i[H^A<\infty] = \sum_{j\in I} \underbrace{\mathbb{P}_i[H^A<\infty, X_1=j]}_{\underbrace{\mathbb{P}_i[H^A<\infty \mid X_1=j]}_{h_j^A}\,\underbrace{\mathbb{P}_i[X_1=j]}_{p_{ij}}} = \sum_j h_j^A p_{ij}.$$

$\Rightarrow h^A$ is a solution to (*).

Step 2. h^A is minimal.
Let x be any nonnegative solution to (*). If $i\in A$ clearly $h_i^A = 1 = x_i$. So suppose $i\notin A$. Then

$$x_i = \sum_{j\in I} p_{ij}x_j = \sum_{j\in A} p_{ij} \underbrace{\cancel{x_j}}_{1} + \sum_{j\notin A} p_{ij}x_j$$

$$= \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij}\left(\sum_{k\in A} p_{jk} + \sum_{k\notin A} p_{jk}x_k\right)$$

$$= \mathbb{P}_i[X_1\in A] + \mathbb{P}_i[X_1\notin A, X_2\in A] + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}x_k$$

By repeated substitution,

$$x_i = \underbrace{\mathbb{P}_i[X_1\in A] + \mathbb{P}_i[X_1\notin A, X_2\in A] + \mathbb{P}[X_1\notin A, X_2\notin A, X_3\in A] + \cdots + \mathbb{P}[X_1\notin A, \ldots, X_{n-1}\notin A, X_n\in A]}_{\mathbb{P}_i[H^A\le n]}$$
$$+ \underbrace{\sum_{j_1\notin A}\cdots\sum_{j_n\notin A} p_{ij_1}p_{j_1j_2}\cdots p_{j_{n-1}j_n}x_{j_n}}_{\ge 0 \text{ since } x \text{ is nonneg.}}.$$

$\Rightarrow x_i \ge \mathbb{P}_i[H^A\le n]$ for all n

$\Rightarrow x_i \ge \lim_{n\to\infty} \mathbb{P}_i[H^A\le n] = \mathbb{P}_i[H^A<\infty] = h_i^A$

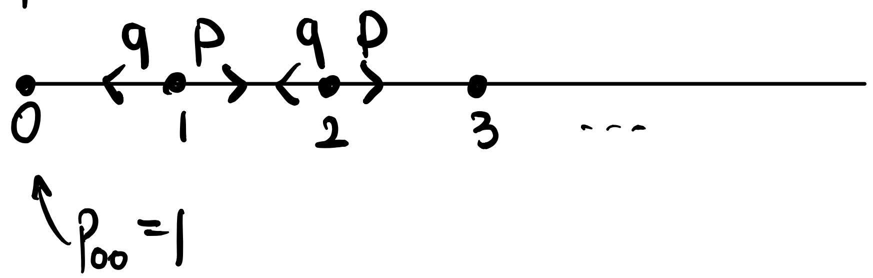
$\Rightarrow$ H^A is minimal.

<u>Example</u> (cont'd). Recall that $h = h^A$ with $A = \{4\}$.

$$(*) \begin{cases} h_1 = h_1 \\ h_4 = 1 \\ h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3 \\ h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4 \end{cases}$$

The system $(*)$ does <u>not</u> determine h_1, but by the minimality condition we must choose $h_1 = 0$ (as last lecture). So we find the same solution.

<u>Example</u> (Gambler's ruin).



$0 < p = 1 - q < 1$

Starting with a fortune of i£, what is the probability of leaving broke? I.e., what is $h_i = \mathbb{P}_i[\text{hit } 0]$.

By the theorem, $\begin{cases} h_0 = 1 \\ h_i = p\,h_{i+1} + q\,h_{i-1} \quad (i = 1, 2, 3, \dots). \end{cases}$

Assume $p \neq q$. The general solution to the recursion is
$$h_i = A + B\left(\frac{q}{p}\right)^i.$$

If $p<q$ (most casinos). $0 \le h_i \le 1$ for all $i \Rightarrow B=0, A=1$
$\Rightarrow h_i = 1$ for all i.

If $p>q$: $h_0 = 1 \Rightarrow B = 1-A \Rightarrow h_i = \left(\frac{q}{p}\right)^i + A\left(1-\left(\frac{q}{p}\right)^i\right)$

$h_i \ge 0$ for all $i \Rightarrow A \ge 0$

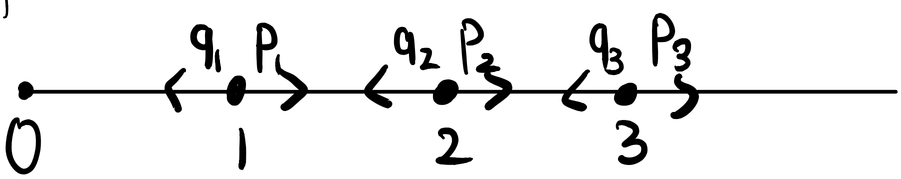
minimality $\Rightarrow A = 0 \Rightarrow h_i = \left(\frac{q}{p}\right)^i$.

If $p=q$ (fair casino), the general solution to the recursion is

$$h_i = A + Bi.$$

$$\left.\begin{array}{l} 0 \le h_i \le 1 \Rightarrow B=0 \\ h_0 = 1 \Rightarrow A = 1 \end{array}\right\} \Rightarrow h_i = 1 \text{ for all } i.$$

<u>Example</u> (Birth and death chain).



$h_i = \mathbb{P}_i[\text{hit } 0]$ is the extinction probability from i.

$$(*)\begin{cases} h_0 = 1 \\ h_i = p_i h_{i+1} + q_i h_{i-1} \qquad (i=1,2,\dots) \end{cases}$$

Consider $u_i = h_{i-1} - h_i$. Then

$$p_i u_{i+1} - q_i u_i = p_i h_i \underbrace{- p_i h_{i+1} - q_i h_{i-1}}_{-h_i} + q_i h_i$$

typo in video

$$= (p_i + q_i - 1) h_i = 0.$$

$$\Rightarrow u_{i+1} = \frac{q_i}{p_i} u_i = \underbrace{\left(\frac{q_i q_{i-1} \cdots q_1}{p_i p_{i-1} \cdots p_1}\right)}_{\gamma_i} u_1 = \gamma_i u_1 .$$

$$\Rightarrow h_i = 1 - (\underbrace{\overset{1}{\overset{\shortparallel}{h_0}} - h_i}_{u_1 + \cdots + u_i}) = 1 - \underset{\text{unknown}}{A}(\gamma_0 + \cdots + \gamma_{i-1}), \quad \gamma_0 = 1, \quad A = u_1$$

If $\sum_{i=0}^{\infty} \gamma_i = \infty$: $0 \le h_i \le 1 \Rightarrow A = 0 \Rightarrow h_i = 1$ for all i

If $\sum_{i=0}^{\infty} \gamma_i < \infty$: minimal solution is $A = \left(\sum_{i=0}^{\infty} \gamma_i\right)^{-1}$

$$\Rightarrow h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j} .$$

Since for any i, we have $h_i < 1$, the population survives with positive probability.

Thm. The vector of mean hitting times $k^A = (k_i^A)_{i \in I}$ is the minimal nonnegative solution to

$$(+) \begin{cases} k_i^A = 0 & (i \in A) \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & (i \notin A) \end{cases}$$

Proof. Step 1. k^A satisfies (+).

If $X_0 = i \in A$, then $H^A = 0$ so clearly $k_i^A = \mathbb{E}_i[H^A] = 0$.
If $X_0 = i \notin A$, then $H^A \geq 1$, so by the Markov prop.,

$$\mathbb{E}_i[H^A \mid X_1 = j] = 1 + \mathbb{E}_j[H^A] = 1 + k_j^A$$

$$\Rightarrow k_i^A = \mathbb{E}_i[H^A] = \sum_{j \in I} \underbrace{\mathbb{E}_i[H^A \mid X_1 = j]}_{(1+k_j^A)} \underbrace{\mathbb{P}_i[X_1 = j]}_{p_{ij}} = 1 + \sum_{j \notin A} p_{ij} k_j^A$$

Step 2. k^A is minimal.

Suppose x is any nonneg. soln. to (+). Then $x_i = k_i^A = 0$ for all $i \in A$. For $i \notin A$,

$$x_i = 1 + \sum_{j \notin A} p_{ij} x_j = 1 + \sum_{j \notin A} p_{ij} \Big(1 + \sum_{k \notin A} p_{jk} x_k \Big)$$

$$= \mathbb{P}_i[H^A \geq 1] + \mathbb{P}_i[H^A \geq 2] + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} x_k$$

Again, by repeated substitution, for any n,

$$x_i = \mathbb{P}_i[H^A \geq 1] + \cdots + \mathbb{P}_i[H^A \geq n] + \underbrace{\sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} \cdots p_{j_{n-1} j_n} x_{j_n}}_{\geq 0}$$

$$\Rightarrow x_i \geq \sum_{n=1}^{\infty} \mathbb{P}_i[H^A \geq n] = \mathbb{E}_i[H^A] = k_i^A$$

Thus k^A is the minimal solution.

4. Strong Markov Property

Defn. A random variable $T : \Omega \to \{0,1,2,\dots\} \cup \{+\infty\}$ is a stopping time if the event $\{T=n\}$ only depends on $X_0, \dots, X_n$ for $n=0,1,2,\dots$.

Examples. (a) The first passage time

$$T_j = \inf\{n \geq 1 : X_n = j\}$$

is a stopping time since $\{T_j = n\} = \{X_1 \neq j, \dots, X_{n-1} \neq j, X_n = j\}$.

(b) The hitting time H^A of a set A is a stopping time $\{H^A = n\} = \{X_0 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}$.

(c) The last exit time of a set A

$$L^A = \sup\{n \geq 0 : X_n \in A\}$$

is in general not a stopping time because $\{L^A = n\}$ depends on whether $(X_{n+m})_{m \geq 1}$ visits A or not.

Thm (Strong Markov Property). Let $(X_n)_{n \geq 0}$ be Markov(λ, P) and let T be a stopping time for (X_n). Then, conditional on $T < \infty$ and $X_T = i$, $(X_{T+n})_{n \geq 0}$ is Markov(δ_i, P) and independent of $X_0, \dots, X_T$.

Proof. Let B be an event determined by $X_0, \dots, X_T$. Then $B \cap \{T=m\}$ is determined by $X_0, \dots, X_m$. So by the (usual) Markov property

$$\mathbb{P}[\{X_T = j_0, \dots, X_{T+n} = j_n\} \cap B \cap \{T = m\} \cap \{X_T = i\}]$$
$$= \mathbb{P}[X_0 = j_0, \dots, X_n = j_n]\, \mathbb{P}[B \cap \{T=m\} \cap \{X_T = i\}]$$

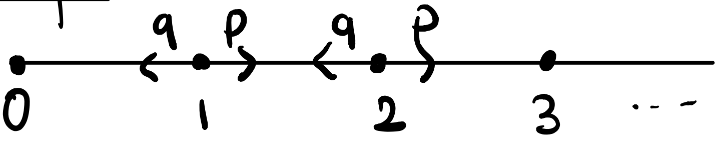
Summing over m gives

$$\mathbb{P}[\{X_T = j_0, \dots, X_{T+n} = j_n\} \cap B \cap \{T < \infty\} \cap \{X_T = i\}]$$
$$= \mathbb{P}[X_0 = j_0, \dots, X_n = j_n]\, \mathbb{P}[B \cap \{T<\infty\} \cap \{X_T = i\}]$$

Dividing by $\mathbb{P}[T<\infty, X_T = i]$ (if it is positive) gives

$$\mathbb{P}[\{X_T = j_0, \dots, X_{T+n} = j_n\} \cap B \mid T < \infty, \quad X_T = i]$$
$$= \mathbb{P}[X_0 = j_0, \dots, X_n = j_n]\, \mathbb{P}[B \mid T<\infty, \quad X_T = i]$$

<u>Example</u> (Gambler's ruin cont'd).



$0 < p = 1-q < 1$

We have previously found $\mathbb{P}_i[\text{hit } 0]$. We now find the distribution of time to hit 0 starting from 1.

Let $H_j = \inf\{n \geq 0 : X_n = j\}$

$$\phi(s) = \mathbb{E}_1[s^{H_0}] = \mathbb{E}_1[s^{H_0} \mathbb{1}_{H_0 < \infty}] \qquad (s \in [0,1))$$
$$= \sum_{n=0}^{\infty} s^n\, \mathbb{P}[H_0 = n]$$

Claim: $\mathbb{E}_2[s^{H_0}] = \phi(s)^2$.

Conditional on $H_1 < \infty$ under $\mathbb{P}_2$, we can write $H_0 = H_1 + \tilde{H}_0$ where $\tilde{H}_0$ is time it takes after time H_1 to reach state 0. Since H_1 is a stopping time, by the strong Markov property at H_1, $\tilde{H}_0$ is independent of H_1 (as it only depends on $(X_{H_1+n})_{n\geq 0}$).

$$\Rightarrow \mathbb{E}_2[s^{H_0}] = \mathbb{E}_2[s^{H_1} \mid H_1 < \infty]\, \mathbb{E}_2[s^{\tilde{H}_0} \mid H_1 < \infty]\, \mathbb{P}[H_1 < \infty]$$

($\tilde{H}_0$ is cond. ind. from H_1)

$$= \underbrace{\mathbb{E}_2[s^{H_1} 1_{H_1 < \infty}]}_{\mathbb{E}_1[s^{H_0}]}\, \underbrace{\mathbb{E}_2[s^{\tilde{H}_0} \mid H_1 < \infty]}_{\mathbb{E}_1[s^{H_0}]} = \phi(s)^2$$

Claim: $ps\,\phi(s)^2 - \phi(s) + qs = 0$

Conditional on $X_1 = 2$, we have $H_0 = 1 + \bar{H}_0$ where $\bar{H}_0$ is the it takes after 1 step to reach 0. By Markov property, $\bar{H}_0$ under $\mathbb{P}[\,\cdot \mid X_2 = 2]$ has the same distribution as H_0 under $\mathbb{P}_2$.

$$\Rightarrow \phi(s) = \mathbb{E}_1[s^{H_0}] = p\,\mathbb{E}_1[s^{H_0} \mid X_1 = 2] + q\,\mathbb{E}_1[s^{H_0} \mid X_1 = 0].$$

$$= p\,\mathbb{E}_1[s^{1+\bar{H}_0} \mid X_1 = 2] + qs$$

typo in videos

$$= ps \underbrace{\mathbb{E}_1[s^{\bar{H}_0} \mid X_1 = 2]}_{\mathbb{P}_2[s^{H_0}] = \phi(s)^2} + qs$$

$$= ps\phi(s)^2 + qs$$

$$\Rightarrow \phi(0) = 0 \text{ and } \phi(s) = \frac{1 \pm \sqrt{1-4pqs^2}}{2ps} \text{ for } s > 0.$$

Since $\phi(s) \le 1$ and since $\phi(s)$ is continuous, only the negative root is possible for all $s \in [0,1)$.

$$\Rightarrow \phi(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}$$

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots$$

$$= \frac{1}{2ps}\left[\cancel{1} - \left(\cancel{1} + \tfrac{1}{2}(-4pqs^2) - \tfrac{1}{8}(4pqs^2)^2 + \cdots\right)\right]$$

$$= qs + pq^2s^3 + \cdots$$

$$= s\,\mathbb{P}[H_0 = 1] + s^2\,\mathbb{P}[H_0 = 2] + s^3\,\mathbb{P}[H_0 = 3] + \cdots$$

$$\Rightarrow \mathbb{P}[H_0 = 1] = q$$

$$\mathbb{P}[H_0 = 2] = 0$$

$$\vdots$$

As $s \uparrow 1$, we have $\phi(s) \to \mathbb{P}_1[H_0 < \infty]$.

$$\Rightarrow \mathbb{P}_1[H_0<\infty] = \frac{1-\sqrt{1-4pq}}{2p} = \begin{cases} 1 & \text{if } p\le q \\ \frac{q}{p} & \text{if } p>q. \end{cases}$$

$$\sqrt{1-4pq} = |1-2q|$$

Also, if $p \le q$,

$$\mathbb{E}_1[H_0] = \mathbb{E}_1[H_0 \mathbb{1}_{H_0<\infty}] = \lim_{s\uparrow 1} \phi'(s)$$

Differentiating the quadratic equation gives

$$2ps\,\phi(s)\,\phi'(s) + p\phi(s)^2 + \phi'(s) + q = 0$$

$$\Rightarrow \phi'(s) = \frac{p\phi(s)^2+q}{1-2ps\phi(s)} \xrightarrow{s\uparrow 1} \frac{1}{1-2p} = \frac{1}{q-p}$$

$$\Rightarrow \mathbb{E}_1[H_0] = \frac{1}{q-p}$$

5. Recurrence and transience

<u>Defn.</u> Let (X_n) be a Markov Chain. A state $i \in I$ is

- **recurrent** if $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 1$
- **transient** if $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 0$.

First passage time to j: $T_j = \inf\{n \geq 1 : X_n = j\}$.

<u>Thm.</u> The following dichotomy holds:

(a) If $\mathbb{P}_i[T_i < \infty] = 1$ then i is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

(b) If $\mathbb{P}_i[T_i < \infty] < 1$ then i is transient and $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$.

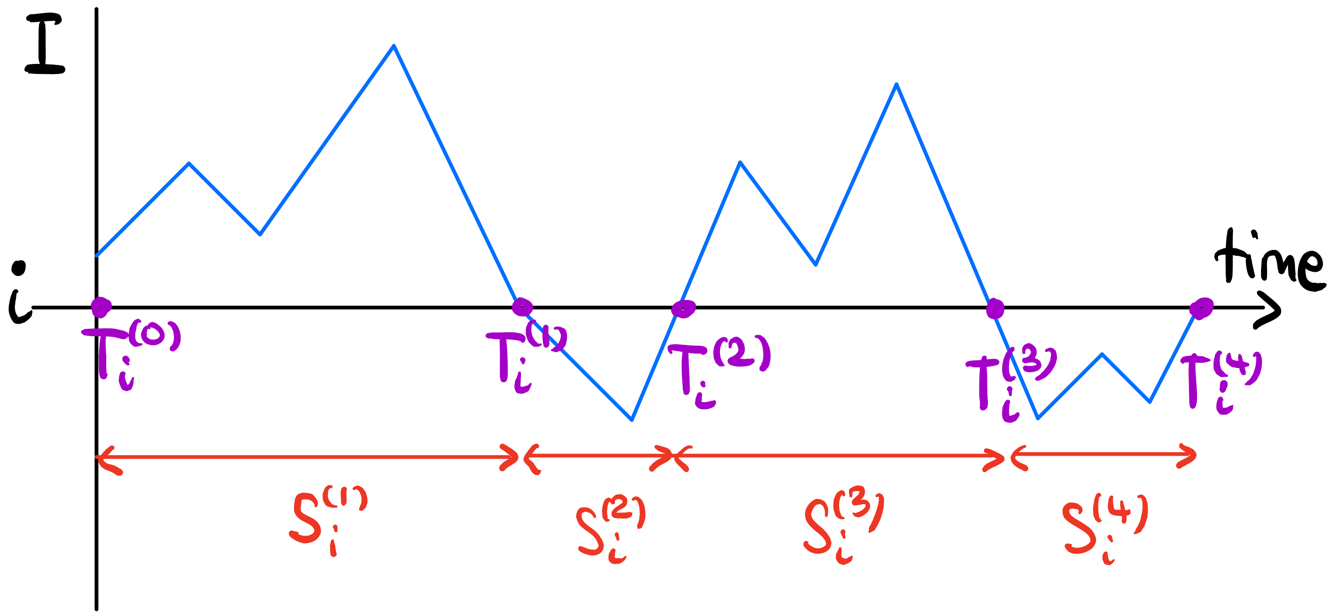
In particular, every state is either recurrent or transient.

Step 1. Inductively, define the **rth passage time** to j:

$$T_j^{(0)} = 0, \quad T_j^{(1)} = T_j, \quad T_j^{(r+1)} = \inf\{n \geq T_j^{(r)} + 1 : X_n = j\}.$$

The length of the **rth excursion** is defined by

$$S_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)} & \text{if } T_i^{(r-1)} < \infty \\ 0 & \text{otherwise.} \end{cases}$$



<u>Lemma</u>. For $r=2,3,\dots$, conditional on $T_i^{(r-1)}<\infty$, the length of the r^{th} excursion $S_i^{(r)}$ is independent of $\{X_m : m < T_i^{(r-1)}\}$ and

$$\mathbb{P}[S_i^{(r)}=n \mid T_i^{(r-1)}<\infty] = \mathbb{P}_i[T_i=n].$$

<u>Proof</u>. By the strong Markov property, conditional on $T_i^{(r-1)}<\infty$, $(X_{T_i^{(r-1)}+n})_{n\geq 0}$ is Markov(δ_i, P) and is independent of $X_0,\dots,X_{T_i^{(r-1)}}$. Now

$$S_i^{(r)} = \inf\{n\geq 1 : X_{T_i^{(r-1)}+n}=i\}$$

is the first passage time of $(X_{T_i^{(r-1)}+n})_{n\geq 0}$ to state i.

Step 2. Let V_i denote the number of visits to i:

$$V_i = \sum_{n=0}^{\infty} 1_{X_n=i}.$$

Then

$$\mathbb{E}_i V_i = \mathbb{E}_i\left[\sum_{n=0}^{\infty} 1_{X_n = i}\right] = \sum_{n=0}^{\infty} \mathbb{P}_i[X_n = i] = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

Let f_i be the return probability to i:

$$f_i = \mathbb{P}_i[T_i < \infty].$$

Lemma. For $r = 0, 1, 2, \ldots$, we have $\mathbb{P}_i[V_i > r] = f_i^r$

Proof. Note that $\{V_i > r\} = \{T_i^{(r)} < \infty\}$ if $X_0 = i$. Also note that $\mathbb{P}_i[V_i > 0] = 1$. By induction,

$$\begin{aligned}\mathbb{P}_i[V_i > r+1] &= \mathbb{P}_i[T_i^{(r+1)} < \infty] \\ &= \mathbb{P}_i[T_i^{(r)} < \infty,\ S_i^{(r+1)} < \infty] \\ &= \underbrace{\mathbb{P}_i[T_i^{(r)} < \infty]}_{f_i^r}\ \underbrace{\mathbb{P}[S_i^{(r+1)} < \infty \mid T_i^{(r)} < \infty]}_{f_i} = f_i^{r+1}\end{aligned}$$

Proof of theorem. (a) If $\mathbb{P}_i[T_i < \infty] = 1$, then by the last lemma,

$$\mathbb{P}_i[V_i = \infty] = \lim_{r \to \infty} \mathbb{P}_i[V_i > r] = 1.$$

So i is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i V_i = \infty$.

(b) If $\mathbb{P}_i[T_i < \infty] < 1$, then

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i V_i = \sum_{r=0}^{\infty} \mathbb{P}_i[V_i > r] = \sum_{r=0}^{\infty} f_i^r = \frac{1}{1-f_i} < \infty.$$

So $\mathbb{P}_i[V_i = \infty] = 0$, so is transient.

Thm. Recurrence and transience are class properties: For any communicating class C, either all states $i \in C$ are recurrent or all are transient.

Proof. Let $i, j \in C$ and assume that i is transient. Since i and j communicate, there exist n, m s.t.

$$p_{ij}^{(n)} > 0 \quad \text{and} \quad p_{ji}^{(m)} > 0.$$

For all $r \geq 0$, then

$$p_{ii}^{(n+m+r)} \geq p_{ij}^{(n)} p_{jj}^{(r)} p_{ji}^{(m)}$$

$$\Rightarrow \sum_{r=0}^{\infty} p_{jj}^{(r)} \leq \frac{1}{p_{ij}^{(n)} p_{ji}^{(m)}} \sum_{r=0}^{\infty} p_{ii}^{(n+m+r)} < \infty$$

So j is transient as well.

$i \in C, i \to j \Rightarrow j \in C$

Thm. Every recurrent class is closed.

Proof. Let C be a class that is not closed, i.e., there is $i \in C$, $j \notin C$ and $m \geq 1$ s.t.

$$\mathbb{P}_i[X_m = j] > 0.$$

Since C is a communicating class and $j \notin C$,

$$\mathbb{P}_i[\{X_m = j\} \cap \{X_n = i \text{ for infinite many } n\}] = 0$$

$\Rightarrow \mathbb{P}_i[X_n = i \text{ for infinitely many } n]$

$$= \sum_{j\in I} \mathbb{P}_i[X_n = i \text{ for infinitely many } n, X_m = j]$$

$$< \sum_{j\in I} \mathbb{P}_i[X_m = j] = 1$$

Thus i is not recurrent and since recurrence is a class property this means that C is not recurrent (i.e. transient).

<u>Thm</u>. Every finite closed class is recurrent.

Careful: Infinite closed classes may be transient.

<u>Proof</u>. Let C be a finite closed class and suppose $X_0 \in C$.

$\Rightarrow 0 < \mathbb{P}[X_n = i \text{ for infinitely many } n]$ for some $i \in C$

$$= \mathbb{P}[X_n = i \text{ for some } n] \, \mathbb{P}_i[X_n = i \text{ for infinitely many } n].$$

by the strong Markov prop.

$\Rightarrow \mathbb{P}_i[X_n = i \text{ for infinitely many } n] > 0$

$\Rightarrow i$ is not transient $\Rightarrow i$ is recurrent

<u>Cor</u>. Finite classes are recurrent iff closed.

<u>Thm.</u> Suppose P is irreducible and recurrent. Then for all $j \in I$, $\mathbb{P}[T_j < \infty] = 1$.

<u>Proof.</u> It suffice to show that $\mathbb{P}_i[T_j < \infty] = 1$ for all $i \in I$ since then

$$\mathbb{P}[T_j < \infty] = \sum_i \mathbb{P}[X_0 = i] \mathbb{P}_i[T_j < \infty] = 1$$

Since P is irreducible, there is m s.t. $p_{ji}^{(m)} > 0$.

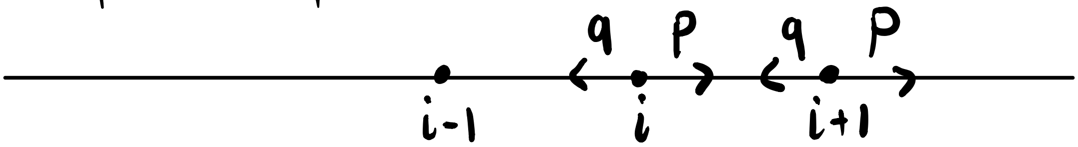
Since P is recurrent,

$$\begin{aligned} 1 &= \mathbb{P}_j[X_n = j \text{ for infinitely many } n] \\ &= \mathbb{P}_j[X_n = j \text{ for some } n \geq m+1] \\ &= \sum_{k \in I} \underbrace{\mathbb{P}_j[X_n = j \text{ for some } n \geq m+1 \mid X_m = k]}_{\substack{\mathbb{P}_k[X_n = j \text{ for some } n \geq 1] \\ = \mathbb{P}_k[T_j < \infty]}} \underbrace{\mathbb{P}_j[X_m = k]}_{p_{jk}^{(m)}} \end{aligned}$$

$\Rightarrow \mathbb{P}_i[T_j < \infty] = 1$ since $\sum_k p_{jk}^{(m)} = 1$ and $p_{ji}^{(m)} > 0$

6. Recurrence and transience of random walks

Example (Simple random walk on $\mathbb{Z}$).



$0<p=1-q<1$

$p_{00}^{(2n+1)}=0$ since the walk cannot return to 0 after an odd number of steps

$$p_{00}^{(2n)}=\binom{2n}{n}p^nq^n=\frac{(2n)!}{(n!)^2}p^nq^n$$

Stirling's formula: $n! \sim \sqrt{2\pi n}\, e^{-n} n^n$

where $A_n \sim B_n$ if $\lim_{n\to\infty} \frac{A_n}{B_n}=1$.

$$\Rightarrow p_{00}^{(2n)} \sim \frac{\sqrt{4\pi n}}{2\pi n}\frac{(2n)^{2n}}{n^{2n}}(pq)^n=\frac{c}{\sqrt{n}}(4pq)^n$$

Case $p=q=\frac{1}{2}$. $p_{00}^{(2n)} \sim \frac{c}{\sqrt{n}} \Rightarrow p_{00}^{(2n)} \geq \frac{c}{2\sqrt{n}}$ for $n \geq n_0$

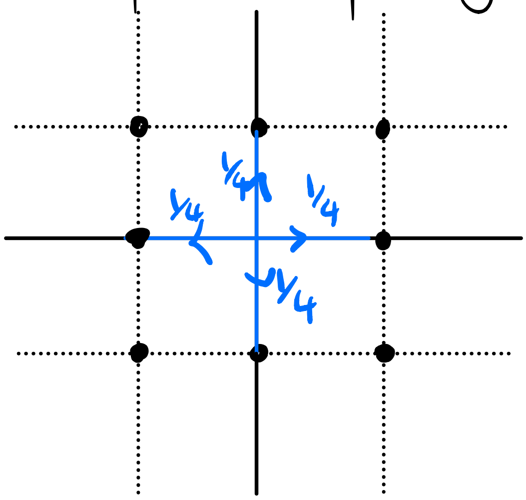
$$\Rightarrow \sum_{n=0}^{\infty} p_{00}^{(n)} \geq \sum_{n=n_0}^{\infty} p_{00}^{(2n)} \geq \frac{c}{2}\sum_{n=n_0}^{\infty} n^{-1/2}=\infty$$

$\Rightarrow$ Random walk is recurrent.

Case $p\neq q$. $r=4pq<1 \Rightarrow p_{00}^{(2n)} \leq r^n$ for $n \geq n_0$

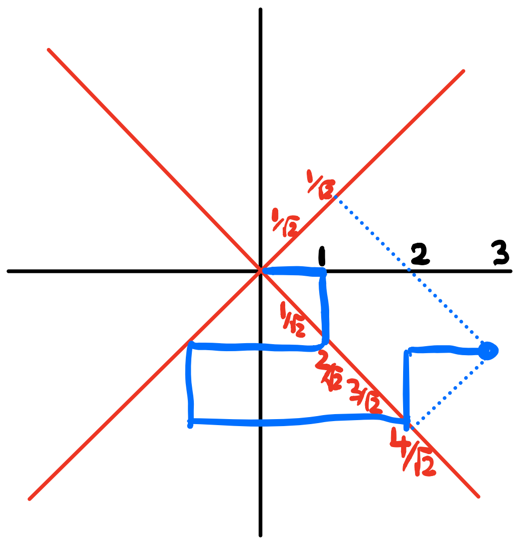
$$\Rightarrow \sum_{n=0}^{\infty} p_{00}^{(2n)}<\infty$$ $\Rightarrow$ Random walk is transient.

Example (Simple symmetric random walk on $\mathbb{Z}^2$)



$$p_{ij} = \begin{cases} 1/4 & \text{if } |i-j|=1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X_0 = 0$ and write $X_n^{\pm}$ for the orthogonal projections onto the lines $y = \pm x$.



Observation: $X_n^{\pm}$ are independent simple symmetric random walks on $\frac{1}{\sqrt{2}}\mathbb{Z}$ and $X_0 = 0$ iff $X_0^{\pm} = 0$.

$$\Rightarrow p_{00}^{(2n)} = \left(\binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right)^2 \sim \frac{c}{n}$$

both X^+ and X^- must take $2n$ steps if X does and must return to 0

$$\Rightarrow \sum_{n=0}^{\infty} p_{00}^{(2n)} = \infty \Rightarrow \text{The random walk is recurrent.}$$

<u>Example</u> (Simple symmetric random walk on $\mathbb{Z}^3$).

$$p_{ij} = \begin{cases} 1/6 & \text{if } |i-j|=1 \\ 0 & \text{otherwise} \end{cases}$$

We will show the random walk is transient.

Again $p_{00}^{(2n+1)} = 0$.

All walks from 0 to 0 must take the same number of steps in direction $(1,0,0)$ as in direction $(-1,0,0)$, and analogously for the other two coordinates.

$$\Rightarrow p_{00}^{(2n)} = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \underbrace{\frac{2n!}{i!\,i!\,j!\,j!\,k!\,k!}}_{\binom{2n}{i\ i\ j\ j\ k\ k}} \left(\frac{1}{6}\right)^{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \Bigg(\underbrace{\frac{n!}{i!\,j!\,k!}}_{\binom{n}{i\ j\ k}}\Bigg)^2 \left(\frac{1}{3}\right)^{2n}$$

<u>Fact 1.</u> If $n = 3m$ then $\binom{n}{i\ j\ k} \leq \binom{n}{m\ m\ m}$ for i,j,k.

(Suppose the maximal $\binom{n}{i\ j\ k}$ has $i > j+1$.
Then $i!\,j! > (i-1)!\,(j+1)!$.
Thus $\binom{n}{i\ j\ k} < \binom{n}{i-1\ \ j+1\ \ k}$ so $\binom{n}{i\ j\ k}$ wasn't max.)

<u>Fact 2.</u> $\displaystyle\sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!}\left(\frac{1}{3}\right)^n = 1$

(The LHS is the total prob. of distributing three balls in three bins.)

$$\Rightarrow p_{00}^{(2n)} \le \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\binom{3m}{m\ m\ m}\left(\frac{1}{3}\right)^{3m} \underset{\uparrow}{\sim} C\frac{\sqrt{n}}{\sqrt{n}^2}\frac{\sqrt{n}}{\sqrt{n}^3} = Cn^{-\frac{3}{2}}$$

provided $n = 3m$ Stirling

Since $p_{00}^{(2n)} \ge \left(\frac{1}{6}\right)^2 p_{00}^{(2n-2)}$ up to changing C,

$p_{00}^{(2n)} \le Cn^{-3/2}$ for all n.

$$\Rightarrow \sum_n p_{00}^{(2n)} \le C\sum_n n^{-3/2} < \infty.$$

$\Rightarrow$ The random walk is transient.

7. Invariant measures

$\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i \geq 0$ for all $i \in I$

Defn. A measure λ is invariant (or stationary or in equilibrium) if

$$\lambda P = \lambda$$

Thm Let $(X_n)_{n \geq 0}$ be Markov(λ, P) and suppose that λ is invariant for P. Then $(X_{n+m})_{n \geq 0}$ is also Markov(λ, P).

Proof.

$$\mathbb{P}[X_m = i] = (\lambda P^m)_i = \lambda_i \quad \text{for all } i \in I$$

so the initial distribution of $(X_{n+m})_{n \geq 0}$ is λ.

Also, conditional on $X_{n+m} = i$, by the Markov prop. for (X_n), X_{n+m+1} is independent $X_m, X_{m+1}, \dots, X_{n+m}$ and it has distribution $(p_{ij})_{j \in I}$.

Thm. Suppose I is finite. For some $i \in I$, suppose

$$p_{ij}^{(n)} \longrightarrow \pi_j \quad \text{as } n \to \infty, \quad \text{for all } j \in I.$$

Then $(\pi_j)_j$ is an invariant distribution.

Proof. (π) is a distribution:

$$\sum_{j\in I} \pi_j = \sum_{j\in I} \lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} \sum_{j\in I} p_{ij}^{(n)} = 1$$

I is finite!

(π) is invariant:

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n+1)} = \lim_{n\to\infty} \sum_{k\in I} p_{ik}^{(n)} p_{kj} = \sum_{k\in I} \underbrace{\lim_{n\to\infty} p_{ik}^{(n)}}_{\pi_k} p_{kj} = (\pi P)_j$$

Rk. For the simple symmetric random walk on $\mathbb{Z}^d$, we have $p_{ij}^{(n)} \to 0$ as $n\to\infty$, for all $i,j\in\mathbb{Z}^d$. The limit 0 is invariant, but not a distribution.

Example
$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

We found earlier that

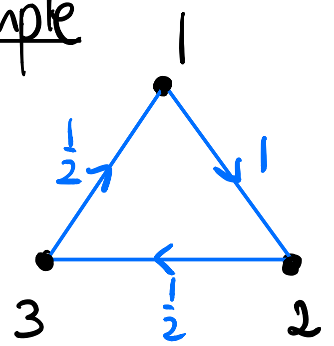
$$p_{11}^{(n)} = \begin{cases} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \text{if } \alpha+\beta>0 \\ 1 & \text{otherwise} \end{cases}$$

So if $\alpha+\beta \notin \{0,1\}$, we have $p_{11}^{(n)} \to \frac{\beta}{\alpha+\beta}$. Similarly,

$$P^n \longrightarrow \begin{bmatrix} \beta/\alpha+\beta & \alpha/\alpha+\beta \\ \beta/\alpha+\beta & \alpha/\alpha+\beta \end{bmatrix}.$$

So by the theorem, $(\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$ is an invariant distribution.

Example



$$\pi P = \pi \iff \begin{cases} \pi_1 = \frac{1}{2}\pi_3 \\ \pi_2 = \pi_1 + \frac{1}{2}\pi_2 \\ \pi_3 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 \end{cases}$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \pi_3 = \frac{2}{5}$$
$$\pi_1 = \frac{1}{5}$$
$$\pi_2 = \frac{2}{5}$$

For each state $k \in I$, let γ_i^k be the expected time spent in the state i between two visits to k:

$$\gamma_i^k = \mathbb{E}_k \sum_{n=0}^{T_k - 1} 1_{X_n = i}$$
$$= \mathbb{E}_k \sum_{n=1}^{T_k} 1_{X_n = i} \qquad \text{if } k \neq i$$

Thm. Let P be irreducible and recurrent. Then

(a) $\gamma_k^k = 1$;

(b) $\gamma^k = (\gamma_i^k)_{i\in I}$ is an invariant measure: $\gamma^k P = \gamma^k$.

(c) $0 < \gamma_i^k < \infty$ for all $i \in I$.

Proof. (a) obvious from definition.

(b) Since P is recurrent, $\mathbb{P}_k[T_k < \infty, X_0 = X_{T_k} = k] = 1$.

$$\Rightarrow \gamma_j^k = \mathbb{E}_k \sum_{n=1}^{T_k} 1_{X_n = j}$$

$$= \mathbb{E}_k \sum_{n=1}^{\infty} 1_{X_n = j \text{ and } n \le T_k}$$

$$= \sum_{n=1}^{\infty} \mathbb{P}_k[X_n = j, n \le T_k]$$

$$= \sum_{i\in I} \sum_{n=1}^{\infty} \underbrace{\mathbb{P}_k[X_{n-1} = i, X_n = j, n \le T_k]}_{\mathbb{P}_k[X_{n-1}=i,\, n\le T_k]\, \underbrace{\mathbb{P}[X_n = j \mid X_{n-1} = i]}_{p_{ij}}}$$

by the Markov prop. since $n \le T_k$ only depends on $X_0, \dots, X_{n-1}$.

$$\gamma_j^k = \sum_{i\in I} p_{ij} \sum_{n=1}^{\infty} \underbrace{\mathbb{P}_k[X_{n-1} = i, n \le T_k]}_{\mathbb{E}_k[1_{X_{n-1} = i,\, n \le T_k}]}$$

$$= \sum_{i \in I} p_{ij} \underbrace{\mathbb{E}_k\left[\sum_{n=0}^{T_k - 1} 1_{X_n = i}\right]}_{\gamma_i^k} = \sum_{i \in I} p_{ij} \gamma_i^k = (\gamma^k P)_j$$

(c) P irreducible $\Rightarrow \exists n, m \geq 0$ s.t. $p_{ik}^{(n)} > 0$, $p_{ki}^{(m)} > 0$.

$$\Rightarrow \gamma_i^k \overset{(b)}{\geq} \gamma_k^k p_{ki}^{(m)} \overset{(a)}{=} p_{ki}^{(m)} > 0$$

$$1 \overset{(a)}{=} \gamma_k^k \overset{(b)}{\geq} \gamma_i^k p_{ik}^{(n)} \Rightarrow \gamma_i^k \leq \frac{1}{p_{ik}^{(n)}} < \infty.$$

<u>Thm.</u> Let P be irreducible and λ be an invariant measure for P with $\lambda_k = 1$. Then $\lambda_i \geq \gamma_i^k$ for all i. If in addition P is recurrent, then $\lambda = \gamma^k$.

<u>Proof.</u> Since λ is invariant,

$$\lambda_j = \sum_{i_1 \in I} \lambda_{i_1} p_{i_1 j} = \sum_{i_1 \neq k} \lambda_{i_1} p_{i_1 j} + p_{kj}$$

$$= \sum_{i_1 \neq k} \left(\sum_{i_2 \neq k} \lambda_{i_2} p_{i_2 i_1} + p_{k i_1} \right) p_{i_1 j} + p_{kj}$$

$$= \cdots$$

$$\lambda_j \quad = \overbrace{\sum_{i_1, \ldots, i_n \neq k} \lambda_{i_n} p_{i_n i_{n-1}} \cdots p_{i_1 j}}^{\geq 0}$$

$$+ \left(p_{kj} + \sum_{i_1 \neq k} p_{k i_1} p_{i_1 j} + \cdots + \sum_{i_1, \ldots, i_{n-1} \neq k} p_{k i_{n-1}} \cdots p_{i_2 i_1} p_{i_1 j} \right)$$

$\Rightarrow$ For $j \neq k$, $\lambda_j \geq \mathbb{P}_k[X_1 = j, T_k \geq 1] + \mathbb{P}_k[X_2 = j, T_k \geq 2]$

$$+ \cdots + \mathbb{P}_k[X_n = j, T_k \geq n]$$

$$= \mathbb{E}_k\left[\sum_{m=1}^{\min(n,T_k)} 1_{X_m = j}\right] = \mathbb{E}_k\left[\sum_{m=0}^{\min(n,T_k-1)} 1_{X_m=j}\right] \xrightarrow{n\to\infty} \gamma_j^k$$

$\Rightarrow \lambda_j \geq \gamma_j^k$.

If P is recurrent, γ^k is invariant, so $\mu = \lambda - \gamma^k \geq 0$ is invariant.

P is irreducible $\Rightarrow \forall i\ \exists n$ s.t. $p_{ik}^{(n)} > 0$.

$$\Rightarrow 0 = \mu_k = \sum_{j\in I} \mu_j p_{jk}^{(n)} \geq \mu_i p_{ik}^{(n)} \Rightarrow \mu_i = 0$$

$\Rightarrow \mu = 0 \Rightarrow \lambda = \gamma^k$.

<u>Example</u>. The simple symmetric random walk on $\mathbb{Z}$ is irreducible and we have also seen that it is recurrent. The measure $\pi = (\pi_i)$ where

$\pi_i = 1$ for all $i \in \mathbb{Z}$

is invariant:

$$\pi = \pi P \Leftrightarrow \pi_i = \tfrac{1}{2}\pi_{i-1} + \tfrac{1}{2}\pi_{i+1} \quad \checkmark$$

By the theorem, every invariant measure is a multiple of π. Since $\sum_{i\in\mathbb{Z}} \pi_i = +\infty$, there is no invariant distribution.

<u>Example</u>. The simple symmetric random walk on $\mathbb{Z}^3$ has an invariant measure, but it is not recurrent.

Recall: i is recurrent if $\mathbb{P}_i[X_n = i \text{ for inf. many } n] = 1$

$$\Leftrightarrow \mathbb{P}_i[T_i < \infty] = 1$$

This does not imply that the expected return time

$$m_i = \mathbb{E}_i[T_i]$$

is finite.

<u>Defn</u>. • i is positive recurrent if $m_i < \infty$

• i is null recurrent if i is recurrent but $m_i = \infty$.

<u>Thm</u>. Let P be irreducible. Then the following are equivalent:

(a) Every state is positive recurrent.

(b) Some state is positive recurrent.

(c) P has an invariant distribution π

Moreover, when (c) holds then $m_i = 1/\pi_i$.

Proof. (a) $\Rightarrow$ (b) clear.

(b) $\Rightarrow$ (c) If i is positive recurrent, it is recurrent in particular. Therefore γ^i is invariant. Since

$$\sum_{j\in I} \gamma^i_j = m_i < \infty$$

Thus $\pi_j = \gamma^i_j / m_i$ defines an invariant distrib.

(c) $\Rightarrow$ (a). First note that, for every $k \in I$, $\pi_k > 0$. Indeed, since π is invariant and P irreducible,

$$\pi_k = \sum_{i\in I} \pi_i \, p^{(n)}_{ik} > 0 \quad \text{for some } n$$

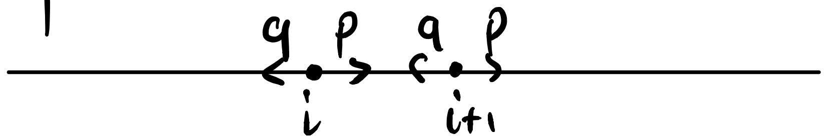
Now set $\lambda_i = \pi_i / \pi_k$. Then λ is an invariant measure with $\lambda_k = 1$. Therefore $\lambda \geq \gamma^k$.

$$\Rightarrow m_k = \sum_{i\in I} \gamma^k_i \leq \sum_{i\in I} \frac{\pi_i}{\pi_k} = \frac{1}{\pi_k} < \infty \qquad (*)$$

Thus k is positive recurrent.

Finally, knowing that P is recurrent, we have previously seen that every invariant measure λ with $\lambda_k = 1$ must satisfy $\lambda = \gamma^k$. Thus we have $=$ in $(*)$

Example



$q < p$

Invariant measure equation: $\pi_i = \sum_j \pi_j p_{ji}$

$$= \pi_{i-1} p + \pi_{i+1} q$$

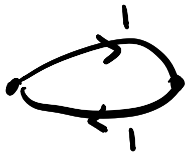
This recurrence relation has the following general solution:

$$\pi_i = A + B(p/q)^i$$

So there is a two-parameter family of invariant measures. Uniqueness up to multiples does not hold.

8. Convergence to equilibrium

Example $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$\Rightarrow P^2 = 1 \Rightarrow P^{2n} = 1$ and $P^{2n+1} = P$

$\Rightarrow P^n$ does not converge

But note that P has invariant distribution $\pi = (\frac{1}{2}, \frac{1}{2})$.

Defn A state $i \in I$ is aperiodic if $p_{ii}^{(n)} > 0$ for n sufficiently large. P is aperiodic if all states are aperiodic.

Lemma. Let P be irreducible and have an aperiodic state i. Then for all $j,k \in I$,

$$p_{jk}^{(n)} > 0 \text{ for } n \text{ sufficiently large.}$$

In particular, all states are aperiodic.

Proof. P irreducible $\Rightarrow \exists r,s$ s.t. $p_{ji}^{(r)} > 0$, $p_{ik}^{(s)} > 0$.

$\Rightarrow p_{jk}^{(r+n+s)} \geq p_{ji}^{(r)} p_{ii}^{(n)} p_{ik}^{(s)} > 0$ for n suff. large.

since i is aperiodic.

Thm. Let P be irreducible and aperiodic and suppose π is an invariant distribution for P. Let λ be any distribution, and suppose that (X_n) is Markov(λ, P). Then for all $j \in I$,

$$\mathbb{P}[X_n = j] \longrightarrow \pi_j \quad \text{as } n \to \infty.$$

In particular,

$$p_{ij}^{(n)} \longrightarrow \pi_j \quad \text{as } n \to \infty, \text{ for all } i, j \in I.$$

Proof. The proof is by coupling.

Let (Y_n) be Markov(π, P) and independent of (X_n). Fix a reference state $b \in I$ and set

$$T = \inf\{n \geq 1 : X_n = Y_n = b\}.$$

Claim: $\mathbb{P}[T < \infty] = 1$.

$W_n = (X_n, Y_n)$ is a Markov Chain on state space $I \times I$ and

- transition probabilities $\tilde{p}_{(i,k)(j,\ell)} = p_{ij} \, p_{k\ell}$
- initial distribution $\tilde{\lambda}_{(i,k)} = \lambda_i \pi_k$.

Since P is aperiodic, the lemma implies that, for all $i, j, k, \ell \in I$,

$\tilde{p}^{(n)}_{(i,k)(j,e)} > 0$ for n sufficiently large

$\Rightarrow \tilde{P}$ is irreducible.

$\tilde{P}$ has invariant distribution $\tilde{\pi}_{(i,k)} = \pi_i \pi_k$

$\Rightarrow \tilde{P}$ is positive recurrent.

T is the first passage time of (W_n) to (b,b).

Since P is irreducible and recurrent, therefore

$$\mathbb{P}[T<\infty] = 1.$$

From the claim, it follows that

$$\mathbb{P}[X_n = i] = \mathbb{P}[X_n = i, n<T] + \underbrace{\mathbb{P}[X_n = i, n \geq T]}_{\text{strong Markov } \mathbb{P}[Y_n = i, n \geq T]}$$

$$= \underbrace{\mathbb{P}[Y_n = i]}_{\pi_i} - \mathbb{P}[Y_n = i, n > T].$$

$$\Rightarrow |\mathbb{P}[X_n = i] - \pi_i| = |\mathbb{P}[X_n = i, n<T] - \mathbb{P}[Y_n = i, n > T]|$$

$$\leq \mathbb{P}[n<T] \xrightarrow[n\to\infty]{} 0.$$

<u>Example</u> (cont'd) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\pi = (\frac{1}{2}, \frac{1}{2})$

If X is Markov(δ_0, P) and Y is Markov(π, P) then with probability $1/2$ one has $Y_0 = 1$ but $X_0 = 0$ and X and Y will never meet.

What happens when (X_n) is periodic?

<u>Lemma</u>. Let P be irreducible. There exists an integer $d \geq 1$ (the period) and a partion

$$I = C_0 \cup \cdots \cup C_{d-1}$$

such that, setting $C_{nd+r} = C_r$,

(i) $p_{ij}^{(n)} > 0$ only if $i \in C_r$ and $j \in C_{r+n}$ for some r

(ii) $p_{ij}^{(nd)} > 0$ for sufficiently large n, for all $i, j \in C_r$, for all r.

<u>Proof</u>. $\to$ Norris.

<u>Thm</u>. Let P be irreducible of period d with the corresponding $C_0, \ldots, C_{d-1}$ as in the lemma. Let λ be a distribution with $\sum_{i \in C_0} \lambda_i = 1$. Suppose (X_n) is Markov(λ, P). Then for $r = 0, \ldots, d-1$, $j \in C_r$,

$$\mathbb{P}[X_{nd+r} = j] \longrightarrow d/m_j \quad (n \to \infty)$$

where m_j is the expected return time to j.

<u>Proof.</u> → Norris.

9. Time Reversal

<u>Thm.</u> Let P be irreducible and have invariant distribution π. Suppose $(X_n)_{0\le n\le N}$ is Markov(π, P) and set $Y_n = X_{N-n}$. Then $(Y_n)_{0\le n\le N}$ is Markov$(\pi, \hat{P})$, where

$$\pi_j \hat{p}_{ji} = \pi_i p_{ij} \qquad (*)$$

and $\hat{P}$ is irreducible with invariant distribution π.

<u>Proof.</u> $\hat{P}$ is well-defined by $(*)$ and is a stochastic matrix since

$$\sum_{i\in I} \hat{p}_{ji} \underset{\substack{\uparrow \\ \pi_j > 0 \text{ since } P \text{ is irreducible and } \pi \text{ invariant}}}{=} \frac{1}{\pi_j} \sum_{i\in I} \pi_i p_{ij} \overset{\pi \text{ is inv. for } P}{=} \frac{\pi_j}{\pi_j} = 1$$

π is invariant for $\hat{P}$:

$$\sum_{j\in I} \pi_j \hat{p}_{ij} = \sum_{j\in I} \pi_i p_{ij} \underset{\substack{\uparrow \\ P \text{ is a stochastic matrix}}}{=} \pi_i$$

(Y_n) is Markov $(\pi, \hat{P})$:

$$\mathbb{P}[Y_0 = i_0, \dots, Y_N = i_N] = \mathbb{P}[X_0 = i_N, \dots, X_N = i_0]$$

$$\overset{X \text{ is Markov}(\pi, P)}{=} \pi_{i_N} p_{i_N i_{N-1}} \cdots p_{i_1 i_0}$$

$$(*) \to \quad = \pi_{i_{N-1}} \hat{p}_{i_{N-1} i_N} p_{i_{N-1} i_{N-2}} \cdots p_{i_1 i_0}$$
$$= \cdots$$
$$= \pi_{i_0} \hat{p}_{i_0 i_1} \cdots \hat{p}_{i_{N-1} i_N}$$

$\hat{P}$ is irreducible since by irreducibility of P, for all $i,j \in I$,

$$p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0 \quad \text{for some } i_0, \dots, i_n \text{ with } i_0 = i,\ i_n = j.$$

$$\Rightarrow \hat{p}_{i_0 i_1} \cdots \hat{p}_{i_{n-1} i_n} = \frac{\pi_{i_0}}{\pi_{i_n}} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0.$$

<u>Defn.</u> A stochastic matrix P and a measure λ are in detailed balance if

$$\lambda_i p_{ij} = \lambda_j p_{ji} \qquad \text{for all } i,j \in I.$$

<u>Lemma.</u> If P and λ are in detailed balance then λ is invariant for P.

<u>Proof.</u> $(\lambda P)_i = \sum_{j \in I} \lambda_j p_{ji} = \sum_{j \in I} \lambda_i p_{ij} = \lambda_i$

<u>Defn.</u> Let P be irreducible and (X_n) be Markov(λ, P). Then (X_n) is reversible if, for all N, $(X_{N-n})_{0 \le n \le N}$ is also Markov(λ, P).

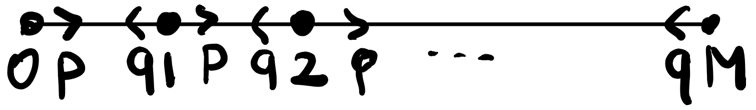
Thm. Let P be irreducible and let λ be a distribution. Suppose (X_n) is Markov(λ, P). Then the following are equivalent:

(a) (X_n) is reversible

(b) P and λ are in detailed balance.

Proof. Both (a) and (b) imply that λ is invariant. By the previous theorem thus both are equivalent to $P = \hat{P}$.

Example.



$0 < p = 1-q < 1$

λ and P are in detailed balance

$\Leftrightarrow \lambda_i p_{i,i+1} = \lambda_{i+1} p_{i+1,i}$ for $i = 0, \dots, M-1$

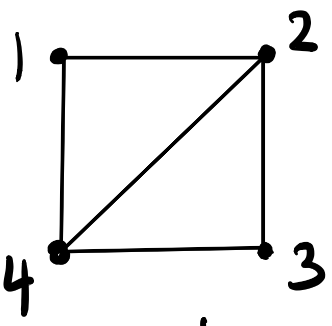
$\Leftrightarrow \lambda_i p = \lambda_{i+1} q$

$\Leftrightarrow \lambda_i = C\left(\frac{p}{q}\right)^i$ for some constant C

Thus $\pi_i = \dfrac{\lambda_i}{\sum_j \lambda_j} = \tilde{C}\left(\frac{p}{q}\right)^i$ for some suitable $\tilde{C}$ is an invariant distr.

Hence the chain started from π is reversible.

<u>Example</u> (Random walk on graph)

$G =$ 

graph

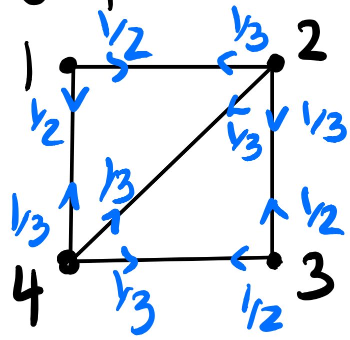


diagram obtained by choosing neighbours with uniform probabilities

Let v_i be the valency (or degree) of vertex i, i.e., the number of edges incident to i.

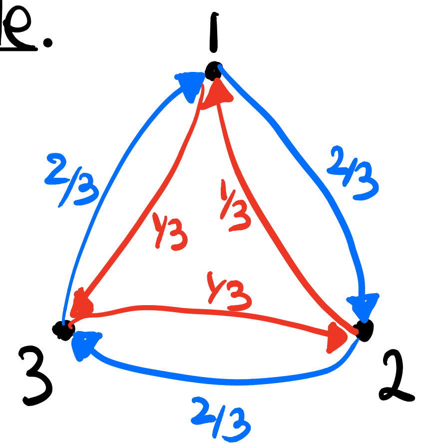
$$p_{ij} = \begin{cases} 1/v_i & \text{if } \{i,j\} \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

G connected $\Rightarrow$ P irreducible

P is in detailed balance with $v = (v_i)_{i \in I}$:

$$v_i \, p_{ij} = 1 = v_j \, p_{ji} .$$

Example.



$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$$

$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$\hat{P} = P^T$$

10. Ergodic theorem

Thm (Strong law of large numbers). Let $(Y_i)_{i=0,\dots}$ be a sequence of i.i.d. non-negativ random variables with $\mathbb{E} Y_i = \mu \in [0,\infty]$. Then

$$\mathbb{P}\left[\frac{Y_0 + \dots + Y_{n-1}}{n} \longrightarrow \mu \text{ as } n \to \infty \right] = 1.$$

Let $V_i(n) = \sum_{k=0}^{n-1} 1_{X_k = i}$ = # of visits to i before n.

Thm (ergodic theorem). Let P be irreducible and let λ be any distribution. If (X_n) is Markov(λ, P) then

$$\mathbb{P}\left[\frac{V_i(n)}{n} \longrightarrow \frac{1}{m_i} \text{ as } n \to \infty \right] = 1.$$

$= \mathbb{E} T_i$ = expected return time to i

In particular, if P is positive recurrent (with invariant distribution $\pi_i = 1/m_i$) then

$$\mathbb{P}\left[\frac{V_i(n)}{n} \longrightarrow \pi_i \text{ as } n \to \infty \right] = 1.$$

<u>Proof</u>. Case 1: P is transient.

$\Rightarrow \mathbb{P}[V_i < \infty] = 1$, $V_i = \sum_{k=0}^{\infty} 1_{X_n = i}$ is the total number of visits to i

$$\Rightarrow \mathbb{P}\left[\frac{V_i(n)}{n} \le \frac{V_i}{n} \longrightarrow 0 = \frac{1}{m_i}\right] = 1$$

as claimed.

Case 2: P is recurrent and $\lambda = \delta_i$, i.e.,

$$\mathbb{P}_i\left[\frac{n}{V_i(n)} \longrightarrow m_i \text{ as } n \to \infty\right] = 1.$$

Let $S_i^{(r)}$ be the rth excursion length between visits to i. We have seen that:

- the $S_i^{(1)}, S_i^{(2)}, \dots$ are independent
- the $S_i^{(r)}$ are identically distributed with $\mathbb{E} S_i^{(r)} = m_i$.

$$\Rightarrow \mathbb{P}_i\left[\frac{S_i^{(1)} + \dots + S_i^{(n)}}{n} \longrightarrow m_i \text{ as } n \to \infty\right] = 1. \quad (*)$$

Strong LNN

To get the claim, note:

$$S_i^{(1)} + \cdots + S_i^{(V_i(n))} \geq n$$

$$S_i^{(1)} + \cdots + S_i^{(V_i(n)-1)} \leq n-1$$

$$\Rightarrow \frac{S_i^{(1)} + \cdots + S_i^{(V_i(n))}}{V_i(n)} \geq \frac{n}{V_i(n)}$$

$$\frac{S_i^{(1)} + \cdots + S^{(V_i(n-1))}}{V_i(n)} \leq \frac{n}{V_i(n)}$$

Since $\mathbb{P}[V_i(n) \longrightarrow \infty] = 1$, by (*), thus

$$\mathbb{P}\left[\frac{n}{V_i(n)} \longrightarrow m_i\right] = 1.$$

Case 3. P is recurrent with general initial distr. λ.

By recurrence, $\mathbb{P}[T_i < \infty] = 1$.

By the strong Markov property $(X_{T_i+n})_{n\geq 0}$ is Markov (δ_i, P) and independent of $X_0, \dots, X_{T_i}$.

The general claim now follows since

$$\lim_n \frac{V_i(n)}{n}$$

remains the same if $(X_n)_{n\geq 0}$ is replaced by $(X_{T_i+n})_{n\geq 0}$.

Cor. In the positive recurrent case, for any bounded function $f: I \longrightarrow \mathbb{R}$,

$$P\left[\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \longrightarrow \bar{f} \text{ as } n\to\infty\right] = 1$$

where

$$\bar{f} = \sum_{i\in I} \pi_i f_i$$

Proof. WLOG, $|f| \le 1$. Then, for any $J \subset I$,

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) - \bar{f}\right| = \left|\sum_{i\in I}\left(\frac{V_i(n)}{n} - \pi_i\right) f_i\right|$$

$$= \sum_{i\in J}\left|\frac{V_i(n)}{n} - \pi_i\right| + \underbrace{\sum_{i\notin J}\left(\frac{V_i(n)}{n} + \pi_i\right)}_{1 - \sum_{i\in J}\frac{V_i(n)}{n} + \sum_{i\notin J}\pi_i \;=\; -\sum_{i\in J}\left(\frac{V_i(n)}{n} - \pi_i\right) + 2\sum_{i\notin J}\pi_i}$$

$$\le 2\sum_{i\in J}\left|\frac{V_i(n)}{n} - \pi_i\right| + 2\sum_{i\notin J}\pi_i$$

Choose $J \subset I$ finite such that $\sum_{i\notin J}\pi_i < \varepsilon$.

Choose $N = N(\omega)$ large enough such that

$$\mathbb{P}\left[\sum_{i \in J} \left| \frac{V_i(n)}{n} - \pi_i \right| < \varepsilon \text{ for } n \geq N \right] = 1$$

Therefore

$$\mathbb{P}\left[\left| \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \bar{f} \right| < 4\varepsilon \text{ for } n \geq N \right] = 1 .$$

11. Another application and outlook

Question: From the observations of a Markov Chain, how can you estimate the transition matrix?

Suppose $(X_i)_{i=0,\dots,n}$ is given (observations).
For any $\tilde{P} = (\tilde{p}_{ij})$ define

$$\ell(\tilde{P}) = \log(\tilde{p}_{X_0 X_1} \tilde{p}_{X_1 X_2} \cdots \tilde{p}_{X_{n-1} X_n})$$

$$= \sum_{i,j \in I} N_{ij}(n) \tilde{p}_{ij}$$

where

$$N_{ij}(n) = \sum_{m=0}^{m-1} 1\{X_m = i, X_{m+1} = j\} = \#\text{transitions from } i \text{ to } j.$$

The maximum likelihood estimator $\hat{P} = \hat{P}(n)$ is the maximiser of $\ell = \ell_n$.

$$V_i(n) = \sum_{k=0}^{n-1} 1_{X_k = i}$$

Exercise. $\hat{p}_{ij}(n) = \dfrac{N_{ij}(n)}{V_i(n)}$

(Hint: Use Lagrange multipliers, i.e., first maximize

$$\ell(\tilde{P}) + \sum_{i,j} \mu_i \tilde{p}_{ij} \quad \text{for any given } \mu$$

and then fit (μ_i) using constraints $\sum_j \tilde{p}_{ij} = 1$ for all i.)

<u>Claim:</u> If P is positive recurrent, then

$$\mathbb{P}\left[\hat{p}_{ij}(n) \longrightarrow p_{ij} \text{ as } n\to\infty\right] = 1.$$

<u>Proof.</u> $N_{ij} = \sum_{m=1}^{V_i} Y_m$ where $Y_m = 1$ if the m-th transition from i is to j and $Y_m = 0$ otherwise.

By strong Markov property, the Y_i are i.i.d with mean p_{ij} and independent from $V_i(n)$.

Markov Chain pos. rec. $\Rightarrow$ $\mathbb{P}[V_i(n) \longrightarrow \infty \text{ as } n\to\infty] = 1$.

Strong LLN $\Rightarrow$ $\mathbb{P}\left[\hat{p}_{ij}(n) = \frac{\sum_{i=1}^{V_i(n)} Y_i}{V_i(n)} \longrightarrow p_{ij} \text{ as } n\to\infty\right] = 1$.

<u>Outlook:</u> For an aperiodic irreducible finite state Markov Chain, we have seen that

$$\mathbb{P}[X_n = i] \longrightarrow \pi_i \quad (n\to\infty).$$

Thus, conversely, to sample from a given distr. π (on say N states), one may try to find a M.C. as above with π as its invariant distribution, and then run it for a long time.
(Markov Chain Monte Carlo — MCMC)
— Metropolis & Ulam.

There are different ways to find such a Markov Chain. The most famous is the Metropolis algorithm (→ Wikipedia).
— Metropolis, Rosenbluth, Rosenbluth, Teller & Teller. (1953).

Question of theoretical and practical relevance: how fast is "$n \to \infty$"? E.g.

$$\min\{ n : \sum_i | \mathbb{P}[X_n = i] - \pi_i | < \varepsilon \} = ?$$

Depends very much on the particular structure of the Markov Chain. It is a subject of current reseach interest.