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1. Let $X = (X_n)_{n\geq 0}$ be a Markov chain. Show that, conditioned on $X_m = i$, $Z = (Z_n)_{n\geq 0}$ given by $Z_n = X_{n+m}$ is a Markov chain with starting state i.

2. Let $X = (X_n)_{n\geq 0}$ be a sequence of independent random variables. Show that X is a Markov chain. Under what condition is this chain homogeneous?

3. Let $X = (X_n)_{n\geq 0}$ be a sequence of fair coin tosses (with the two possible outcomes interpreted as 0 and 1) and set $M_n = \max_{k\leq n} X_k$. Show that $(M_n)_{n\geq 0}$ is a Markov chain and find the transition probabilities.

4. (Harder) Let $S = (S_n)_{n\geq 0}$ be a simple (possibly asymmetric) random walk on $\mathbb{Z}$ with $S_0 = 0$. Show that $X_n = |S_n|$ defines a Markov chain and find its transition probabilities. Let $M_n = \max_{k\leq n} S_k$ and show that $Y_n = M_n - S_n$ defines a Markov chain.

5. Let $X = (X_n)_{n\geq 0}$ be a Markov chain and let $(n_r)_{r\geq 0}$ be an unbounded increasing sequence of positive integers. Show that $Y_r = X_{n_r}$ defines a (possibly inhomogeneous) Markov chain. Find the transition probabilities of Y when $n_r = 2r$ and X is a simple random walk.

6. Let $X = (X_n)_{n\geq 0}$ and $Y = (Y_n)_{n\geq 0}$ be Markov chains on the integers $\mathbb{Z}$. Is $Z_n = X_n + Y_n$ necessarily a Markov chain. Justify your answer.

7. A flea hops about at random on the vertices of a triangle where each hop is from the currently occupied vertex of one of the other two vertices each with probability $1/2$. Find the probability that after n hops the flea is back where it started.

Now suppose that the flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops the flea is back where it started now? [Hint: $1/2 \pm i/(2\sqrt{3}) = (1/\sqrt{3})e^{\pm i\pi/6}$.]

8. A die is 'fixed' so that when it is rolled the score cannot be the same as the previous score, all other scores having probability $1/5$. If the first score is 6, what is the probability p that the nth score is 6? What is the probability that the nth score is j, where $j \neq 6$?

Suppose instead that the die cannot score one greater (mod 6) than the previous score, all other five scores having equal probability. What is the new value of p? [Hint: Think about the relationship between the two dice.]

9. Let $X = (X_n)_{n\geq 0}$ be a Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{bmatrix}.$$

Find $\mathbb{P}[X_n = 1 | X_0 = 1]$ in each of the following cases: (a) $p = 1/16$, (b) $p = 1/6$, (c)* $p = 1/12$.

10. Identify the communicating classes of the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Which of the classes are closed?

11. Show that every transition matrix on a finite state space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

12. A gambler has £2 and needs to increase it to £10 in a hurry. She can play a game with the following rules: a fair coin is tossed; if a player bets on the side which actually turns up, she wins a sum equal to her stage, and her stake is returned; otherwise she loses her stake. The gambler decides to use a bold strategy in which she stakes all her money if she has £5 or less and otherwise stages just enough to increase her capital, if she wins, to £10.

Let $X_0 = 2$ and $X = (X_n)_{n\geq 0}$ be her capital after n throws. Prove that the gambler will achieve her aim with probabilty $1/5$. What is the expected number of tosses until she either achieves her aim or loses her capital?

13. (Optional) Let $X = (X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, \dots\}$ with transition probabilities given by

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad (i \geq 1).$$

Show that if $X_0 = 0$ then the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.

14. (Optional) Let $Y_1, Y_2, \dots$ be i.i.d. random variables with $\mathbb{P}[Y_1 = 1] = \mathbb{P}[Y_1 = -1] = 1/2$ and set $X_0 = 1$, $X_n = X_0 + Y_1 + \dots Y_n$ for $n \geq 1$. Define

$$H_0 = \inf\{n \geq 0 : X_n = 0\}.$$

Find the probability generating function $\phi(s) = \mathbb{E}[s^{H_0}]$.

Suppose the common distribution of the Y_i is changed to $\mathbb{P}[Y_1 = 2] = \mathbb{P}[Y_1 = -1] = 1/2$. Show that the probability generating function ϕ now satisfies

$$s\phi^3 - 2\phi + s = 0.$$