Ferromagnetic spin systems

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Preface

The different sections are inspired by various references, in particular (but not exclusively): Section 1 by [43, 28], Section 2 by [43, 6, 5, 30], Section 3 by [8, 21, 28], Section 4 by [11, 27], Section 5 by [2, 23], Section 6 by [2, 20, 19].

Warning: These notes are in progress and change from week to week. They are likely to contain errors!

1 Introduction

Spin systems are collections of many random vectors, *spins*, associated to the vertices of a graph, and typically with locally specified dependence. They were invented as simple models for ferromagnetism, but have now become fundamental models for phase transitions, of which many aspects can be understood rather well. Part of the fascination of the subject results from their rich structure and the many different tools that have turned out useful in their study, and also from their connections to other areas such as quantum field theory.

1.1 Spin systems

Spin systems are formulated on graphs (which will always be simple undirected graphs). Thus a graph $G = (V, E)$ is given by a vertex set V and a set of edges E which are subsets of V with two elements.

Definition 1.1 (Spin system on finite graph)**.** *Let G be a finite graph, and $n > 0$ an integer. Let μ be a finite positive measure on $\mathbb{R}^n$ with $\int e^{t|s|^2}\mu(ds) < \infty$ for any $t > 0$. The spin system on G with spin-spin coupling $J = (J_e) \in \mathbb{R}_+^E$, single spin measure μ, and external field $h = (h_x) \in (\mathbb{R}^n)^V$ is the probability measure*

$$P(d\varphi) = \frac{1}{Z} e^{-H(\varphi)} \mu^{\otimes V}(d\varphi) \tag{1.1}$$

where

$$H(\varphi) = -\sum_{xy\in E} J_{xy}(\varphi_x \cdot \varphi_y) - \sum_{v\in V} h_x \cdot \varphi_x. \tag{1.2}$$

The expectation with respect to P is written as $\langle \cdot \rangle$, and we sometimes denote the support of μ by $\Omega \subset \mathbb{R}^n$. We write the covariance (or truncated expectation) as $\langle F; G\rangle = \langle FG\rangle - \langle F\rangle\langle G\rangle$.

The graph of primary interest for spin systems is the infinite graph $\mathbb{Z}^d$ and its finite approximations. The definition on infinite graphs is more subtle, and requires the specification of *boundary conditions*. For any subset $\Lambda \subset \mathbb{Z}^d$, the outer (vertex) boundary is defined by

$$\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda : x \sim y \text{ for some } y \in \Lambda\}, \tag{1.3}$$

where $x \sim y$ denotes that $xy \in E$. The closure of Λ is $\bar\Lambda = \Lambda \cup \partial\Lambda$.

Definition 1.2 (Gibbs measures)**.** (a) *Given $\Lambda \subset \mathbb{Z}^d$ finite, the spin system on Λ with boundary condition $\omega \in (\mathbb{R}^n)^{\partial\Lambda}$ is the probability measure (finite volume Gibbs measure)*

$$P_{\Lambda,\omega}(d\varphi) = \frac{1}{Z} e^{-H(\bar\varphi)} \mu^{\otimes\Lambda}(d\varphi), \tag{1.4}$$

where $\bar\varphi_x = \varphi_x$ for $x \in \Lambda$ and $\bar\varphi_x = \omega_x$ for $x \in \partial\Lambda$, and H is defined by (1.2) with G replaced by $\bar\Lambda$.

(b) *A probability measure P on $(\mathbb{R}^n)^{\mathbb{Z}^d}$ is an infinite volume spin system on $\mathbb{Z}^d$ (infinite volume Gibbs measure) if for all finite $\Lambda \subset \mathbb{Z}^d$ and P-almost every $\varphi|_{\partial\Lambda}$, it obeys the* DLR condition

$$P(\cdot|\varphi|_{\partial\Lambda}) = P_{\Lambda,\varphi|_{\partial\Lambda}}(\cdot). \tag{1.5}$$

In practice, the study of infinite volume Gibbs measures requires study of their finite approximations, with sufficient uniformity in Λ and the boundary conditions. Throughout the course, we almost exclusively focus on methods to study finite volume Gibbs measures. Moreover, a particularly useful approximation of $\mathbb{Z}^d$ that is not exactly an example of the above definition is the torus $\mathbb{T}_m^d = \mathbb{Z}^d/m\mathbb{Z}^d$.

Remark 1.3. *(a)* The assumption $J > 0$ means that the spin system is *ferromagnetic*; antiferromagnetic spin systems, for which $J < 0$, are also of interest, but this is a different story.

(b) In more generality, the single spin measure μ might be space dependent, i.e., μ is replaced by measures μ_x. By replacing $\mu(ds)$ by $\mu_x(ds) = \mathrm{e}^{\sum_{y\sim x} J_{xy}|s|^2}\mu(ds)$, one can then replace the interaction part of the Hamiltonian by the (discrete) Dirichlet energy since

$$- \sum_{xy\in E} J_{xy}(\varphi_x \cdot \varphi_y) = \frac{1}{2} \sum_{xy\in E} J_{xy}|\varphi_x - \varphi_y|^2 + \sum_{x\in V} \Big(\sum_{y\sim x} J_{xy}\Big)|\varphi_x|^2. \tag{1.6}$$

The second can on the right-hand side is irrelevant if $|\varphi_x|$ is constant (which is the case for the $O(n)$ models; see below), and can otherwise be included in the single spin measure(s).

(c) It is customary to include a parameter $\beta > 0$ (inverse temperature) in (1.4), but we will usually omit it. It can either be included in J or (by rescaling the field) in the single spin measure.

1.2 Models

$O(n)$ symmetric models. The following are important examples of spin systems with $O(n)$ symmetry.

(a) In the *$O(n)$ model*, μ is the surface measure on the unit sphere in $\mathbb{R}^n$ and J is constant, i.e., $J_e = \beta$. Equivalently, by rescaling, it is given by the surface measure on the n-sphere of radius $\sqrt{\beta}$ and with $J = 1$.

For $n = 1, 2, 3$, the $O(n)$ model is known as the *Ising* ($n = 1$), *XY* or *rotator model* ($n = 2$), and *classical Heisenberg model*, respectively. For example, the Ising model is given by

$$J_{xy} = \beta, \quad \mu(d\varphi) = \delta_{-1}(d\varphi) + \delta_{+1}(d\varphi), \qquad (\beta > 0), \tag{1.7}$$

or (after rescaling)

$$J_{xy} = 1, \quad \mu(d\varphi) = \delta_{-\sqrt{\beta}}(d\varphi) + \delta_{+\sqrt{\beta}}(d\varphi), \qquad (\beta > 0). \tag{1.8}$$

For the $O(n)$ models, it is customary to denote the spin variable by σ (with $|\sigma| = 1$).

(b) The *Gaussian model*, or *Gaussian Free Field*, is given by the probability measure

$$P(d\varphi) = \frac{1}{Z}\mathrm{e}^{-S(\varphi)}\, d\varphi, \quad S(\varphi) = \frac{1}{2}\sum_{x\sim y} |\varphi_x - \varphi_y|^2 + \frac{1}{2}m^2 \sum_x |\varphi_x|^2, \tag{1.9}$$

where $d\varphi$ on the right-hand side denotes the Lebesgue measure on $\mathbb{R}^V$ and $m > 0$ is called the *mass*. This is a spin system with single spin measure

$$J_{xy} = 1, \quad \mu_x(d\varphi) = \mathrm{e}^{-(J_x + \frac{1}{2}m^2)|\varphi|^2}\, d\varphi, \tag{1.10}$$

where $J_x = \sum_{y\sim x} J_{xy}$.

(c) The *Ginzburg–Landau–Wilson $|\varphi|^4$ model* is given by the probability measure

$$P(d\varphi) = \frac{1}{Z}\mathrm{e}^{-S(\varphi)}\, d\varphi, \quad S(\varphi) = \frac{1}{2}\sum_{x\sim y} |\varphi_x - \varphi_y|^2 + \sum_x \left(\frac{1}{2}\nu|\varphi_x|^2 + \frac{1}{4}g|\varphi_x|^4\right). \tag{1.11}$$

where $g > 0$ and $\nu \in \mathbb{R}$. It is a spin system with single spin measure

$$J_{xy} = 1, \quad \mu_x(d\varphi) = \mathrm{e}^{-\frac{1}{4}g|\varphi|^4 - (J_x + \frac{1}{2}\nu)|\varphi|^2}\, d\varphi. \tag{1.12}$$

In general, if the single spin measure μ is absolutely continuous with some density $e^{-U(t)}$ with respect to the Lebesgue measure on $(\mathbb{R}^V)^n$, we also write the spin measure as

$$P(d\varphi) = \frac{1}{Z} \mathrm{e}^{-S(\varphi)}\, d\varphi, \tag{1.13}$$

where $d\varphi$ is the $n|V|$-dimensional Lebesgue measure on $(\mathbb{R}^V)^n$ and

$$S(\varphi) = H(\varphi) + U(\varphi). \tag{1.14}$$

Potts models. The Potts models are examples of spin models with discrete symmetry.
(a) Let $\vartheta^1, \dots, \vartheta^q$ in $\mathbb{R}^{q-1}$ be q unit vectors with

$$\vartheta^i \cdot \vartheta^j = \begin{cases} 1 & (i = j) \\ \frac{-1}{q-1} & (i \neq j). \end{cases} \tag{1.15}$$

The *q-state Potts model* is given by μ the uniform measure on $\vartheta^1, \dots, \vartheta^q$. Identifying $\varphi_x \in \{\vartheta^1, \dots, \vartheta^q\}$ with $\sigma_x \in \{1, \dots, q\}$, the energy becomes

$$-\sum_{xy} J_{xy}(\varphi_x \cdot \varphi_y) = -\sum_{xy} J_{xy}(1_{\sigma_x=\sigma_y} - \frac{1}{q-1} 1_{\sigma_x \neq \sigma_y}) = -\sum_{xy} \frac{q}{q-1} J_{xy} 1_{\sigma_x=\sigma_y} + \frac{1}{q-1} \sum_{xy} J_{xy}. \tag{1.16}$$

Since the second term is a constant, and rescaling J to $K = (q/(q-1))J$, the Hamiltonian can also be represented as

$$-\sum_{xy} K_{xy} 1_{\sigma_x=\sigma_y}. \tag{1.17}$$

(b) The *$\mathbb{Z}_q$-clock Potts model* is given by single-spin measure the uniform measure on $\mathbb{Z}_q \subset S^1$.

Exercise 1.4. Show that the $|\varphi|^4$ model interpolates between the $O(n)$ model and the Gaussian model in the sense that, on any finite graph, the probability measures of the $O(n)$ model and the Gaussian model are limits of that of the $|\varphi|^4$ model.

Exercise 1.5. Interpret the Ising model as a lattice gas.

1.3 Mean field theory for the Ising model

Most of the course will focus on the study of spin systems on approximations to $\mathbb{Z}^d$. However, to get some intuition, we first consider *mean field theory*, which is obtained by replacing the spin φ by its mean

$$m = m(\varphi) = \frac{1}{|V|} \sum_{x \in V} \varphi_x. \tag{1.18}$$

Then the interaction becomes

$$-\sum_{xy} J_{xy} \varphi_x \varphi_y \rightsquigarrow -\frac{\beta|V|}{2} \frac{1}{|V|} \sum_x \varphi_x \frac{1}{|V|} \sum_y \varphi_y = -\frac{\beta}{2}|V||m|^2, \quad \beta = \frac{2}{|V|} \sum_{xy} J_{xy}, \tag{1.19}$$

$$-\sum_x h_x \cdot \varphi_x \rightsquigarrow -|V| h \cdot \frac{1}{|V|} \sum_x \varphi_x = -|V| h \cdot m, \quad h = \frac{1}{|V|} \sum_x h_x. \tag{1.20}$$

This is equivalent to a spin model on the complete graph on $N = |V|$ vertices with $J_{xy} = \beta/N$ for all x, y and $h_x = h$ for all x. (The complete graph K_N consists of N vertices with all possible edges between the vertices.) Thus mean field theory is given by $G = K_N$ with interaction $J_{xy} = \beta/N$ for all $x, y \in [N]$ and $h_x = h$ for all $x \in [N]$, i.e.,

$$H_N(\varphi) = -\frac{N}{2}\beta|m|^2 - Nh \cdot m. \tag{1.21}$$

Much is exactly computable in mean field theory and provides intuition for the general case.

In the remainder of this section, we study the mean field Ising model, the *Curie–Weiss model*. Then the possible values that m can assume are

$$M_N = \left\{-1, -1 + \frac{2}{N}, -1 + \frac{4}{N}, \dots, +1\right\}, \tag{1.22}$$

and each of these values of m, the number of x such that $\varphi_x = +1$ is $\frac{1+m}{2}N$, which means that the number of possible configurations φ with $m(\varphi) = m$ is

$$\frac{N!}{(\frac{1+m}{2}N)!(\frac{1-m}{2}N)!}. \tag{1.23}$$

Thus:

$$Z = \sum_{m \in M_N} \frac{N!}{(\frac{1+m}{2}N)!(\frac{1-m}{2}N)!} \mathrm{e}^{\frac{N}{2}\beta m^2 + Nhm} = \sum_{m \in M_N} \mathrm{e}^{-Nf_{\beta,h}(m) + o(N)} \tag{1.24}$$

where, using Stirling's formula, $\log n! = n(\log n - 1) + o(n)$,

$$f_{\beta,h}(m) = \underbrace{\left(-\frac{\beta}{2}m^2 - hm\right)}_{\text{energy}} - \underbrace{\left(-\frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2}\right)}_{\text{entropy}} = -hm + f_\beta(m). \tag{1.25}$$

Proposition 1.6.

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = -\min_{m \in [-1,1]} f_{\beta,h}(m) = \max_{m \in [-1,1]} (hm - f_\beta(m)). \tag{1.26}$$

Proof. Let m_0 be an absolute minimum of f. Since f is continuous,

$$\min_{m \in M_N} f(m) = f(m_0) + o(1), \quad \text{as } N \to \infty. \tag{1.27}$$

Therefore

$$Z_N = \mathrm{e}^{-Nf(m_0)} \sum_{m \in M_N} \mathrm{e}^{-N(f(m) - f(m_0)) + o(N)} \tag{1.28}$$

and since

$$\mathrm{e}^{o(N)} \leqslant \sum_{m \in M_N} \mathrm{e}^{-N(f(m) - f(m_0)) + o(N)} \leqslant N\mathrm{e}^{o(N)} \leqslant \mathrm{e}^{o(N)} \tag{1.29}$$

the claim follows. □

Exercise 1.7. For any interval $J \subset [-1, 1]$ show that the magnetization concentrates around the minima of f, in the sense that

$$\Pr\left(\frac{1}{N}\sum_x \sigma_x \in J\right) = \mathrm{e}^{-NI(J) + o(N)}, \quad I(J) = \min_{m \in J} f(m) - \min_{m \in [-1,1]} f(m). \tag{1.30}$$

Proposition 1.8. *The local minima of* $f_{\beta,h}$ *satisfy the* self-consistent equation

$$m = \tanh(\beta m + h), \quad m \in (-1,1). \tag{1.31}$$

Let $\beta_c = 1$. *Then*

(a) *For* $\beta \leqslant \beta_c$, *the function* $f_{\beta,h}$ *has one absolute minimum* $m_0(\beta,h)$, *and* $m_0(\beta,0) = 0$.

(b) *For* $\beta > \beta_c$ *and* $h = 0$, *the function* $f_{\beta,0}$ *has two absolute minima* $\pm m_0(\beta,0)$.

(c) *For* $\beta > \beta_c$ *and* $h \neq 0$, *the function* $f_{\beta,h}$ *has one absolute minimum* $m_0(\beta,h) \neq 0$, *and* $m_0(\beta,0_+) = -m_0(\beta,0_-)$.

Proof.

$$f'(m) = -\beta m - h + \frac{1}{2}\log\frac{1+m}{1-m} = -(\beta m + h) + \operatorname{artanh}(m) \tag{1.32}$$

Since $f'(m) \to \pm\infty$ as $m \to \pm 1$ the minima lie in $(-1,1)$ and obey $f'(m) = 0$. The rest is an exercise. □

Two important observables for spin systems are the *(mean) magnetisation* and the *(magnetic) susceptibility*, defined respectively by

$$M = \lim_{|V|\to\infty} \frac{1}{|V|}\sum_{x\in V}\langle\varphi_x\rangle, \qquad \chi = \lim_{|V|\to\infty}\frac{1}{|V|}\sum_{x,y\in V}\langle\varphi_x;\varphi_y\rangle, \tag{1.33}$$

defined along a suitable sequence of graphs (and possibly boundary conditions), and regarded as functions of the parameters of the Hamiltonian, here (β,h).

Exercise 1.9. Show that for $(\beta,h) \neq (\beta_c,0)$, it holds that

$$M = \frac{\partial}{\partial h}\lim_{N\to\infty}\frac{1}{N}\log Z_N, \quad \chi = \frac{\partial^2}{\partial h^2}\lim_{N\to\infty}\frac{1}{N}\log Z_N. \tag{1.34}$$

It follows that

$$M = -\frac{\partial}{\partial h} f_{\beta,h}(m_0(\beta,h)) = -\underbrace{f'_{\beta,h}(m_0)}_{0}\frac{\partial m}{\partial h} - \frac{\partial}{\partial h} f_{\beta,h}(m_0) = m_0(\beta,h) \tag{1.35}$$

For the susceptibility, note that $0 = f'_{\beta,h}(m_0(\beta,h))$ implies

$$0 = \underbrace{\frac{\partial^2}{\partial h\partial m} f_{\beta,h}(m_0(\beta,h))}_{-1} + \frac{\partial^2}{\partial m^2} f_{\beta,h}(m_0(\beta,h))\underbrace{\frac{\partial m_0}{\partial h}}_{\chi} \tag{1.36}$$

and therefore

$$\chi = \frac{1}{f''_{\beta,h}(m_0(\beta,h))} = \frac{1}{-\beta + (1 - m_0(\beta,h)^2)^{-1}}. \tag{1.37}$$

Proposition 1.10 (Critical exponents)**.** (a) *The magnetisation obeys*

$$M(\beta,0_+)\begin{cases} > 0 & (\beta > \beta_c) \\ = 0 & (\beta \geqslant \beta_c), \end{cases} \tag{1.38}$$

and

$$M(\beta,0_+) \sim C(\beta - \beta_c)^{\frac{1}{2}} \quad (\beta \downarrow \beta_c). \tag{1.39}$$

(b) *The susceptibility is finite for $\beta \neq \beta_c$ and (Curie–Weiss law)*

$$\chi(\beta, 0) \sim C_\pm |\beta - \beta_c|^{-1} \quad (\beta - \beta_c \to \pm 0). \tag{1.40}$$

Proof. (a) $\tanh(x) = x - \frac{1}{3}x^3 + o(x^3)$ and $m_0(\beta, 0_+) \to 0$ as $\beta \to \beta_c$ implies

$$m_0(\beta, 0_+) = \tanh(\beta m(\beta, 0_+)) = \beta m_0(\beta, 0_+) - \frac{1}{3}(\beta m_0(\beta, 0_+))^3 + o(\beta m_0(\beta, 0_+))^3 \tag{1.41}$$

and therefore

$$(\beta - 1)m = \frac{1}{3}(\beta m)^3 + o(\beta m)^3. \tag{1.42}$$

The claim follows by dividing by $m/3$ and taking the square root.
(b)

$$\chi = \frac{1}{-\beta + (1 - m_0(\beta, 0_+)^2)^{-1}} = \frac{1}{1 - \beta} = \frac{1}{\beta_c - \beta} \quad (\beta < \beta_c), \tag{1.43}$$

$$\chi \sim \frac{1}{-\beta + (1 - 3(\beta - 1))^{-1}} \sim \frac{1}{1 - \beta + 3(\beta - 1)} = \frac{1}{2(\beta - \beta_c)} \quad (\beta > \beta_c), \tag{1.44}$$

as claimed. □

Much of the theory of phase transition concerns proving similar results for $\mathbb{Z}^d$, and understanding the behaviour of the system. Spin systems are a paradigm for other systems that undergo phase transitions.

The problem can roughly be divided into the high temperature regime ($\beta \ll 1$ or $\nu \ll 0$), the low temperature regime, as well as the critical temperature and its approach. The regime that is by far easiest to understand is the high temperature (or phase uniqueness) regime. The low temperature regime is well understood for the Ising model, and should be similar for models with discrete symmetry ($n = 1$), but is much more difficult to understand for $n > 1$ (and many aspects are open). The abelian case $n = 2$ is more tractable than the nonabelian case $n \geqslant 3$ (which includes the Heisenberg model).

The critical point has the most delicate and most interesting behaviour. However, even is uniqueness is rather nontrivial and in general not known (it is a priori conceivable that high and low temperature phases alternate in some way, though it is highly unplausible). The exponents $\frac{1}{2}$ and 1 in (1.39) and (1.40) are examples of *critical exponents*. They characterize the shape of the phase diagram, and are believed to be *universal* with very general scope. For example, experimentally, certain fluids should are found to have the same exponents as spin models. The universality of phase transitions is open except in special cases.

1.4 Mean field theory by Hubbard–Strononovich transform

Exercise 1.11. Compute the Laplace transform of the standard Gaussian measure:

$$e^{\frac{1}{2}t^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 + tx} \, dx. \tag{1.45}$$

This implies

$$e^{-H(\varphi)} = e^{\frac{\beta}{2N}\sum_{x,y}\varphi_x\varphi_y + h\sum_x \varphi_x} = \left(\frac{\beta N}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{1}{2}N\beta\psi^2 + (\psi\beta + h)\sum_x \varphi_x} \, d\psi, \tag{1.46}$$

and thus

$$Z = \int \mathrm{e}^{-H(\sigma)}\, \mu^{\otimes H}(d\varphi) = \left(\frac{\beta N}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}N\beta\psi^2} \underbrace{\left(\int \prod_{x=1}^{N} \mathrm{e}^{(\psi\beta+h)\cdot\varphi_x} \mu^{\otimes N}(d\varphi)\right)}_{\prod_{x=1}^N \int \mathrm{e}^{(\psi\beta+h)\cdot s}\mu(ds) = \mathrm{e}^{Nv(\psi\beta+h)}} d\psi, \tag{1.47}$$

with the definition $\mathrm{e}^{v(y)} = \int \mathrm{e}^{y\cdot s}\mu(ds)$. (For example, for the Ising model, $\mathrm{e}^{v(y)} = 2\cosh(y)$.) Therefore the partition function is equal to the integral

$$Z = \left(\frac{N\beta}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-NS(\psi)}\, d\psi, \qquad S(\psi) = \frac{1}{2}\beta\psi^2 - v(\psi\beta + h). \tag{1.48}$$

Theorem 1.12 (Laplace's Principle)**.** *Let $S : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded below, and assume that $\{\psi \in \mathbb{R}^n : S(\psi) \leqslant \min S + 1\}$ is compact and that $\int \mathrm{e}^{-S}\, d\psi < \infty$.*

(a) *For any bounded continuous $f : \mathbb{R}^n \to \mathbb{R}_+$ with $f(\psi_0) \neq 0$ for some ψ_0 with $S(\psi_0) = \min S$, we have*

$$\int f(\psi)\mathrm{e}^{-tS(\psi)}\, d\psi = \mathrm{e}^{-t\min S + o(t)}, \quad (t \to \infty). \tag{1.49}$$

(b) *Assume that S is in C^2 and has a unique global minimum at ψ_0. Then for any bounded continuous $f : \mathbb{R}^n \to \mathbb{R}$ with $f(\psi_0) \neq 0$, we have*

$$\int f(\psi)\mathrm{e}^{-tS(\psi)}\, d\psi = \sqrt{\frac{(2\pi)^n}{t\det(\nabla^2 S(\psi_0))}} f(\psi_0)\mathrm{e}^{-tS(\psi_0)}(1 + o(1)), \quad (t \to \infty). \tag{1.50}$$

Proof. Assume without loss of generality that $\min S = S(0)$. First, it suffices to show the claim with the integral replaced by the integral over $K = [-\delta, \delta]^n$ with arbitrary $\delta > 0$. Indeed, since S is continuous and $\{\psi : S(\psi) \leqslant \min S + 1\}$ is compact, there is $\alpha > 0$ such that $S(\psi) \geqslant S(0) + \alpha$ for $\psi \notin K$. Thus

$$\left|\int_{K^c} f(\psi)\mathrm{e}^{-tS(\psi)}\, d\psi\right| \leqslant \|f\|_\infty \mathrm{e}^{-(t-1)(S(0)+\alpha)} \int \mathrm{e}^{-S}\, d\psi = O\left(\mathrm{e}^{-tS(0)}\mathrm{e}^{-\alpha t}\right), \tag{1.51}$$

which is smaller than the right-hand sides.

(a) Clearly, for any compact K, we have

$$\int_K f(\psi)\mathrm{e}^{-tS(\psi)}\, d\psi \leqslant \|f\|_\infty |K| \mathrm{e}^{-t\min S} = \mathrm{e}^{-t\min S + O(1)}. \tag{1.52}$$

On the other, if we choose $\delta > 0$ sufficiently small that for $\psi \in K$ we have

$$S(\psi) \leqslant \min S + \varepsilon, \quad f(\psi) \geqslant \frac{1}{2} f(\psi_0), \tag{1.53}$$

it follows that

$$\int_K f(\psi)\mathrm{e}^{-tS(\psi)}\, d\psi \geqslant \frac{1}{2} f(\psi_0)|K|\mathrm{e}^{-t(\min S+\varepsilon)} = \mathrm{e}^{-t(\min S+\varepsilon)+O(1)}. \tag{1.54}$$

The claim follows since $\varepsilon > 0$ is arbitrary.

(b) Choose $\delta > 0$ such that for $\psi \in K$ we have

$$|\nabla^2 S(0) - \nabla^2 S(\psi)| \leqslant \varepsilon, \quad |f(0) - f(\psi)| \leqslant \varepsilon. \tag{1.55}$$

Then

$$\int_{-\delta}^{\delta} f(\psi)\mathrm{e}^{-tS(\psi)}\,d\psi \leqslant \cdots \leqslant (f(0)+\varepsilon)\mathrm{e}^{-tS(0)}\sqrt{\frac{2\pi}{t(S''(0)-\varepsilon)}}. \tag{1.56}$$

Since

$$\frac{\int_{-\delta}^{\delta} \mathrm{e}^{-tx^2}\,dx}{\int_{-\infty}^{\infty} \mathrm{e}^{-tx^2}\,dx} = \frac{\int_{-\sqrt{t}\delta}^{\sqrt{t}\delta} \mathrm{e}^{-y^2}\,dy}{\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\,dy} \to 1 \quad \text{as } t\to\infty, \tag{1.57}$$

similarly

$$\int_{-\delta}^{\delta} f(\psi)\mathrm{e}^{-tS(\psi)}\,d\psi \geqslant \cdots \geqslant (1+o(1))(f(0)-\varepsilon)\mathrm{e}^{-tS(0)}\sqrt{\frac{2\pi}{t(S''(0)+\varepsilon)}}, \tag{1.58}$$

as claimed. □

Using Laplace's Principle, we can now study (1.48). Clearly,

$$S(\psi) = \tfrac{1}{2}\beta\psi^2 - v(\psi\beta+h), \tag{1.59}$$

$$\nabla S(\psi) = \beta\psi - \beta\nabla v(\psi\beta+h), \tag{1.60}$$

$$\nabla^2 S(\psi) = \beta - \beta^2\nabla^2 v(\psi\beta+h). \tag{1.61}$$

Example 1.13. For the Ising model, we have

$$v(y) = \log\cosh(y) + \log 2, \quad v'(y) = \tanh(y)\in[-1,1], \quad v''(y) = \frac{1}{\cosh(t)^2}\in(0,1]. \tag{1.62}$$

In particular, S is strictly uniformly convex if $\beta > \beta^2$, i.e., if $\beta<1$; see Proposition 1.8.

Exercise 1.14. Assume that μ is non-degenerate and $\int \mathrm{e}^{c|s|^2}\mu(ds)<\infty$. Then $\nabla^2 v(y)$ is strictly positive definite for each y, and $\nabla v(y)$ is bounded in y. Compute the critical temperature for the mean field $O(n)$ model.

Sketch. The strict convexity follows from the fact that $[\nabla^2 v(y)](h,h)$ is the variance of $s\cdot h$ under the probability $\mathrm{e}^{-v(y)+y\cdot s}\mu(ds)$ which by assumption always has more than one point in its support. $\nabla v(y)$ is the mean of $s\cdot h$ under $\mathrm{e}^{-v(y)+y\cdot s}\mu(ds)$ and therefore bounded. □

Since $-\nabla_h S(\psi) = \nabla v(\psi\beta+h)$, the magnetization is given by

$$M = \lim_{N\to\infty}\frac{\partial}{\partial h}\frac{1}{N}\log Z = -\lim_{N\to\infty}\frac{\int \nabla_h S(\psi)\,\mathrm{e}^{-NS(\psi)}\,d\psi}{\int \mathrm{e}^{-NS(\psi)}\,d\psi} = \lim_{N\to\infty}\frac{\int \nabla v(\psi\beta+h)\,\mathrm{e}^{-NS(\psi)}\,d\psi}{\int \mathrm{e}^{-NS(\psi)}\,d\psi}. \tag{1.63}$$

Laplace's Principle shows that $M=\nabla v(\psi_0\beta+h)$ if S has a global minimum at ψ_0. Since then ψ_0 must satisfy $\nabla S(\psi_0)=0$ this gives $\psi_0=\nabla v(\psi_0\beta+h)$, and we again obtain a self-consistent equation for the magnetization:

$$M = \nabla v(M\beta+h). \tag{1.64}$$

Laplace's Principle also implies

$$\lim_{N\to\infty}\frac{1}{N}\log Z_N = -\min S = -\frac{1}{2}\beta M^2 + v(\beta M+h), \tag{1.65}$$

where the second inequality only holds if the global minimum is unique.

Let

$$p_{\mathrm{MF}} = -\inf_{\psi\in\mathbb{R}^n} S(\psi) = \sup_{\psi\in\mathbb{R}^n}\left(-\frac{\beta}{2}|\psi|^2 + v(\beta\psi+h)\right). \tag{1.66}$$

Then we have already shown the following proposition.

Proposition 1.15. *For the mean field model with single spin measure μ, it holds that*

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = p_{MF}. \tag{1.67}$$

Exercise 1.16 (Kac model)**.** Consider, say, the torus $\mathbb{T}_m^d$, and spin-spin interaction

$$J_{xy} = m^{-d}\eta((x-y|)/m), \tag{1.68}$$

for a continuous function η that is compactly supported in $(-\frac{1}{2}, \frac{1}{2})^d$. Show that the pressure of the Kac model converges to the mean field pressure,

$$m^{-d} \log Z \longrightarrow p_{\mathrm{MF}} \quad (m \to \infty), \tag{1.69}$$

with mean field model defined by $\beta = \int_{\mathbb{R}^d} \eta(x)\, dx$. The Kac model is essentially a mean field model.

The following proposition shows that mean field theory always provides a lower bound on the pressure.

Proposition 1.17 (Mean field bound for partition function)**.** *For any spin model on a graph, the partition function is bounded below by*

$$\frac{1}{|V|} \log Z \geqslant p_{MF}, \tag{1.70}$$

with the mean field model defined by the same single spin measure and

$$\beta = \frac{2}{|V|} \sum_{xy} J_{xy}, \quad h = \frac{1}{|V|} \sum_{x} h_x. \tag{1.71}$$

Lemma 1.18.

$$p_{MF} = \sup_{\psi \in \mathbb{R}^n} \left(-S(\psi) + \frac{1}{2\beta} |\nabla S(\psi)|^2 \right). \tag{1.72}$$

Proof. Let $T(\psi) = -S(\psi) + \frac{1}{2\beta}|\nabla S(\psi)|^2$. Then

$$\begin{aligned} \nabla T(\psi) = -\nabla S(\psi) + \frac{1}{\beta} \nabla S(\psi) \cdot \nabla^2 S(\psi) &= -\nabla S(\psi) + \frac{1}{\beta} \nabla S(\psi) \cdot (\beta - \beta^2 \nabla^2 v(\psi\beta + h)) \\ &= -\beta \nabla S(\psi) \cdot \nabla^2 v(\psi\beta + h) \\ &= -\beta^2 (\psi - \nabla v(\beta\psi + h)) \cdot \nabla^2 v(\psi\beta + h). \end{aligned} \tag{1.73}$$

Now, by Exercise 1.14, $\nabla^2 v(y)$ is strictly positive definite for each y, and thus $\nabla T(\psi) = 0$ if and only if $\nabla S(\psi) = 0$. Moreover, $\nabla v(y)$ is bounded, and thus thus $(\psi - \nabla v(\beta\psi + h)) \cdot u \to \infty$ if $|\psi| \to \infty$, for any $u \in \mathbb{R}^n$, and $\nabla T \cdot u < 0$ outside a compact set. Thus T assumes its maximum in a compact set. □

Lemma 1.19.

$$\sup_{\psi \in \mathbb{R}^n} \left(-S(\psi) + \frac{1}{2\beta} |\nabla S(\psi)|^2 \right) = \sup_{\psi \in \mathbb{R}^n} \left(v(\psi) - \psi \cdot \nabla v(\psi) + \frac{1}{2} \beta |\nabla v(\psi)|^2 + h \cdot \nabla v(\psi) \right) \tag{1.74}$$

Proof. Indeed, by direct computation,

$$|\nabla S(\psi)|^2 = \beta^2 \, |\psi - \nabla v(\beta\psi + h)|^2 = \beta^2 \Big(|\psi|^2 - 2\psi \cdot \nabla v(\beta\psi + h) + |\nabla v(\beta\psi + h)|^2 \Big), \tag{1.75}$$

and therefore

$$-S(\psi)+\frac{1}{2\beta}|\nabla S(\psi)|^2 = v(\beta\psi+h)-\beta\psi\cdot\nabla v(\beta\psi+h)+\frac{1}{2}\beta|\nabla v(\beta\psi+h)|^2$$
$$= v(\beta\psi+h)-(\beta\psi+h)\cdot\nabla v(\beta\psi+h)+\frac{1}{2}\beta|\nabla v(\beta\psi+h)|^2+h\cdot\nabla v(\beta\psi+h). \tag{1.76}$$

The claim now follows by replacing $\beta\psi+h$ by ψ. □

Proof of Proposition 1.17 [43]. For any $F(\varphi)>0$, since $t\mapsto \mathrm{e}^t$ is convex, Jensen's inequality applied to the measure $F(\varphi)\,\mu^{\otimes V}(d\varphi)$ implies

$$\log Z = \log\int \mathrm{e}^{-H(\varphi)-\log F(\varphi)}F(\varphi)\,\mu^{\otimes V}(d\varphi) \geqslant -\int (H(\varphi)F(\varphi)+F(\varphi)\log F(\varphi))\,\mu^{\otimes V}(d\varphi). \tag{1.77}$$

Let $F(\varphi)=\prod_x(\mathrm{e}^{-v(y)}\mathrm{e}^{y\cdot\varphi_x})$ for an arbitrary $y\in\mathbb{R}^n$. Then since $\nabla v(y)=\int s\mathrm{e}^{-v(y)+y\cdot s}\,\mu(ds)$ and $1=\int \mathrm{e}^{-v(y)+y\cdot s}\,\mu(ds)$, we have

$$-\int H(\varphi)F(\varphi)\,\mu^{\otimes V}(d\varphi) = \int\left(\sum_{xy\in E}J_{xy}(\varphi_x\cdot\varphi_y)+\sum_{x\in V}h_x\cdot\varphi_x\right)F(\varphi)\,\mu^{\otimes V}(d\varphi)$$
$$=\left(\sum_{xy\in E}J_{xy}\right)\int (s\cdot s')\mathrm{e}^{-v(y)+y\cdot s}\mu(ds)\,\mathrm{e}^{-v(y)+y\cdot s'}\mu(ds')+\left(\sum_{x\in V}h_x\right)\cdot\int s\,\mathrm{e}^{-v(y)+y\cdot s}\mu(ds)$$
$$=\left(\sum_{xy\in E}J_{xy}\right)|\nabla v(y)|^2+\left(\sum_{x\in V}h_x\right)\cdot\nabla v(y) = |\Lambda|\left(\frac{1}{2}\beta|\nabla v(y)|^2+h\cdot\nabla v(y)\right), \tag{1.78}$$

and

$$-\int F(\varphi)\log F(\varphi)\,\mu^{\otimes V}(d\varphi) = -\sum_x\int F(\varphi)(-v(y)+y\cdot\varphi_x)\,\mu^{\otimes V}(d\varphi) = |\Lambda|(v(y)-y\cdot\nabla v(y)). \tag{1.79}$$

Together we have

$$-\int (H(\varphi)F(\varphi)+F(\varphi)\log F(\varphi))\,\mu^{\otimes V}(d\varphi) = |\Lambda|\left(\frac{1}{2}\beta|\nabla v(y)|^2+h\cdot\nabla v(y)+(v(y)-y\cdot\nabla v(y))\right). \tag{1.80}$$

Since y was arbitrary, by Lemmas 1.18–1.19, we obtain

$$\log Z \geqslant |V|\sup_\psi\left(\frac{1}{2}\beta|\nabla v(\psi)|^2+h\cdot\nabla v(\psi)+(v(\psi)-y\cdot\nabla v(\psi))\right) = |V|p_{\mathrm{MF}}. \tag{1.81}$$

This was the claim. □

2 High temperature by inequalities

For sufficiently high temperature, there are various techniques to show that the *spontaneous magnetization* (the limit of the magnetization as $h \downarrow 0$) vanishes, and that the correlation functions decay exponentially. Due to its interpretation in quantum field theory, exponential decay of the two-point function is often called a *mass gap*, and the rate of exponential decay is called the *mass*. In this section, we consider *non-expansion methods* that are effective at high temperatures.

2.1 Preparation: Griffith/GKS inequalities

The is a zoo of correlation inequalities for *ferromagnetic* spin systems ($J > 0$), which are very useful for their understanding. The number of available inequalities decreases with n, and this is one of the reasons that in many respects n-component models with larger n become more difficult to study.

Notation. For any $A \subset V \times [n]$, where $[n] = \{1, \dots, n\}$, write

$$\varphi^A = \prod_{(x,a)\in A} \varphi_x^a. \tag{2.1}$$

Moreover, we write $J \geqslant 0$ if $J_e \geqslant 0$ for all $e \in E$ (which we always assume), and similarly $h \geqslant 0$ if $h_x^a \geqslant 0$ for all $x \in V$ and $a \in [n]$ (which we do not always assume, but which is often a helpful assumption).

Proposition 2.1 (First Griffith/GKS inequality)**.** *Assume that the single spin measure μ is $O(n)$ invariant, and that $h \geqslant 0$. Then for any A, it holds that*

$$\langle \varphi^A \rangle \geqslant 0. \tag{2.2}$$

Proof. Let $A = \{(x_1, a_1), \dots, (x_k, a_k)\}$. By expanding $e^{-H(\varphi)}$ (which can be justified by the assumption that $\int e^{t|s|^2} \mu(ds) < \infty$ for any t), it follows that

$$\langle \varphi^A \rangle \propto \sum_{l=0}^{\infty} \frac{1}{l!} \int \varphi_{x_1}^{a_1} \cdots \varphi_{x_k}^{a_k} \left(\sum_{x,y,a} J_{xy} \varphi_x^a \varphi_y^a + \sum_x h_x^a \varphi_x^a \right)^l \mu^{\otimes V}(d\varphi). \tag{2.3}$$

For any rotation invariant single spin measure μ on $\mathbb{R}^n$, and any components $a_1, \dots, a_k$, it holds that

$$\int s^{a_1} \cdots s^{a_k} \, \mu(ds) \geqslant 0, \tag{2.4}$$

because the integral is 0 if any component appears an odd number of times, and is nonnegative otherwise. This and $J \geqslant 0$ and $h \geqslant 0$ imply the claim. ☐

Proposition 2.2 (Second Griffith/GKS inequality for one-component spins)**.** *Let $n = 1$. Then for $h \geqslant 0$ and any A, B, it holds that*

$$\langle \varphi^A; \varphi^B \rangle \geqslant 0. \tag{2.5}$$

In particular, $\langle \varphi^A \rangle$ is increasing in each of the spin-spin couplings J_{xy} and in the h_x. The same holds for any Hamiltonian that is a polynomial in the spins with nonnegative coefficients.

Proof. Let φ' be an independent copy of φ. Then

$$\langle \varphi^A; \varphi^B \rangle = \frac{1}{2} \langle (\varphi^A - \varphi'^A)(\varphi^B - \varphi'^B) \rangle, \tag{2.6}$$

and

$$-H(\varphi) - H(\varphi') = \sum_{xy} J_{xy}(\varphi_x\varphi_y + \varphi'_x\varphi'_y) + h_x(\varphi_x + \varphi_y). \tag{2.7}$$

By expanding the exponential, it suffices to prove

$$\int \prod_A (\varphi^A \pm \varphi'^A)\, \mu^{\otimes V}(d\varphi)\, \mu^{\otimes V}(d\varphi') \geqslant 0, \tag{2.8}$$

where the product ranges over a finite number of sets of vertices. By the identities

$$ab + a'b' = \frac{1}{2}(a+a')(b+b') + \frac{1}{2}(a-a')(b-b') \tag{2.9}$$

$$ab - a'b' = \frac{1}{2}(a+a')(b-b') + \frac{1}{2}(a-a')(b+b'), \tag{2.10}$$

the claim reduces to

$$\int \prod (s \pm s')\, \mu(ds)\, \mu(ds') \geqslant 0. \tag{2.11}$$

As a consequence, then

$$\frac{\partial}{\partial J_{xy}} \langle \varphi^A \rangle = \langle \varphi^A; \varphi_x\varphi_y \rangle \geqslant 0, \quad \frac{\partial}{\partial h_x} \langle \varphi^A \rangle = \langle \varphi^A; \varphi_x \rangle \geqslant 0, \tag{2.12}$$

and the claim about monotonicity follows immediately. □

The analysis of two-component spin system (such as the XY model) is often made possible by Fourier analysis. In polar coordinates, $\varphi_x = (r_x \cos\theta_x, r_x \sin\theta_x)$ for $h_x = (h_x^1, 0, \ldots, 0) \geqslant 0$,

$$H(\varphi) = -\sum_{xy} J_{xy} r_x r_y (\sigma_x \cdot \sigma_y) - \sum_x h_x \cdot \sigma_x = -\sum_{xy} J_{xy} r_x r_y \cos(\theta_x - \theta_y) - \sum_x h_x^1 r_x \cos\theta_x. \tag{2.13}$$

In particular, with r fixed, $-H$ is *positive definite* on S^1, i.e., it has nonnegative Fourier coefficients. For $m \in \mathbb{Z}^k$ and $\theta \in \mathbb{T}^k$ we abbreviate $\cos(m\theta) = \cos(m_1\theta_1 + \cdots + m_k\theta_k)$.

Proposition 2.3 (First and second Griffith inequality for two-component spins; Ginibre inequalities)**.** *Let $n = 2$ and assume that μ is $O(2)$-invariant or that θ is uniform on $2\pi[q]/q$. Then for $h = (h^1, 0) \geqslant 0$ and any $a, b \in \mathbb{Z}^V$,*

$$\langle \cos(a\theta) \rangle \geqslant 0, \qquad \langle \cos(a\theta); \cos(b\theta) \rangle \geqslant 0. \tag{2.14}$$

In particular, $\langle \cos(a\theta) \rangle$ is increasing in J and h. The same holds for any Hamiltonian H that is negative definite on $(S^1)^V$.

Remark 2.4. Since

$$\cos(m\theta)\cos(m'\theta) = \frac{1}{2}(\cos((m+m')\theta) + \cos((m-m')\theta)), \tag{2.15}$$

the inequalities (2.14) extend to any product of cosines with positive coefficients.

It does not seem to be known whether the second Griffith inequality holds for $n \geqslant 3$; see [47].

Proof [26]. For simplicity of notation, assume that $r_x = 1$. Moreover, we only give the proof for $O(2)$ symmetry; the $\mathbb{Z}_q$ case is analogous (but requires some care in the change of variables below).

First, for any $m \in \mathbb{Z}^N$, one has $(2\pi)^{-N} \int_{\mathbb{T}^N} \mathrm{e}^{im\theta}\, d\theta \in \{0,1\}$. This and $\cos t = \frac{1}{2}(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})$ imply that for any $m \in \mathbb{Z}^k$, one also has the inequality

$$\int_{\mathbb{T}^N} \prod \cos(m\theta)\, d\theta \geqslant 0. \tag{2.16}$$

The first inequality in (2.14) follows from (2.16), expanding the exponential as

$$\mathrm{e}^{-H(\sigma)} = \sum_{k=0}^{\infty} \frac{1}{k!}(-H(\sigma))^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{xy} J_{xy} \cos(\theta_x - \theta_y) + h^1 \sum_x \cos\theta_x \right)^k \tag{2.17}$$

and using (2.15) since all coefficients are by assumption nonnegative.

To show the second inequality in (2.14), denote by σ' and independent copy of σ. Then

$$\begin{aligned} \mathrm{e}^{-H(\sigma)-H(\sigma')} &= \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{xy} J_{xy} \big(\cos(\theta_x - \theta_y) + \cos(\theta'_x - \theta'_y) \big) + h^1 \sum_x \Big(\cos\theta_x + \cos\theta'_x \Big) \Big)^k \\ &\propto \sum \prod (\cos(m\theta) + \cos(m\theta')), \end{aligned} \tag{2.18}$$

where the sum and product range over collections of $m \in \mathbb{Z}^N$. Therefore, with some sequence of signs $\pm$,

$$\begin{aligned} \langle \cos(a\theta); \cos(b\theta) \rangle &= \frac{1}{2} \langle (\cos(a\theta) - \cos(a\theta'))(\cos(b\theta) - \cos(b\theta')) \rangle \\ &\propto \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \sum \prod (\cos(m\theta) \pm \cos(m\theta'))\, d\theta\, d\theta'. \end{aligned} \tag{2.19}$$

It remains to show that the right-hand side is nonnegative. Since

$$\cos(m\theta) + \cos(m\theta') = 2\cos(m(\theta+\theta')/2)\cos(m(\theta-\theta')/2) \tag{2.20}$$

$$\cos(m\theta) - \cos(m\theta') = 2\sin(m(\theta+\theta')/2)\sin(m(\theta-\theta')/2), \tag{2.21}$$

for any sequence of signs $\pm$, there exists $F : \mathbb{T}^N \to \mathbb{R}$ such that

$$\begin{aligned} \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \prod_{i=1}^{n} (\cos(m_i\theta) \pm \cos(m_i\theta'))\, d\theta\, d\theta' &= \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} F((\theta+\theta')/2) F((\theta-\theta')/2)\, d\theta\, d\theta' \\ &= \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} F(\alpha) F(\beta)\, d\alpha\, d\beta = \left(\int_{\mathbb{T}^N} F(\alpha)\, d\alpha \right) \geqslant 0, \end{aligned} \tag{2.22}$$

where we used the change of variables $\alpha = (\theta+\theta')/2$ and $\beta = (\theta-\theta')/2$. □

Proposition 2.5 (FKG inequality)**.** *Let P be a probability measure on $\mathbb{R}^N$ given by*

$$P(d\varphi) = \frac{1}{Z} \mathrm{e}^{-H(\varphi)}\, \mu^{\otimes N}(d\varphi)\,. \tag{2.23}$$

with $\partial_{\varphi_i}\partial_{\varphi_j} H(\varphi) \leqslant 0$ whenever $i \neq j$. Then P is positively correlated*, i.e., for all increasing functions $F, G : \mathbb{R}^N \to \mathbb{R}$,*

$$\langle F; G \rangle \geqslant 0\,. \tag{2.24}$$

Here $F : \mathbb{R}^N \to \mathbb{R}$ is increasing if $F(\varphi) \leqslant F(\varphi')$ if $\varphi_i \leqslant \varphi'_i$ for all i.

Proof [7]. The proof is by induction in N. For $N = 1$, denote by $\tilde{\varphi}$ an independent copy of φ. Then

$$\langle F; G\rangle = \frac{1}{2}\langle (F(\varphi) - F(\tilde{\varphi}))(G(\varphi) - G(\tilde{\varphi}))\rangle \geqslant 0, \tag{2.25}$$

where the last inequality follows since F and G are increasing. Now assume the claim on $\mathbb{R}^{N-1}$ and write $\varphi = (\varphi', \varphi_N)$. Then

$$\langle FG\rangle = \langle\langle FG|\varphi_N\rangle\rangle, \tag{2.26}$$

where $\langle\cdot|\varphi_N\rangle$ is the conditional expectation on φ_N. The latter is proportional to $\mathrm{e}^{-H(\varphi',\varphi_N)}\mu^{\otimes(N-1)}(d\varphi')$, which clearly satisfies the hypothesis of the proposition. Thus, by the FKG inequality on $\mathbb{R}^{N-1}$,

$$\langle FG\rangle = \langle\langle FG|\varphi_N\rangle\rangle \geqslant \langle\langle F|\varphi_N\rangle\langle G|\varphi_N\rangle\rangle. \tag{2.27}$$

Next, we show that $\langle F|\varphi_N\rangle$ and $\langle G|\varphi_N\rangle$ are increasing in φ_N. Indeed,

$$\frac{\partial}{\partial\varphi_N}\langle F|\varphi_N\rangle = \langle -\frac{\partial H}{\partial\varphi_N}F|\varphi_N\rangle + \langle F|\varphi_N\rangle\langle\frac{\partial H}{\partial\varphi_N}|\varphi_N\rangle = \langle -\frac{\partial H}{\partial\varphi_N}; F|\varphi_N\rangle \geqslant 0, \tag{2.28}$$

where we used that the assumption implies that $-\frac{\partial H}{\partial\varphi_N}$ is increasing in φ' and that the FKG inequality holds for $\langle\cdot|\varphi_N\rangle$. Thus $\langle F|\varphi_N\rangle$ and $\langle G|\varphi_N\rangle$ are increasing on $\mathbb{R}$, and the one-dimensional FKG inequality implies that

$$\langle FG\rangle \geqslant \langle\langle F|\varphi_N\rangle\langle G|\varphi_N\rangle\rangle \geqslant \langle\langle F|\varphi_N\rangle\rangle\langle\langle G|\varphi_N\rangle\rangle = \langle F\rangle\langle G\rangle, \tag{2.29}$$

which was the claim. □

2.2 Mean field bounds on correlations

Throughout this section, the single spin measure μ is the surface measure on $S^{n-1} \subset \mathbb{R}^n$. In particular, for constant coupling $J > 0$, this is the $O(n)$ model at inverse temperature J. Proposition 1.17 shows that the partition function can always be estimated from below by the corresponding mean field approximation. The goal now is to prove that if $\frac{1}{n}\sum_y J_{xy} < 1$ (the critical temperature of the mean field approximation) then the model is in the high temperature phase, in the sense that:

(a) The spontaneous magnetization at $h = 0$ vanishes.

(b) There is a mass gap, i.e., there exists $m > 0$ such that $\langle\sigma_x \cdot \sigma_y\rangle \leqslant \mathrm{e}^{-md(x,y)}$.

Theorem 2.6. *Consider the $O(n)$-model with $n \geqslant 1$ and general spin-spin couplings J.*

(a) *For constant external field $h = (h, 0, \dots, 0)$,*

$$\langle\sigma_x^1\rangle \leqslant \frac{1}{n}\sum_y J_{xy}\langle\sigma_y^1\rangle + h. \tag{2.30}$$

(b) *For any $x \neq z$, for $h = 0$,*

$$\langle\sigma_x^1\sigma_z^1\rangle \leqslant \frac{1}{n}\sum_y J_{xy}\langle\sigma_y^1\sigma_z^1\rangle\,. \tag{2.31}$$

In fact, if $n = 1$, (a) holds with h replaced by $\tanh h \leqslant h$, *and (b) holds with J_{xy} replaced by* $\tanh J_{xy} \leqslant J_{xy}$.

In particular, on the torus $\mathbb{T}_m^d$ with translation invariant J, by (a) the function $m(h) = \langle\sigma_x^1\rangle_h$ obeys $m(h) \leqslant J^{MF}m(h) + h$ where $J^{\mathrm{MF}} = \frac{1}{n}\sum_y J_{xy}$, and thus $m(0_+) = 0$ if $J^{\mathrm{MF}} < 1$. Thus the *spontaneous magnetization* vanishes. Morever, (b) implies that the two-point function decays exponentially if $J^{\mathrm{MF}} < 1$, as shown by the following proposition and the fact that, by translation invariance and continuity, on the torus the assumption $J^{\mathrm{MF}} < 1$ implies that (2.32) below holds for some $\mu > 0$.

Proposition 2.7. *Let $d : V \times V \to [0,\infty)$ be a metric and assume that (2.31) holds. Assume further that*

$$\sup_x \frac{1}{n} \sum_y J_{xy} \mathrm{e}^{\mu d(x,y)} < 1 \tag{2.32}$$

for some $\mu > 0$. Then there exists $m > 0$ such that

$$\langle \sigma_x \cdot \sigma_y \rangle \leqslant \mathrm{e}^{-md(x,y)}. \tag{2.33}$$

Proof [6]. Let $\alpha > 0$ be such that

$$\sup_x \frac{1}{n} \sum_y J_{xy} \mathrm{e}^{\mu d(x,y)} = \mathrm{e}^{-\alpha}. \tag{2.34}$$

Let X_t be a discrete-time random walk with transition probability

$$\mathbb{P}[X_{t+1} = y | X_t = x] = \frac{1}{n} J_{xy} \mathrm{e}^{\mu d(x,y) + \alpha_x}, \tag{2.35}$$

where $\alpha_x \geqslant \alpha$ is chosen such that the right-hand side is a probability distribution. Then the assumption (2.31) implies

$$\begin{aligned} \mathbb{E}[\langle \sigma_{X_{t+1}} \sigma_z \rangle \mathrm{e}^{-\alpha_x - \mu d(x, X_{t+1})} | X_t = x] &= \sum_z \mathbb{P}[X_{t+1} = y | X_t = x] \langle \sigma_z \sigma_z \rangle \mathrm{e}^{-\alpha_x - \mu d(x,y)} \\ &= \sum_z \frac{1}{n} J_{xy} \langle \sigma_y \sigma_z \rangle \geqslant \langle \sigma_x \sigma_z \rangle. \end{aligned} \tag{2.36}$$

Since $|\sigma_x| \leqslant 1$, it therefore follows that

$$\langle \sigma_{X_t} \sigma_z \rangle \leqslant \begin{cases} \mathbb{E}[\langle \sigma_{X_{t+1}} \sigma_y \rangle \mathrm{e}^{-\alpha_{X_t} - \mu d(X_{t+1}, X_t)} | X_t] & (X_t \neq z) \\ 1 & (X_t = z). \end{cases} \tag{2.37}$$

Define $\tau_N = \min\{t \leqslant N : X_t = z\}$ with $\tau_N = N$ if the minimum is not assumed. Then τ_N is a stopping time for the random walk. Using $\alpha_x \geqslant \alpha$ and the triangle inequality for d, we obtain

$$\mathbb{E}\langle \sigma_{X_0} \sigma_z \rangle \leqslant \mathbb{E}[\langle \sigma_{X_1} \sigma_z \rangle \mathrm{e}^{-\alpha - \mu d(X_1, X_0)}] \leqslant \cdots \leqslant \mathbb{E}[\langle \sigma_{X_{\tau_N}} \sigma_z \rangle \mathrm{e}^{-\alpha \tau_N - \mu d(X_{\tau_N}, X_0)}]. \tag{2.38}$$

Finally, with $X(0) = x$ and $N \to \infty$,

$$\mathbb{E}\langle \sigma_x \sigma_z \rangle \leqslant \langle \sigma_z \sigma_z \rangle \mathrm{e}^{-\mu d(x,z)} \lim_{N \to \infty} \mathbb{E}[\mathrm{e}^{-\alpha \tau_N}] \leqslant \mathrm{e}^{-\mu d(x,z)}, \tag{2.39}$$

as claimed. □

Theorem 2.6 is proved differently for $n = 1$ and $n \geqslant 2$.

Proof of Theorem 2.6 ($n = 1$). The inequality (b) follows most easily from the random current representation, and its proof will therefore be deferred. To show (a), fix $x \in V$, and define

$$J_e(\lambda) = \begin{cases} \lambda J_e & (x \in e) \\ J_e & (x \notin e), \end{cases} \tag{2.40}$$

and consider the spin system with spin-spin coupling $J(\lambda)$. For $\lambda = 0$, the spin σ_x completely decouples from all other spins, and it is therefore clear that

$$\langle \sigma_x \rangle_{\lambda=0} = \frac{\sum_{s=\pm 1} s\, e^{sh}}{\sum_{s=\pm 1} e^{sh}} = \frac{\mathrm{e}^h - \mathrm{e}^{-h}}{\mathrm{e}^h + \mathrm{e}^{-h}} = \tanh h. \tag{2.41}$$

Next,

$$\frac{\partial}{\partial\lambda}\langle\sigma_x\rangle_\lambda = \lambda\sum_y J_{xy}\langle\sigma_x;\sigma_x\sigma_y\rangle_\lambda = \lambda\sum_y J_{xy}\Big(\langle\sigma_y\rangle_\lambda - \langle\sigma_x\rangle_\lambda\langle\sigma_x\sigma_y\rangle_\lambda\Big) \leqslant \sum_y J_{xy}\langle\sigma_y\rangle_{\lambda=1}, \tag{2.42}$$

where the last inequality follows from the positivity of correlation functions and their monotonicity in λ (first and second Griffith inequality). Integrating over $0 \leqslant \lambda \leqslant 1$ gives

$$\langle\sigma_x\rangle_{\lambda=1} - \langle\sigma_x\rangle_{\lambda=0} \leqslant \sum_y J_{xy}\langle\sigma_y\rangle_{\lambda=1}, \tag{2.43}$$

and thus

$$\langle\sigma_x\rangle \leqslant \sum_y J_{xy}\langle\sigma_y\rangle + \tanh h \leqslant \sum_y J_{xy}\langle\sigma_y\rangle + h, \tag{2.44}$$

as claimed. □

For $n \geqslant 2$, the proof of Theorem 2.6 uses the idea of *local Ward identity*.

Exercise 2.8. Let Ω be the support of the single spin measure μ and $\gamma_t : \Omega^V \to \Omega^V$ be a smooth family of automorphisms that preserve the a priori measure $\mu^{\otimes V}$. For a smooth observable $A : \Omega^V \to \mathbb{R}$, set

$$\dot{A} = \frac{\partial}{\partial t}\gamma_t(A)\Big|_{t=0}. \tag{2.45}$$

Show that the following local Ward identity holds:

$$\langle\dot{A}\rangle = \langle A\dot{H}\rangle. \tag{2.46}$$

We also require the following correlation inequalities for the $O(n)$ model, which are similar to the first Griffith inequality.

Lemma 2.9. *For the $O(n)$-model, for all constant $h = (h^1, 0, \dots, 0) \geqslant 0$,*

$$\langle((\sigma_x^1)^2 - (\sigma_x^2)^2)\sigma_y^1\rangle \geqslant 0, \quad \langle((\sigma_x^1)^2 - (\sigma_x^2)^2)\sigma_y^1\sigma_z^1\rangle \geqslant 0. \tag{2.47}$$

Proof. We show the first inequality; the second one is analogous. Use polar coordinates for the first two components of σ:

$$\sigma_x = (\cos\theta_x, \sin\theta_x, \sigma^3, \dots, \sigma^n). \tag{2.48}$$

Then

$$((\sigma_x^1)^2 - (\sigma_x^2)^2)\sigma_y^1 = ((\cos\theta_x)^2 - (\sin\theta_x)^2)^2\cos\theta_y = (\cos 2\theta_x)^2\cos\theta_y, \tag{2.49}$$

and the inequality follows from the first inequality in (2.14), at least for $n = 2$. However, the essentially same proof carries through for $n > 2$ (with the other components handled as in Proposition 2.1). □

Proof of Theorem 2.6 ($n \geqslant 2$) [17, 6]. Let $R(t) \in SO(n)$ be a rotation by angle t in the first two components. Fix $x \in V$, and define $\gamma_t : (S^{n-1})^\Lambda \to (S^{n-1})^\Lambda$ by

$$[\gamma_t(\varphi)]_y = \begin{cases} R(t)\varphi_x & (y = x) \\ \varphi_y & (y \neq x). \end{cases} \tag{2.50}$$

Then

$$\dot{H} = -\sum_y J_{xy}(\sigma_x^2\sigma_y^1 - \sigma_x^1\sigma_y^2). \tag{2.51}$$

(a) Since $\dot\sigma_x^1 = \sigma_x^2$ and $\dot\sigma_x^2 = -\sigma_x^1$, by (2.46) we have

$$\langle \sigma_x^1 \rangle = -\langle \dot\sigma_x^2 \rangle = -\langle \sigma_x^2 \dot H \rangle = \sum_y J_{xy} \langle \sigma_x^2 (\sigma_x^2 \sigma_y^1 - \sigma_x^1 \sigma_y^2) \rangle = \sum_y J_{xy} \langle (\sigma_x^2)^2 \sigma_y^1 \rangle. \tag{2.52}$$

Now (2.47) implies

$$\langle \sigma_y^1 \rangle = \langle (\sigma_i \cdot \sigma_i) \sigma_y^1 \rangle = (n-1) \langle (\sigma_x^2)^2 \sigma_y^1 \rangle + \langle (\sigma_x^1)^2 \sigma_y^1 \rangle \geqslant n \langle (\sigma_x^2)^2 \sigma_y^1 \rangle, \tag{2.53}$$

and it follows that

$$\langle \sigma_x^1 \rangle \leqslant \frac{1}{n} \sum_y J_{xy} \langle \sigma_y^1 \rangle. \tag{2.54}$$

(b) Set $A = \sigma_x^2 \sigma_y^1$. Then $\dot A = \sigma_x^1 \sigma_y^1$ and (2.46) implies

$$\langle \sigma_x^1 \sigma_z^1 \rangle = \langle \dot A \rangle = \sum_y J_{xy} \langle (\sigma_x^2)^2 \sigma_z^1 \sigma_y^1 - \sigma_x^2 \sigma_y^1 \sigma_x^1 \sigma_y^2 \rangle \leqslant \sum_y J_{xy} \langle (\sigma_x^2)^2 \sigma_z^1 \sigma_y^1 \rangle \tag{2.55}$$

Since by (2.47), analogously, we have $\langle \sigma_y^1 \sigma_z^1 \rangle \geqslant n \langle (\sigma_x^2)^2 \sigma_y^1 \sigma_z^1 \rangle$, it follows that

$$\langle \sigma_x^1 \sigma_z^1 \rangle \leqslant \frac{1}{n} \sum_y J_{xy} \langle \sigma_z^1 \sigma_y^1 \rangle, \tag{2.56}$$

as claimed. □

It is also possible to bound the two-point function of the XY model in terms of that of the Ising model. This is of particular interest in $d = 2$ since there the critical temperature of the Ising model can be computed explicitly.

Theorem 2.10 (Aizenman–Simon)**.** *For $h = 0$, it holds that $\langle \sigma_x \cdot \sigma_y \rangle_{2J}^{XY} \leqslant \langle \sigma_x \sigma_y \rangle_J^{Ising}$.*

Proof [5]. Let $\langle \cdot \rangle_J^{\mathrm{P}}$ be the two-component spin model taking values in $\{(1,0), (0,1), (-1,0), (0,-1)\}$, i.e., the $\mathbb{Z}_4$ clock/Potts model, and consider more generally $H_\lambda = H + \lambda \sum_x \cos(4\theta_x)$ with expectation $\langle \cdot \rangle^\lambda$. Then $\langle \cdot \rangle^{\mathrm{P}}$ is the limit $\lambda \to \infty$ and $\langle \cdot \rangle^{\mathrm{XY}}$ is $\lambda = 0$. By invariance under rotations by $\pi/2$, clearly

$$\langle \sigma_x \cdot \sigma_y \rangle_J^\lambda = 2 \langle \sigma_x^1 \sigma_y^1 \rangle_J^\lambda. \tag{2.57}$$

Note that the second Griffith inequality for two-component spins (Proposition 2.3) applies to $\langle \cdot \rangle^\lambda$. Therefore the right-hand side is increasing in λ, and

$$\langle \sigma_x \cdot \sigma_y \rangle_{2J}^{\mathrm{XY}} \leqslant 2 \langle \sigma_x^1 \sigma_y^1 \rangle_J^{\mathrm{P}}. \tag{2.58}$$

To estimate the right-hand side, set $s_x = \sigma_x^1 + \sigma_x^2$ and $t_x = \sigma_x^1 - \sigma_x^2$. Then

$$2 \sum_{xy} J_{xy} \sigma_x \cdot \sigma_y = \sum_{xy} J_{xy} (s_x s_y + t_x t_y), \tag{2.59}$$

and thus s and t are independently distributed according to $\langle \cdot \rangle^{\mathrm{Ising}}$ under $\langle \cdot \rangle^P$. It follows that

$$2 \langle \sigma_x^1 \sigma_y^1 \rangle^P = \frac{1}{2} \langle (s_x + t_x)(s_y + t_y) \rangle^P = \langle s_x s_y \rangle^P + \langle s_x t_y \rangle^P = \langle s_x s_y \rangle^P = \langle \sigma_x \sigma_y \rangle^{\mathrm{Ising}} \tag{2.60}$$

as claimed. □

2.3 Convexity: Helffer–Sjöstrand representation and Brascamp–Lieb inequality

Mean field theory is an example of an exactly solvable spin system and thus provides a very useful reference for comparision, as we saw in the previous subsections. An equally important reference system is given by the *Gaussian free field*, which is given by the probability measure

$$P(d\varphi) = \frac{1}{Z} \mathrm{e}^{-S(\varphi)}\, d\varphi, \quad S(\varphi) = \frac{1}{2}\sum_{xy} |\varphi_x - \varphi_y|^2 + \frac{1}{2} m^2 \sum_x |\varphi_x|^2, \tag{2.61}$$

where $d\varphi$ is the Lebesgue measure on $(\mathbb{R}^n)^V$. For notational convenience, we restrict to $n = 1$ throughout this section, though the content does not rely on it. The *action* S is quadratic, and its generator (which is now the same as its constant Hessian) is

$$D^2 S = -\Delta + m^2, \tag{2.62}$$

where Δ is the graph Laplacian defined by $(-\Delta f)_x = \sum_{y\sim x}(f_x - f_y)$. Indeed,

$$\frac{1}{2}\sum_{xy} |\varphi_x - \varphi_y|^2 = \frac{1}{4}\sum_x \sum_{y\sim x} |\varphi_x - \varphi_y|^2 = \frac{1}{4}\sum_x \sum_{y\sim x} \Big(|\varphi_x|^2 + |\varphi_y|^2 - 2\varphi_x \cdot \varphi_y\Big) \tag{2.63}$$

$$= \frac{1}{2}\sum_x \sum_{y\sim x} \varphi_x \cdot (\varphi_x - \varphi_y) = \frac{1}{2}\sum_x \varphi_x \cdot (-\Delta\varphi)_x. \tag{2.64}$$

Much can now be calculated using the Gaussian structure. We can return to this in detail later. For now, we only note that two-point function is always summable if $m^2 > 0$, and thus the spin system is in a high temperature phase if $m^2 > 0$.

Exercise 2.11. Let $m^2 > 0$. Show that

$$\langle \varphi_x \varphi_y \rangle = (-\Delta + m^2)^{-1}_{xy}, \qquad \sum_y \langle \varphi_x \varphi_y \rangle = \frac{1}{m^2} < \infty, \tag{2.65}$$

and that on the torus, $(-\Delta + m^2)^{-1}_{xy}$ actually decays exponentially in $|x - y|$.

Given an *action* $S : \mathbb{R}^N \to \mathbb{R}$, we now consider the general probability measure (assuming it is defined)

$$P(d\varphi) = \frac{1}{Z} \mathrm{e}^{-S(\varphi)}\, d\varphi, \tag{2.66}$$

and denote its expectation by $\langle \cdot \rangle$ or $\langle \cdot \rangle_S$ (if another expectation appears). Compared to mean field theory, where the dimension n was the number of spin components and therefore fixed (for example $n = 1$ for the Ising model), now we are interested in the case that the dimension is $N = n|V|$ is unbounded (proportional to the number of vertices).

Notation. For any smooth enough functions $F : \mathbb{R}^N \to \mathbb{R}$, write $D_x F = \frac{\partial}{\partial \varphi_x} F$, $DF = (D_1 F, \dots, D_N F)$, and $D^2 F$ for the Hessian of F. For two quadratic forms Q and Q' (such as $D^2 F(\varphi)$), we write $Q \leqslant Q'$ if $Q(h, h) \leqslant Q'(h, h)$ for any h; in particular $Q \leqslant c$ means $Q(h, h) \leqslant c|h|^2$ for any h.

Definition 2.12. *The Dirichlet form associated to P is*

$$\langle (DF)(DG) \rangle = \sum_x \langle (D_x F)(D_x G) \rangle. \tag{2.67}$$

Its symmetric generator (acting on sufficiently nice test functions) is given by

$$L = -\sum_x D_x^2 + \sum_x (D_x S) D_x. \tag{2.68}$$

For Dirichlet forms, the generator L can always be defined as a self-adjoint linear operator on $L^2(P)$. There are many textbook references treating this in detail; e.g. [41, 39]. In this section, we will not be too careful about such points. For the material of this section, see in particular [30].

Exercise 2.13. Check that L is the generator of (2.67) in the sense that for nice F and G it holds that

$$\langle (DF)(DG)\rangle = \langle F(LG)\rangle. \tag{2.69}$$

Strong results about P relating to the Gaussian case (in which D^2S is constant) exist if V is uniformly strictly convex, which we will assume throughout this section.

Definition 2.14. $S : \mathbb{R}^N \to \mathbb{R}$ *is* uniformly strictly convex *if* $D^2S(\varphi) \geqslant c\,\mathrm{id}$ *as quadratic forms for* $c > 0$ *independent of* φ.

Proposition 2.15 (Helffer–Sjöstrand representation)**.** *Assume that S is uniformly strictly convex (and perhaps that S and its derivatives obey some growth condition). Then for any $F, G : \mathbb{R}^N \to \mathbb{R}$ (sufficiently nice),*

$$\langle F; G\rangle = \langle DF(\varphi) \cdot \mathcal{L}^{-1} DG(\varphi)\rangle, \tag{2.70}$$

where $\mathcal{L}$ is the Witten Laplacian *on $L^2(P) \otimes \mathbb{R}^N$ given by*

$$\mathcal{L} = L \otimes \mathrm{id} + D^2 S. \tag{2.71}$$

For the proof, we require some (perhaps somewhat technical) properties of L. Note that $U : L^2(\mathbb{R}^N) \to L^2(P)$, $F \mapsto UF = e^{\frac{1}{2}S} F$ is a unitary map, and that

$$ULU^{-1} = -\sum_x D_x^2 + \frac{1}{4}\sum_x (D_x S)^2 - \frac{1}{2}\sum_x D_x^2 S. \tag{2.72}$$

Thus L is unitarily equivalent to the Schrödinger operator $-\sum_x D_x^2 + V$ with potential $V = \frac{1}{4}\sum_x (D_x S)^2 - \frac{1}{2}\sum_x D_x^2 S$.

Lemma 2.16. $V \to \infty$ *implies that* $-\sum_x D_x^2 + V$ *(and thus L) have compact resolvent.*

Proof. [40, Theorem XIII.67] □

The assumption of the lemma can be weakened somewhat [30, Proposition 6.8.2]. The typical example of a continuous spin model that we are interested in is the $|\varphi|^4$ model given by $S = \frac{1}{2}(\varphi, -\Delta\varphi) + \sum_x v(\varphi_x)$ with $v(t) = \frac{1}{4}gt^4 + \frac{1}{2}\nu t^2$ and $g > 0$. Then

$$D_x S = (-\Delta\varphi)_x + g\varphi_x^3 + \nu\varphi_x, \quad D_x^2 S = 2d + \nu + 3g\varphi_x^2, \tag{2.73}$$

and the growth assumption $V \to \infty$ is easily satisfied.

Lemma 2.17. *Under the previous condition,* $\mathrm{Im}\, L = \{F \in L^2(P) : \langle F\rangle = 0\}$.

Proof. By the spectral theorem for compact self-adjoint operators, the eigenvalues of L are discrete and since L is positive they are all nonnegative. The lowest eigenvalue 0 is simple because $LF = 0$ implies

$$0 = \langle F(LF)\rangle = \langle (DF)^2\rangle, \tag{2.74}$$

and thus $DF = 0$ almost everywhere, which implies that F is constant. Therefore $\ker L = \{\text{constants}\}$, and $\mathrm{Im}\, L = (\ker L)^\perp$ since L is self-adjoint. □

Proof of Proposition 2.15. By Lemma 2.17, there exists U such that $G - \langle G \rangle = LU$, and therefore

$$DG = DLU = \mathcal{L}DU. \tag{2.75}$$

Since L is a positive operator on $L^2(P)$ and S is uniformly strict convex, so D^2S is strictly positive, the operator $\mathcal{L}$ is invertible on $L^2(P) \otimes \mathbb{R}^N$. Thus (2.75) implies that $DU = \mathcal{L}^{-1}DG$. Thus

$$\langle F; G \rangle = \langle F(G - \langle G \rangle) \rangle = \langle F(LU) \rangle = \langle (DF)(DU) \rangle = \langle DF \cdot \mathcal{L}^{-1} DG \rangle, \tag{2.76}$$

so (2.70) holds as claimed. □

Corollary 2.18 (Brascamp–Lieb inequality)**.** *Assume that S is uniformly strictly convex. Then*

$$\langle F; F \rangle \leqslant \langle DF(\varphi) \cdot [D^2S(\varphi)]^{-1} DF(\varphi) \rangle. \tag{2.77}$$

Proof. We prove the inequality only for S such that the conclusion of Proposition 2.15 holds. By definition, L is a positive operator on $L^2(P)$, and by assumption $D^2S(\varphi)$ is positive and has bounded inverse. This implies the quadratic form inequality

$$\mathcal{L}^{-1} = (L \otimes 1 + D^2S(\varphi))^{-1} \leqslant (D^2S(\varphi))^{-1}. \tag{2.78}$$

The claim then follows from (2.70). □

In particular, the Brascamp–Lieb inequality implies bounds on correlations.

Corollary 2.19. *Assume that $S \in C^2$ is even and that $D^2S(\varphi) \geqslant Q$ for a strictly positive quadratic form Q on $\mathbb{R}^N$, uniformly in φ. Then, for any $h \in \mathbb{R}^N$,*

$$\langle \mathrm{e}^{(h,\varphi)} \rangle \leqslant \mathrm{e}^{\frac{1}{2}(h, Q^{-1}h)}. \tag{2.79}$$

Proof. Set $F(\varphi) = (h, \varphi)$ and $v(t) = \log\langle \mathrm{e}^{tF} \rangle$. Then the claim is equivalent to $v(t) \leqslant \frac{1}{2}t^2(h, Q^{-1}h)$. Let

$$\langle G \rangle_t = \frac{\langle G e^{-tF} \rangle}{\langle e^{-tF} \rangle}. \tag{2.80}$$

Since F is linear in φ, the Brascamp–Lieb inequality (2.77) also applies to $\langle \cdot \rangle_t$ instead of $\langle \cdot \rangle$, and therefore

$$v''(t) = \langle F; F \rangle_t \leqslant \langle DF \cdot [D^2S]^{-1} DF \rangle_t \leqslant \langle DF \cdot Q^{-1} DF \rangle_t = (h, Q^{-1}h). \tag{2.81}$$

Since $v(0) = 0$ and $v'(0) = \langle F \rangle = 0$ (since V is even), this implies

$$v'(t) = \int_0^t v''(s)\, ds \leqslant t(h, Q^{-1}h), \qquad v(t) = \int_0^t v'(s)\, ds \leqslant \frac{t^2}{2}(h, Q^{-1}h), \tag{2.82}$$

as claimed. □

Example 2.20 (Boundedness of the susceptibility)**.** For any continuous spin model with $S(\varphi) = D(\varphi) + \sum_x v(\varphi_x)$ with $D(\varphi) = \frac{1}{2}\sum_{xy} |\varphi_x - \varphi_y|^2$ and $v''(t) \geqslant \delta > 0$, the Brascamp–Lieb inequality implies

$$\chi = \frac{1}{|V|} \sum_{x,y} \langle \varphi_x; \varphi_y \rangle \leqslant \frac{1}{|V|} \sup_{\varphi} [D^2S(\varphi)]^{-1}(1, 1) \leqslant \frac{1}{\delta}. \tag{2.83}$$

Thus a strictly convex action implies that the susceptibility is bounded (as in mean field theory).

Remark 2.21 (Decay of correlations)**.** Using the Helffer-Sjöstrand representation, it can also be shown that the two-point function decays exponentially; see [29].

2.3.1. Application: Hubbard–Stratonovich transform in general spin systems. Let J be the coupling matrix of the Ising or $O(n)$ model. By addition of a constant diagonal matrix, we can assume that J is positive definite; the addition of the constant does does not change the probability measure if the length of the spins is constant.

Exercise 2.22. For any positive definite $N \times N$ matrix J, it holds that

$$\mathrm{e}^{\frac{1}{2}(\sigma,J\sigma)} \propto \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{1}{2}(\psi,J\psi)}\mathrm{e}^{(\sigma,J\psi)}\,d\psi \propto \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{1}{2}(\psi,J^{-1}\psi)}\mathrm{e}^{(\sigma,\psi)}\,d\psi. \tag{2.84}$$

Exactly as in (1.46)–(1.47), with the same definition of the function $v : \mathbb{R}^n \to \mathbb{R}$ as below (1.47), the first identity in (2.84) implies

$$\int_{(\mathbb{R}^n)^V} \mathrm{e}^{\frac{1}{2}(\sigma,J\sigma)+(h,\sigma)}\,\mu^{\otimes V}(d\sigma) \propto \int_{(\mathbb{R}^n)^V} \mathrm{e}^{-\frac{1}{2}(\psi,J\psi)}\prod_x e^{v((J\psi)_x+h_x)}\,d\psi = \mathrm{e}^{\frac{1}{2}(h,J^{-1}h)}\int_{(\mathbb{R}^n)^V} \mathrm{e}^{-S(\psi)+(h,\psi)}\,d\psi, \tag{2.85}$$

with action

$$S(\psi) = \frac{1}{2}(\psi,J\psi) - \sum_x v((J\psi)_x). \tag{2.86}$$

In contrast to (1.47), the integral is now over nN instead of n variables. Regarding the partition function $Z = Z(h)$ as a function of h, the two-point function is given by

$$\langle\sigma_x\sigma_y\rangle = D^2\log Z(0;\delta_x,\delta_y) = [J^{-1}]_{xy} + \langle\psi_x\psi_y\rangle_S. \tag{2.87}$$

An alternative form is obtained using the second inequality of (1.47) (i.e, by rescaling by J^{-1}). Then the action becomes

$$S(\psi) = \frac{1}{2}(\psi,J^{-1}\psi) - \sum_x v(\psi_x). \tag{2.88}$$

In this case, the spin-spin coupling J is not nearest-neighbour in the transformed variables ψ, but with a diagonal added above it typically (say on the torus) has exponential decay; see below. In the following proposition, we assume the form (2.86).

Proposition 2.23. *Assume that* $\sup\nabla^2 v \leqslant \lambda\,\mathrm{id}$ *and that* $\|J\| \leqslant (1-\delta)/\lambda$*. Then*

$$D^2S(\psi) \geqslant \delta J. \tag{2.89}$$

In particular, S is then uniformly strictly convex. Thus (possibly subject to technical growth assumptions on v), the susceptibility is bounded (Example 2.20) and the two-point function decays exponentially (at least if J^{-1} does and perhaps technical conditions to show Remark 2.21 are satisfied).

Proof. By definition,

$$D^2_\varphi\Big(\sum_x v(\varphi_x)\Big) = \mathrm{diag}(\nabla^2 v(\varphi_x)) \leqslant \lambda\,\mathrm{id}, \tag{2.90}$$

where diag is a block diagonal matrix, consisting of $n\times n$ blocks. Thus, with $D^2_\psi F(J\psi) = J^T[D^2F](J\psi)J \leqslant cJ^TJ$ for $F(\varphi) = \sum_x v(\varphi_x)$, it follows that

$$D^2S(\psi) \geqslant J - \lambda J^TJ = (1-\lambda J^T)J, \tag{2.91}$$

and since $\|J\| \leqslant (1-\delta)/\lambda$ the claim (2.89) follows. □

Example 2.24. Consider the nearest-neighbour Ising model on the torus $\mathbb{T}_m^d$, i.e., with $J_{xy} = \beta\mathbf{1}(x \sim y) + \beta(2d + \lambda)\mathbf{1}(x = y)$ the coupling matrix with a constant diagonal term added, making J strictly positive definite for $\lambda > 0$. Indeed, its Fourier transform is

$$\hat{J}(k)/\beta = 2\sum_{i=1}^{d} \cos(k_i) + 2d + \lambda > 0 \quad \text{if } \lambda > 0, \tag{2.92}$$

and $\|J\| \leqslant (4d + \lambda)\beta$. Moreover, since $1/\hat{J}$ is analytic in a strip around the real axis, if $\lambda > 0$, it can be shown that there exists some $m > 0$ such that

$$[J^{-1}]_{xy} \leqslant \mathrm{e}^{-m|x-y|}. \tag{2.93}$$

By Example 1.13, for the Ising model,

$$v(t) = \log\cosh(t) + \log 2, \quad v'(t) = \tanh(t) \in [-1, 1], \quad v''(t) = \frac{1}{\cosh^2(t)} \leqslant 1. \tag{2.94}$$

Thus $\lambda = 1$ and the condition of Proposition 2.23 is satisfied if $(4d+\lambda)\beta \leqslant 1-\delta$. Hence, for any $\beta < 1/4d$, there are some $\delta, \lambda > 0$ so that the condition holds. (Can this be improved to give the mean field bound $\beta < 1/2d$ on the transition temperature? The loss of the factor $1/2$ might be technical.)

The growth assumption $V \to \infty$ should also be satisfied. (Verify that $(t+v'(t))^2 \to \infty$ as $|t| \to \infty$ since $|v'| \leqslant 1$, and $\|J\psi\| \to \infty$ as $\|\psi\| \to \infty$ since $\|\psi\| \leqslant \frac{1}{\lambda}\|J\psi\|$. Thus $(D_x S)^2 = ((J\varphi)_x + v'((J\varphi)_x))^2 \to \infty$ and $D_x^2 S$ is bounded, so that $V \to \infty$.)

Example 2.25. Instead of nearest-neighbour spin-spin coupling, one can of course consider more general J_{xy} such as finite range distributions (e.g., constant in a band), or with exponential decay in $|x-y| \to \infty$. In these cases, the behaviour of the model is expected to remain identical. In particular, one can consider

$$J = \beta(1 - w^2\Delta)^{-1}. \tag{2.95}$$

Then $0 \leqslant J_{xy}/\beta \approx C\mathrm{e}^{-c|x-y|/w}$ (so w is a measure of the bandwidth) and $\sum_y J_{xy} = (J1)_x = \beta$ (so β is the mean field temperature). In particular, as $w \to 0$, J becomes diagonal (independent spins), while in the limit $w \to \infty$, J_{xy} becomes constant for all $x, y \in V$ (mean field theory). Then, with this choice of J, instead of the first equality in (2.84) it is better to use the second one, giving

$$\begin{aligned}\mathrm{e}^{\frac{1}{2}(\sigma,J\sigma)} \propto \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{1}{2}(\psi,J^{-1}\psi)}\mathrm{e}^{(\sigma,\psi)}\,d\psi &\propto \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{1}{2\beta}(\psi,-w^2\Delta\psi)-\frac{1}{2\beta}(\psi,\psi)}\mathrm{e}^{(\sigma,\psi)}\,d\psi \\ &\propto \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{\beta}{2}(\psi,-w^2\Delta\psi)-\frac{\beta}{2}(\psi,\psi)}\mathrm{e}^{(\sigma,\beta\psi)}\,d\psi.\end{aligned} \tag{2.96}$$

Thus the action after Hubbard–Stratonovich transform becomes

$$S(\psi) = \frac{\beta}{2}(\psi, -w^2\Delta\psi) + \sum_x \left(\frac{\beta}{2}|\psi_x|^2 - v(\beta\psi_x)\right), \tag{2.97}$$

which has nearest-neighbour spin-spin coupling in the ψ variables. In particular, for the Ising model, the action becomes strictly convex if $\beta > \beta^2$, i.e., $\beta < 1$ as in mean field theory.

3 Low temperature by inequalities

Low temperatures are much more difficult to understand than high temperatures, and a robust understanding is still lacking for $n > 2$. In this section, we consider non-expansion methods applicable for low temperature.

3.1 Peierls argument

The Peierls argument shows that the Ising model has a phase transition in dimensions $d \geqslant 2$.

Theorem 3.1. *Let $d \geqslant 2$. For any $\Lambda \subset \mathbb{Z}^d$, for the nearest-neighbour Ising model on Λ with $+$ boundary conditions outside Λ, for any $x \in \Lambda$, it holds that $\langle \sigma_x \rangle \to 1$ as $J \to \infty$, uniformly in Λ and x.*

Proof. We restrict to $d = 2$; the same argument applies in $d > 2$, but requires slightly more care there. The energy of a spin configuration σ is

$$H = -J \sum_{xy} \sigma_x \sigma_y, \tag{3.1}$$

Thus the energy difference between a $++$ and $+-$ pair of spins is $2J$. Moreover, the energy cost of an island of $-$ spins surrounded by $+$ spins (or vice-versa) is $2J \times$ (length of the boundary). The configurations σ are in correspondence with contours separating $+$ and $-$ island; denote these contours by Γ. Then

$$H = 2J|\Gamma| - J|E|. \tag{3.2}$$

Decompose Γ into disconnected components γ (breaking ambiguities). The probability of some γ is

$$p(\gamma) = \frac{\sum_{\Gamma \supset \gamma} \mathrm{e}^{-2J|\Gamma|}}{\sum_{\Gamma} \mathrm{e}^{-2J|\Gamma|}}. \tag{3.3}$$

Since, given any γ,

$$\sum_{\Gamma} \mathrm{e}^{-2J|\Gamma|} \geqslant \sum_{\Gamma' = \Gamma \setminus \gamma, \Gamma \supset \gamma} \mathrm{e}^{-2J|\Gamma'|} = \mathrm{e}^{+2J|\gamma|} \sum_{\Gamma \supset \gamma} \mathrm{e}^{-2J|\Gamma|}, \tag{3.4}$$

it holds that

$$p(\gamma) \leqslant \mathrm{e}^{-2J|\gamma|}. \tag{3.5}$$

Since $\langle \sigma_x \rangle = 1 - 2\langle 1_{\sigma_x = -1} \rangle$, and since $\sigma_x = -1$ implies that x is enclosed by some γ, written $x \subset \gamma$, it suffices to estimate

$$\langle 1_{\sigma_x = -1} \rangle \leqslant \sum_{\gamma \supset x} p(\gamma). \tag{3.6}$$

Fix some y and consider contours γ with $\gamma \ni y$ (on the contour). Let $2n$ be the number of horizontal edges and $2m$ the number of vertical edges of γ. Then

$$(\text{number of } \gamma \ni y \text{ with given } n, m) \leqslant 3^{(2n+2m)}. \tag{3.7}$$

Fix some x and a contour $\gamma \supset x$ (enclosing x). Then

$$(\text{number of translates } \gamma' \text{ of } \gamma \text{ such that } \gamma' \supset x) \leqslant nm. \tag{3.8}$$

It follows that

$$\langle 1_{\sigma_x = -1} \rangle \leqslant \sum_{n,m} nm 3^{2n+2m} \mathrm{e}^{-2J(2m+2n)} = \underbrace{\left(\sum_n n \mathrm{e}^{2n(\log 3 - 2J)} \right)^2}_{g(J)}. \tag{3.9}$$

Clearly, $g(J) \to 0$ as $J \to \infty$. It follows that, as $J \to \infty$,

$$\langle \sigma_x \rangle \geqslant 1 - 2g(J)^2 \to 1 \tag{3.10}$$

as claimed. □

3.2 Infrared bound and reflection positivity

The most important result for spin systems in dimensions greater than two is the *infrared bound* [21]. It only applies to the discrete torus $\mathbb{T}_m^d$ with constant spin-spin couplings J (and other interactions that are reflection positive), which we assume throughout this section. For convenience, we normalize $J_e = 1$ for all e, by rescaling μ. The Hamiltonian (without magnetic field) is then

$$H = -\sum_{xy} \varphi_x \cdot \varphi_y = \frac{1}{2}\sum_{xy} |\varphi_x - \varphi_y|^2 - \frac{d}{4}\sum_x |\varphi_x|^2. \tag{3.11}$$

The second term on the right-hand side can be kept or dropped at convenience, by changing the single spin measure μ.

Theorem 3.2 (Infrared bound)**.** *Consider a spin system on $(V,E) = \mathbb{T}_m^d$ with $J_e = 1$ for all $e \in E$ and $O(n)$ invariant single spin measure μ. Then for any $h : V \to \mathbb{R}^n$ with $\sum_x h_x = 0$,*

$$\langle \mathrm{e}^{(h,\varphi)} \rangle \leqslant \mathrm{e}^{\frac{1}{2}(h,(-\Delta)^{-1}h)}. \tag{3.12}$$

In particular, for h with $\sum h = 0$ then

$$\langle (h,\varphi)^2 \rangle \leqslant (h, (-\Delta)^{-1}h). \tag{3.13}$$

In the above normalization, in which $J = 1$, the infrared bound provides a uniform bound on fluctuations, *independent of the measure μ (and thus the temperature)*, as opposed to (2.79), for example, which is only effective at high temperature. On the other hand, the magnitude of the spin field φ_x can be made large by choice of the single-spin measure μ. In particular, in the $O(n)$ model with above normalization, $|\varphi_x| = \sqrt{\beta}$ becomes large for large β (small temperatures).

Proposition 3.3. *Fix a single-spin measure μ_1 and set $\mu_\beta(ds) = \mu_1(ds/\sqrt{\beta})$. As $\beta \to \infty$, it holds that*

$$\lim_{N\to\infty} \langle |\varphi_x|^2 \rangle_\beta \to \infty. \tag{3.14}$$

Proof. Let $p_N = \frac{1}{N}\log Z$ and $p = \lim_{N\to\infty} p_N$. Let $M_\beta = \max\{|s| : s \in \operatorname{supp}\mu_\beta\}$. First, we show

$$p(\beta) \sim 2dM_\beta^2 = 2d\beta M_1^2 \quad (\beta \to \infty). \tag{3.15}$$

Let $A_1 \subset \mathbb{R}^n$ be a with $s \cdot s' \geqslant (1-\delta)M_1^2$ for $s, s' \in A_1$ and $\mu_1(A_1) > 0$, and set $A = \sqrt{\beta}A_1$. Then

$$\mathrm{e}^{2dM^2|V|}\mu(\mathbb{R}^n)^{|V|} \geqslant Z(\beta) \geqslant \mathrm{e}^{2dM^2|V|}\mu(A)^{|V|}, \tag{3.16}$$

and therefore

$$2dM^2 + \log\mu(\mathbb{R}^n) \geqslant p_N \geqslant 2dM^2 + \log\mu(A). \tag{3.17}$$

Since $\mu(\mathbb{R}^n)$ and $\mu(A)$ are independent of β, this shows

$$p(\beta) \sim 2d\beta M_1^2 \quad (\beta \to \infty). \tag{3.18}$$

Finally, the inequality $\mathrm{e}^t - 1 \leqslant t\mathrm{e}^t$ (convexity) implies

$$\langle |\varphi_x|^2 \rangle \geqslant \frac{1}{2d}\sum_{y\sim x} \langle \varphi_x \cdot \varphi_y \rangle \geqslant p(\beta) - p(0) \sim 2d\beta M_1^2, \tag{3.19}$$

and the claim follows. □

Notation. The Fourier transform of a function $f : \mathbb{Z}^d \to \mathbb{C}$ or $f : \mathbb{T}_m^d \to \mathbb{C}$ is given by $\hat{f}(k) = \hat{f}_k = \sum_x e^{ik\cdot x} f_x$, and k takes values in the Fourier dual $(\mathbb{Z}^d)^* = \mathbb{T}^d = [-\pi,\pi]^d$ respectively in $(\mathbb{T}_m^d)^* = [-\pi,\pi]^d \cap m^{-1}\mathbb{Z}^d$. These Fourier space tori are not to be confused with the torus $\mathbb{T}_m^d$ in real space. In the last case, we identify $\hat{f}$ with a function that is constant on cubes of side length $1/m$ such that

$$\int_{\mathbb{T}^d} f(k)\, dk = \frac{1}{m^d} \sum_{k\in(\mathbb{T}_m^d)^*} f(k). \tag{3.20}$$

The Fourier inversion formula is then $f_x = \frac{1}{2\pi}\int e^{ik\cdot x}\hat{f}(k)\,dk$.

Exercise 3.4. Let $\hat{\Delta}(k) = 2\sum_{i=1}^d (\cos k_i - 1)$ be the Fourier multiplier of the discrete Laplace operator Δ. Then

$$\int_{\mathbb{T}^d} \frac{dk}{-\hat{\Delta}(k)} \begin{cases} < \infty & (d > 2) \\ = \infty & (d \leqslant 2), \end{cases} \tag{3.21}$$

where, in the discrete case, $< \infty$ means uniformly bounded and $= \infty$ means unbounded.

The infrared bound, together with observation that $|\varphi_x|$ grows with β, implies that there is *long-range order* for large β and $d \geqslant 3$. Long-range order means that the two-point function $\langle \varphi_x \varphi_y \rangle$ does not decay to 0 as the distance between x and y becomes large (as it does for high temperatures). In different terms, it means that the ground state *spin wave* is *macroscopically occupied*. Without reflection positivity, which is used in the proof of Theorem 3.2, such *continuous symmetry breaking* is very difficult to prove.

Corollary 3.5 (Long-range order, spin wave condensation)**.**

$$\sum_y \langle \varphi_x \cdot \varphi_y \rangle \geqslant N \left[\langle |\varphi_x|^2 \rangle - \int_{(\mathbb{T}_m^d)^* \setminus 0} (-\hat{\Delta})^{-1}(k)\, dk \right], \tag{3.22}$$

and in the infinite volume limit,

$$\limsup_{|x-y|\to\infty} \langle \varphi_x \cdot \varphi_y \rangle \geqslant \langle |\varphi_x|^2 \rangle - \int (-\hat{\Delta})^{-1}(k)\, dk. \tag{3.23}$$

Since, in $d \geqslant 3$, the sum $\int_{(\mathbb{T}_m^d)^*\setminus 0} (-\hat{\Delta})^{-1}(k)$ is uniformly bounded, it follows that the 0-mode is macroscopically occupied whenever $\langle |\varphi_x|^2 \rangle$ is sufficiently large.

Proof. Denoting by $\hat{\varphi}_k = \sum_x e^{ik\cdot x}\varphi_x$ the Fourier transform of φ, and by $\hat{\Delta}(k) = 2\sum_{i=1}^d(\cos k_i - 1)$ the Fourier multiplier of the discrete Laplace operator Δ, the infrared bound (3.13) implies that, for $k \neq 0$,

$$\langle |\hat{\varphi}_k|^2 \rangle \leqslant N(-\hat{\Delta})^{-1}(k). \tag{3.24}$$

On the other hand, by Parseval's identity,

$$\left\langle \frac{1}{N} \sum_k |\hat{\varphi}_k|^2 \right\rangle = \left\langle \sum_x |\varphi_x|^2 \right\rangle = N \langle |\varphi_x|^2 \rangle. \tag{3.25}$$

Thus

$$\begin{aligned} \sum_y \langle \varphi_x \cdot \varphi_y \rangle &= \frac{1}{N} \sum_{x,y} \langle \varphi_x \cdot \varphi_y \rangle = \frac{1}{N} \left\langle \Big| \sum_x \varphi_x \Big|^2 \right\rangle \\ &= \frac{1}{N} \langle |\hat{\varphi}_0|^2 \rangle = N \langle |\varphi_x|^2 \rangle - \frac{1}{N} \sum_{k\neq 0} \langle |\hat{\varphi}_k|^2 \rangle \geqslant N \left[\langle |\varphi_x|^2 \rangle - \frac{1}{N} \sum_{k\neq 0} (-\hat{\Delta})^{-1}(k) \right], \end{aligned} \tag{3.26}$$

as claimed. □

Proof [21]. $\langle \varphi_x \cdot \varphi_y \rangle$ is a positive definite $\mathbb{T}_m^d \times \mathbb{T}_m^d$ matrix. Bochner's theorem implies that there exists a positive measure ω on $(\mathbb{T}_m^d)^*$ such that

$$\langle \varphi_x \cdot \varphi_y \rangle = \int \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\, \omega(dk). \tag{3.27}$$

The infrared bound implies that for any h with $\hat{h}(0) = 0$, it holds that

$$\int |\hat{h}(k)|^2\, \omega(dk) = \langle (h,\varphi)^2 \rangle \leqslant (h, (-\Delta)^{-1}h) = \int \frac{|\hat{h}(k)|^2}{-\hat{\Delta}(k)}\, dk. \tag{3.28}$$

Since h is arbitrary with $\hat{h}(0) = 0$, this implies that

$$\omega = c\delta_0 + g \tag{3.29}$$

for some constant $c \geqslant 0$ and $g : \mathbb{T}^d \to \mathbb{R}$ absolutely continuous with $g(k) \leqslant 1/(-\hat{\Delta}(k))$. Since $d \geqslant 3$, in particular $\int g\, dk$ is bounded independently of β. It follows that

$$\langle |\varphi_x|^2 \rangle = \int d\omega = c + \int g\, dk \tag{3.30}$$

Thus

$$\langle \varphi_x \cdot \varphi_y \rangle = \int \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\, \omega(dk) = c + \int \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\, g(k)\, dk = \langle |\varphi_x|^2 \rangle - \int g(k)\, dk + \int \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\, g(k)\, dk, \tag{3.31}$$

as claimed. □

The proof of Theorem 3.2 relies on the method of *reflection-positivity*. While powerful when applicable, it exploits symmetries of the torus in a crucial way, and fails for more general geometries (for which the infrared bound is believed to continue to hold, but few methods of proof are known).

Let $H(\varphi) = \frac{1}{2}(\varphi, -\Delta\varphi)$. Since, for any h with $\sum h = 0$,

$$H(\varphi) - (h,\varphi) = \frac{1}{2}(\varphi, -\Delta\varphi) - (h,\varphi) = H(\varphi - \Delta^{-1}h) + \frac{1}{2}(h, (-\Delta)^{-1}h), \tag{3.32}$$

we have

$$\int \mathrm{e}^{-H(\varphi)+(h,\varphi)} \mu^{\otimes V}(d\varphi) = \mathrm{e}^{\frac{1}{2}(h,(-\Delta)^{-1}h)} \int e^{-H(\varphi-\Delta^{-1}h)} \mu^{\otimes V}(d\varphi) = \mathrm{e}^{\frac{1}{2}(h,(-\Delta)^{-1}h)} Z(-\Delta^{-1}h) \tag{3.33}$$

where

$$Z(h) = \int \mathrm{e}^{-H(\varphi+h)} \mu^{\otimes V}(d\varphi). \tag{3.34}$$

To prove Theorem 3.2, it therefore suffices to show that $Z(h) \leqslant Z(0)$ for any h.

Consider a plane going through the midpoints of edges (an edge plane) that splits the torus into two halves. (It is also possible to consider planes going through vertices instead of edges.) The plane gives a decomposition $V = V_+ \cup V_-$ and $E = E_+ \cup E_- \cup E_0$, with E_0 the edges between the two halves. Let $\theta : V_\pm \to V_\mp$ be the reflection about this plane, and

$$(\theta\varphi)_x = \varphi_{\theta(x)}, \quad (\theta F)(\varphi) = F(\theta\varphi). \tag{3.35}$$

Definition 3.6 (reflection positivity)**.** *Let* $(V,E) = \mathbb{T}_m^d$*. A measure on* Ω^V *is reflection positive if*

$$\langle F\theta G\rangle = \langle G\theta F\rangle, \qquad \langle F\theta F\rangle \geqslant 0. \tag{3.36}$$

Lemma 3.7. *Any product measure* $\mu^{\otimes V}$ *is reflection positive.*

Proof. Clearly, $\varphi|_{V_+}$ and $\varphi|_{V_-}$ are independent, so

$$\langle F\theta G\rangle = \langle F\rangle\langle \theta G\rangle = \langle F\rangle\langle G\rangle, \tag{3.37}$$

and both conditions for reflection positivity are obvious from this. □

By definition, reflection positivity of $\langle\cdot\rangle$ means that $\langle F\theta G\rangle$ is a symmetric positive semi-definite bilinear form. The importance of reflection positivity results from the Cauchy–Schwarz inequality

$$\langle F\theta G\rangle^2 \leqslant \langle F\theta F\rangle\langle G\theta G\rangle. \tag{3.38}$$

Lemma 3.8. *Let* θ *be a reflection and* $\langle\cdot\rangle$ *reflection positive. Then for any* $A,B,C,D : \Omega^{V_+} \to \mathbb{R}$*,*

$$\langle \mathrm{e}^{A+\theta B + C\theta D}\rangle^2 \leqslant \langle \mathrm{e}^{A+\theta A+C\theta C}\rangle\langle \mathrm{e}^{B+\theta B+D\theta D}\rangle, \tag{3.39}$$

and the measures $\langle(\cdot)\mathrm{e}^{A+\theta A+C\theta C}\rangle$ *and* $\langle(\cdot)\mathrm{e}^{B+\theta B+D\theta D}\rangle$ *are reflection positive. In fact, the same holds with* $C\theta D$*,* $C\theta C$*, and* $D\theta D$ *replaced by sums of such terms.*

Proof. Expand the exponential as

$$\mathrm{e}^{A+\theta B+C\theta D} = \sum_{k=0}^\infty \frac{1}{k!} \underbrace{(\mathrm{e}^A C^k)}_{X_k}\, \theta\, \underbrace{(\mathrm{e}^B D^k)}_{Y_k}. \tag{3.40}$$

Then the Cauchy-Schwarz inequality (twice) implies

$$\langle \mathrm{e}^{A+\theta B+C\theta D}\rangle^2 \leqslant \left[\sum_{k=0}^\infty \frac{1}{k!}\langle X_k\theta X_k\rangle^{1/2}\langle Y_k\theta Y_k\rangle^{1/2}\right]^2 \leqslant \sum_{k=0}^\infty \frac{1}{k!}\langle X_k\theta X_k\rangle \sum_{k=0}^\infty \frac{1}{k!}\langle Y_k\theta Y_k\rangle. \tag{3.41}$$

By (3.40) and reflection positivity of $\langle\cdot\rangle$, we also have

$$\langle (F\theta F)\mathrm{e}^{A+\theta A+C\theta C}\rangle = \sum_{k=0}^\infty \frac{1}{k!}\langle (FX_k)\theta(FX_k)\rangle \geqslant 0. \tag{3.42}$$

This completes the proof. □

Lemma 3.9 (Gaussian domination)**.** *For any* $h : V \to \mathbb{R}$*,*

$$Z(h) \leqslant Z(0). \tag{3.43}$$

Proof. By definition,

$$H(\varphi) = \frac{1}{2}\sum_{E_+} |\varphi_x - \varphi_y|^2 + \frac{1}{2}\sum_{E_-}|\varphi_x-\varphi_y|^2 + \frac{1}{2}\sum_{E_0}|\varphi_x - \varphi_y|^2 = H_+(\varphi) + H_-(\varphi) + H_0(\varphi), \tag{3.44}$$

Since $H_-(\varphi) = \theta H_+(\varphi)$, and since

$$H_0(\varphi) = \frac{1}{2}\sum_{xy\in E_0}|\varphi_x-\varphi_y|^2 = \frac{1}{2}\sum_{x\in V_+\cap E_0}(|\varphi_x|^2 + \theta|\varphi_x|^2 + 2\varphi_x\cdot\theta\varphi_x), \tag{3.45}$$

we see that $H(\varphi)$ is of the form

$$H(\varphi) = A + \theta A + \sum C\theta C, \tag{3.46}$$

with $A, C : \Omega^{V_+} \to \mathbb{R}$. From this, it follows that $H(\varphi + h)$ is of the form

$$H(\varphi + h) = A_{h_+} + \theta A_{h_-} + \sum C_{h_+}\theta C_{h_-}, \tag{3.47}$$

with $A_{h_\pm}, C_{h_\pm} : \Omega^{V_+} \to \mathbb{R}$. Lemma 3.8 implies that

$$Z(h)^2 \leqslant Z(h_+)Z(h_-) \tag{3.48}$$

where $h_+ = h$ on V_+ and $h_+ = \theta h$ on V_- and similarly for h_-. Note that $\nabla_e h_\pm = 0$ for $e \in E_0$ and that $\nabla_e h = 0$ implies $\nabla_e h_\pm$ for any $e \in E$. Thus, by iteration, it follows that

$$Z(h) \leqslant \sup_{f:\nabla f = 0} Z(f) \tag{3.49}$$

where $\nabla f = 0$ means $\nabla_e f = 0$ for all $e \in E$. Thus f is constant and therefore $Z(f) = Z(0)$. □

Reflection positivity extends a number of other very interesting models, but not for example to next-to-nearest neighbour interactions, and is in this sense rather special. It is a major open problem to prove continuous symmetry breaking without reflection positivity.

3.3 Mermin–Wagner Theorem; McBryan–Spencer Theorem

Theorem 3.10 (Mermin–Wagner Theorem)**.** *For the $O(n)$ model with $n \geqslant 2$ on the discrete torus, with magnetic field $h = (h^1, 0, \dots, 0) > 0$, for small k, it holds that*

$$\sum_x e^{ik\cdot x}\langle \sigma_0^2 \sigma_x^2 \rangle \geqslant c\langle \sigma_0^1 \rangle^2 / (\beta |k|^2 + h^1 \langle \sigma_0^1 \rangle). \tag{3.50}$$

In particular, in $d = 2$, since $\int |k|^{-2}\, dk = \infty$, it follows that $\langle \sigma_0^1 \rangle \to 0$ as $h^1 \to 0$ for any $\beta > 0$.

We will only give the proof for $n = 2$. The general case follows by considering the first two components. In preparation, recall that the Hamiltonian is then

$$H = -\sum_{xy} \beta \cos(\theta_x - \theta_y) - \sum_x h^1 \cos\theta_x. \tag{3.51}$$

The Ward identity (2.46) associated to *global* spin rotation (all spins are rotated by the same angle), with observable $A = \sin\theta_0$, is

$$\langle \cos\theta_0 \rangle = \langle \dot{A} \rangle = \langle A\dot{H} \rangle = h^1 \sum_x \langle \sin\theta_0 \sin\theta_x \rangle. \tag{3.52}$$

(Compare this to the Ward identity in random matrix theory: $\operatorname{Im} G_{xx}(z) = (\operatorname{Im} z)\sum_y |G_{xy}(z)|^2$. In fact, many aspect of random matrix theory can be viewed in terms of supersymmetric spin systems [45, 46].)

Proof [24, 45]. Let $D = N^{-1/2}\sum_x e^{-ik\cdot x}\frac{\partial}{\partial\theta_x}$ and $\hat{S}(k) = N^{-1/2}\sum_x e^{ik\cdot x}\sin\theta_x$. By integration by parts and the Cauchy–Schwarz inequality then

$$\langle \cos\theta_0 \rangle = \langle D\hat{S}(k) \rangle = \langle \hat{S}(k)(DH) \rangle \leqslant \langle |\hat{S}(k)|^2 \rangle^{1/2} \langle |DH|^2 \rangle^{1/2}, \tag{3.53}$$

and thus

$$\langle |\hat{S}(k)|^2 \rangle \geqslant \langle \cos\theta_0 \rangle^2 / \langle |DH|^2 \rangle = \langle \sigma_0^1 \rangle^2 / \langle |DH|^2 \rangle. \tag{3.54}$$

By translation invariance, the left-hand side is

$$\langle|\hat{S}(k)|^2\rangle = \frac{1}{N}\sum_{x,y} \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\langle\sin\theta_x\sin\theta_y\rangle = \frac{1}{N}\sum_{x,y}\mathrm{e}^{\mathrm{i}k\cdot(x-y)}\langle\sin\theta_{x-y}\sin\theta_0\rangle = \sum_x \mathrm{e}^{\mathrm{i}k\cdot x}\langle\sin\theta_x\sin\theta_0\rangle$$
$$= \sum_x \mathrm{e}^{\mathrm{i}k\cdot x}\langle\sigma_x^2\sigma_0^2\rangle. \tag{3.55}$$

Since

$$\sum_{u,v}\frac{\partial^2}{\partial\theta_x\partial\theta_y}\cos(\theta_u-\theta_v) = \begin{cases} +2\cos(\theta_x-\theta_y) & (x\neq y)\\ -2\sum_v\cos(\theta_x-\theta_v) & (x=y), \end{cases} \tag{3.56}$$

and by another integration by parts,

$$\begin{aligned}\langle|DH|^2\rangle = \langle D\overline{DH}\rangle &= \frac{1}{N}\sum_{xy}\langle 2\beta\cos(\theta_x-\theta_y)(1-\mathrm{e}^{\mathrm{i}k\cdot(x-y)}) + h^1\cos\theta_x\rangle\\ &\leqslant \frac{1}{N}\sum_{xy}\langle\beta|\cos(\theta_x-\theta_y)||k|^2|x-y|^2 + h^1\cos\theta_x\rangle\\ &\leqslant 2d(\beta|k|^2 + h^1\langle\cos\theta_0\rangle) \leqslant 2d(\beta|k|^2+h^1\langle\sigma_0^1\rangle).\end{aligned} \tag{3.57}$$

This completes the proof. □

There is also a quantitative version of the Mermin–Wagner Theorem for the two-point function.

Theorem 3.11 (McBryan–Spencer Theorem)**.** *For the $O(n)$ model with $n \geqslant 2$ on the discrete torus, with magnetic field $h = 0$, it holds that*

$$\langle\sigma_0\cdot\sigma_x\rangle \leqslant C_{\beta,\varepsilon}(1+|x|)^{-(1-\varepsilon)/(2\pi\beta)}. \tag{3.58}$$

Exercise 3.12. In $d = 2$, define the Green's function on the discrete torus by $C_x = \sum_{k\neq 0}\mathrm{e}^{\mathrm{i}k\cdot x}(-\hat{\Delta}(k))^{-1}$. Show that

$$C_x \sim \frac{1}{2\pi}\log|x| \quad \text{as } |x|\to\infty, \tag{3.59}$$

(in the sense of first taking the infinite volume limit and then $|x|\to\infty$), and that

$$|C_x - C_y| \leqslant 2 \quad \text{for } x\sim y. \tag{3.60}$$

Proof [32]. Again we only provide the proof for $n = 2$; the general case is again analogous by applying the argument to the first two components and using rotational invariance. The point x is fixed throughout the proof. By definition and since $\langle\sin(\theta_0-\theta_x)\rangle = 0$,

$$\langle\sigma_0\cdot\sigma_x\rangle = \langle\cos(\theta_0-\theta_x)\rangle = \langle\mathrm{e}^{\mathrm{i}(\theta_0-\theta_x)}\rangle = \frac{1}{Z}\int \mathrm{e}^{\beta\sum_{yz}\cos(\theta_y-\theta_z)+\mathrm{i}(\theta_0-\theta_x)}\,d\theta. \tag{3.61}$$

Use the complex translation (and periodicity to cancel the vertical parts of the contours)

$$\theta_y \mapsto \theta_y + \mathrm{i}a_y, \quad a_y = \frac{1}{\beta}(C_y - C_{y-x}). \tag{3.62}$$

Then, since $\cos(u+\mathrm{i}v) = \cos(y)\cosh(v) + \mathrm{i}\sin(y)\sinh(v)$, and since

$$\begin{aligned}\langle\cos(\theta_x-\theta_0)\rangle &= \mathrm{e}^{-(a_0-a_x)}\frac{1}{Z}\int \mathrm{e}^{\beta\sum_{yz}\cos(\theta_y-\theta_z+\mathrm{i}a_y-\mathrm{i}a_z)+\mathrm{i}(\theta_0-\theta_x)}\,d\theta\\ &\leqslant \mathrm{e}^{-(a_0-a_x)}\frac{1}{Z}\int \mathrm{e}^{\beta\sum_{yz}\cos(\theta_y-\theta_z)\cosh(a_y-a_z)}\,d\theta \leqslant \mathrm{e}^{-(a_0-a_x)+\beta\sum_{yz}(\cosh(a_y-a_z)-1)}.\end{aligned} \tag{3.63}$$

To estimate the right-hand side, use $|a_y - a_z| \leqslant 4/\beta$ for $|y-z| = 1$ by the exercise, so that for β sufficiently large depending on ε, by Taylor approximation of cosh,

$$\sum_{yz} (\cosh(a_y - a_z) - 1) \leqslant \frac{1}{2}(1+\varepsilon) \sum_{yz} (a_y - a_z)^2 = \frac{1}{2}(1+\varepsilon)(a, -\Delta a) = \frac{1+\varepsilon}{2\beta}(a_0 - a_x). \tag{3.64}$$

Since $a_0 - a_x = -\frac{2}{\beta}(C_x - C_0)$, this gives

$$\langle \cos(\theta_x - \theta_0) \rangle \leqslant \mathrm{e}^{-(1-\varepsilon)(a_0 - a_x)/2} = \mathrm{e}^{(1-\varepsilon)(C_x - C_0)/\beta} = |x|^{-(1-\varepsilon)/(2\pi\beta) + o(1)}, \tag{3.65}$$

as claimed. □

For high temperatures $\beta < 1$, we have already seen that this bound on the two-point function is not sharp: the two-point function actually decays exponentially. It is believed, though unproved, that exponential decay persists in $d = 2$ for all temperatures if $n \geqslant 3$ (in particular for the Heisenberg model). One of the deepest theorems in the theory of spin systems is that for $d = 2$ and $n = 2$, for low temperatures $\beta \gg 1$, power law decay is actually correct. Thus there is a phase transition without long range order, and in fact many interesting properties: the *Kosterlitz–Thouless transition.*

Theorem 3.13 (Fröhlich–Spencer)**.** *For the $O(2)$ model in $d = 2$, for any sufficiently large β, there exists $\beta' > 1/(4\pi)$ such that*

$$\langle \sigma_0 \cdot \sigma_x \rangle \geqslant C_\beta (1 + |x|)^{-1/(2\pi\beta')}. \tag{3.66}$$

The proof [22] of Theorem 3.13 is very interesting, but long and difficult, and will not be given for now. Time permitting, we will come back to it at a later point in class.

What remains?

(a) What happens between high and low temperatures? In particular, how does the transition between the two regimes occur? As in mean field theory, is there a unique transition temperature, are there critical exponents, and what are they?

(b) Are there expansions at high and low temperatures that can give more precise information than the upper and lower bounds presented? Are there expansions that converge near the transition?

(c) How do spin systems with nonabelian continuous symmetry behave in two dimensions? How is the existence of the Kosterlitz–Thouless transition for abelian spin systems proved? How can one prove the existence of a phase transition in spin systems with continuous symmetry *without* reflection positivity?

(d) Is the phase transition universal? In particular, are critical exponents independent of the single spin measure μ?

4 Expansions for high and low temperature

The bounds for high and low temperatures presented in the previous sections are very elegant. Expansion is messier, but they are generally much more robust and can give more complete information when they apply. Their ideas play a role in much of statistical mechanics.

4.1 High temperature expansion

For simplicity, we consider the Ising model, although without difficulties high temperature expansions apply more generally. The partition function of the Ising model is

$$Z = \sum_{\sigma\in\{\pm1\}^V} \prod_{xy\in E} \mathrm{e}^{\beta\sigma_x\sigma_y} \prod_{x\in V} \mathrm{e}^{h_x\sigma_x} = \sum_{\sigma\in\{\pm1\}^V} \prod_{xy\in E} (\mathrm{e}^{\beta\sigma_x\sigma_y} - 1 + 1) \prod_{x\in V} \mathrm{e}^{h_x\sigma_x}. \tag{4.1}$$

Since

$$\prod_{i\in I}(1+a_i) = \sum_{J\subseteq I}\prod_{i\in J} a_i, \tag{4.2}$$

the partition function can be rewritten as

$$\begin{aligned} Z &= \sum_{\sigma\in\{\pm1\}^V} \sum_{X\subset E} \prod_{xy\in X} (\mathrm{e}^{\beta\sigma_x\sigma_y} - 1) \prod_{x\in V(X)} \mathrm{e}^{h_x\sigma_x} \prod_{x\in V\setminus V(X)} \mathrm{e}^{h_x\sigma_x} & (4.3)\\ &= \sum_{X\subset E} \Bigg(\sum_{\sigma\in\{\pm1\}^{V(X)}} \prod_{xy\in X} (\mathrm{e}^{\beta\sigma_x\sigma_y} - 1) \prod_{x\in V(X)} \mathrm{e}^{h_x\sigma_x} \Bigg) \Bigg(\sum_{\sigma\in\{\pm1\}^{V\setminus V(X)}} \prod_{x\in V\setminus V(X)} \mathrm{e}^{h_x\sigma_x} \Bigg) & (4.4) \end{aligned}$$

The second sum is equal to

$$\sum_{\sigma\in\{\pm1\}^{V\setminus V(X)}} \prod_{x\in V\setminus V(X)} \mathrm{e}^{h_x\sigma_x} = \prod_{x\in V\setminus V(X)} \sum_{\sigma\in\{\pm1\}} \mathrm{e}^{h_x\sigma} = \prod_{x\in V\setminus V(X)} 2\cosh(h_x). \tag{4.5}$$

The partition function becomes

$$\begin{aligned} Z &= \Bigg(\prod_{x\in V} 2\cosh(h_x)\Bigg) \sum_{X\subset E} \Bigg(\sum_{\sigma\in\{\pm1\}^{V(X)}} \prod_{xy\in X} (\mathrm{e}^{\beta\sigma_x\sigma_y} - 1) \prod_{x\in V(X)} \mathrm{e}^{h_x\sigma_x} (2\cosh(h_x))^{-1} \Bigg) \\ &= \Bigg(\prod_{x\in V} 2\cosh(h_x)\Bigg) \sum_{X\subset E} K(X), & (4.6) \end{aligned}$$

with

$$K(X) = \sum_{\sigma\in\{\pm1\}^{V(X)}} \prod_{xy\in X} (\mathrm{e}^{\beta\sigma_x\sigma_y} - 1) \prod_{x\in V(X)} \mathrm{e}^{h_x\sigma_x} (2\cosh(h_x))^{-1}. \tag{4.7}$$

Using

$$\mathrm{e}^{\beta\sigma_x\sigma_y} = \cosh(\beta)(1 + \tanh(\beta)\sigma_x\sigma_y), \tag{4.8}$$

an alternative form of the high temperature expansion is given by

$$Z = \Bigg(\prod_{x\in V} 2\cosh(h_x)\Bigg) \cosh(\beta)^{|E|} \sum_{X\subset E} K(X) \tag{4.9}$$

with

$$K(X) = \tanh(\beta)^{|X|} \sum_{\sigma\in\{\pm1\}^{V(X)}} \prod_{xy\in X} \sigma_x\sigma_y \prod_{x\in V(X)} \mathrm{e}^{h_x\sigma_x}(2\cosh(h_x))^{-1}. \tag{4.10}$$

Let $\mathbf{B} = E$ be the edges and $\mathbf{P}$ the set of subgraphs (i.e., subsets of $\mathbf{B}$). Elements of $\mathbf{P}$ are also called *polymers*. Clearly, if $X, Y \in \mathbf{P}$ have no vertices in common, we have $K(X \cup Y) = K(X)K(Y)$, and up to a multiplicative constant the partition function takes the form

$$Z = \sum_{X\in\mathbf{P}} K(X). \tag{4.11}$$

This is an example of a *polymer system*. Correlation functions can also be expressed in terms of polymer gas partition functions. By definition, the *truncated correlation functions* are

$$\langle\sigma_{x_1};\cdots;\sigma_{x_k}\rangle = \frac{\partial^k}{\partial h_{x_1}\cdots\partial h_{x_k}} \log Z. \tag{4.12}$$

In particular,

$$\langle\sigma_{x_1};\sigma_{x_2}\rangle = \langle\sigma_{x_1}\sigma_{x_2}\rangle - \langle\sigma_{x_1}\rangle\langle\sigma_{x_2}\rangle, \tag{4.13}$$

$$\begin{aligned}\langle\sigma_{x_1};\sigma_{x_2};\sigma_{x_3}\rangle &= \langle\sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\rangle - \langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\rangle - \langle\sigma_{x_1}\sigma_{x_3}\rangle\langle\sigma_{x_2}\rangle - \langle\sigma_{x_2}\sigma_{x_3}\rangle\langle\sigma_{x_1}\rangle \\ &\quad + 2\langle\sigma_{x_1}\rangle\langle\sigma_{x_2}\rangle\langle\sigma_{x_3}\rangle.\end{aligned} \tag{4.14}$$

Set $A = (x_1, \dots, x_k)$ and

$$K_A(X) = \left(\prod_{x\in X\cap A} \frac{\partial}{\partial h_x}\right) K(X), \tag{4.15}$$

with empty product interpreted as 1. Then

$$\langle\sigma_{x_1};\cdots;\sigma_{x_k}\rangle = \frac{\sum_{X\in\mathbf{P}} K_A(X)}{\sum_{X\in\mathbf{P}} K(X)} = \exp\left[\log\sum_X K_A(X) - \log\sum_X K(X)\right] \tag{4.16}$$

Thus it is very useful to be able to understand the logarithm of the polymer partition function. This will be done in a more general context. To motivate it, we consider another example of polymer systems.

4.2 Peierls expansion

In this section, we again consider the Ising model. But now this is not merely a question of convenience; the Peierls expansion does not apply to $n > 1$. Drop the last term in

$$H(\sigma) = -\sum_{xy} \beta\sigma_x\sigma_y - \sum_x h_x\sigma_x = -\sum_{xy} \beta(\sigma_x\sigma_y - 1) - \sum_x h_x(\sigma_x - 1) - \left(\sum_{xy}\beta + \sum_x h_x\right) \tag{4.17}$$

Denote by $Z_{\Lambda,\pm}$ the partition functions with $\pm$ boundary conditions on $\partial\Lambda$:

$$Z_{\Lambda,\pm} = \sum_{\sigma\in\{\pm1\}^\Lambda} \mathrm{e}^{-H_{\Lambda,\pm}(\sigma)}. \tag{4.18}$$

As in Section 4.2, denote by Γ the set of phase boundaries and by Γ_0 the outermost boundaries (which are well defined given + or − boundary conditions on $\partial\Lambda$). Then

$$Z_{\Lambda,+} = \sum_{\Gamma_0}\prod_{\gamma\in\Gamma_0}\left(Z_{\mathrm{int}\gamma,-}\mathrm{e}^{-2\beta|\gamma|}\right) = \sum_{\Gamma_0}\prod_{\gamma\in\Gamma_0}\left(Z_{\mathrm{int}\gamma,+}K(\gamma)\right), \qquad K(\gamma) = \mathrm{e}^{-2\beta|\gamma|}\frac{Z_{\mathrm{int}\gamma,-}}{Z_{\mathrm{int}\gamma,+}}. \tag{4.19}$$

Iterating this identity until the γ do not enclose further phase boundaries, it follows that

$$Z_{\Lambda,+} = \sum_{\Gamma} K(\Gamma), \qquad K(\Gamma) = \prod_{\gamma \in \Gamma} K(\gamma). \tag{4.20}$$

Let $\mathbf{B}$ be the set of simple phase boundaries and $\mathbf{P}$ the unions of such. Then again the partition function takes the form of a polymer system

$$\sum_{X \in \mathbf{P}} K(X), \tag{4.21}$$

with $K(X \cup Y) = K(X)K(Y)$ for disjoint X, Y.

4.3 Polymer expansion

The high and low temperature expansions of the Ising model are examples of a *polymer gas*, which have a wide range of other applications in statistical mechanics (see [27, 11, 12] and references).

Definition 4.1 (Polymer system)**.** (a) *Let* $\mathbf{B}$ *be a finite set, with a symmetric, reflexive relation expressing nonintersection of its elements. Let* $\mathbf{P}$ *be the set of subsets of* $\mathbf{B}$*, and extend the notion of intersection to* $\mathbf{P}$*, so that* X *and* Y *intersect if* γ *and* γ' *intersect for some* $\gamma \in X$ *and* $\gamma' \in Y$*. The* $X \in \mathbf{P}$ *are called* polymers*. The set of* connected polymers *is denoted* $\mathbf{C}$*.*

(b) *The* activity *is a function* $K : \mathbf{P} \to \mathbb{C}$ *such that* $K(X \cup Y) = K(X)K(Y)$ *if* X *and* Y *do not intersect, and with* $K(\emptyset) = 1$*. The polymer* partition function *is*

$$Z = \sum_{X \in \mathbf{P}} K(X). \tag{4.22}$$

(c) *Given* $X_1, \dots, X_n \in \mathbf{C}$*, we associate the graph* $G = G(X_1, \dots, X_n)$ *on vertices* $[n]$ *with edges* $E(G) = \{ij : X_i \text{ and } X_j \text{ intersect}\}$*. Then* $X_1, \dots, X_n$ *are* disjoint *if* G *is empty, and* connected *if* G *is connected.*

Remark 4.2. Our connected polymers $\mathbf{C}$ are often simply called polymers (e.g. in [27]). More generally than in the definition above, the activity could take values in a commutative normed algebra (for example regarding the polymer activity as a function of a magnetic field h).

Exercise 4.3. *(a)* Show that the polymer partition function can be written as

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{X_1,\dots,X_n \in \mathbf{C} \\ \text{disjoint}}} K(X_1) \cdots K(X_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{X_1,\dots,X_n \in \mathbf{C}} K(X_1) \cdots K(X_n) \prod_{i<j} U(X_i, X_j), \tag{4.23}$$

where

$$U(X, Y) = 1_{X \cap Y = \emptyset}. \tag{4.24}$$

Thus the polymer partition function takes the form of a grand canonical partition function of a gas of particles X_i which interact via the two-body interaction U (which is a hard-core repulsion). This explains the name polymer gas.

(b) Show that the polymer partition function generalizes the product over $\mathbf{B}$ in the sense that

$$\prod_{\gamma \in \mathbf{B}} (1 + R(\gamma)) = \sum_{X \in \mathbf{P}} K(X) \quad \text{for } K(X) = \prod_{\gamma \in X} R(\gamma). \tag{4.25}$$

For a connected graph G, define the index or Ursell function of the hard core polymer gas by

$$I(G) = \sum_{H \subset G} (-1)^{|E(H)|}. \tag{4.26}$$

The sum is over all spanning subgraphs $H \subset G$ (thus H and G have the same vertex set).

Theorem 4.4 (Formal polymer expansion)**.** *As formal power series in K,*

$$\log Z = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{X_1,\dots,X_n \in \mathbf{C} \\ \textit{connected}}} K(X_1) \cdots K(X_n) I(G(X_1,\dots,X_n)). \tag{4.27}$$

Proof. Given $X_1,\dots,X_n \in \mathbf{C}$, denote by $G = G(X_1,\dots,X_n)$ the connection graph associated to the X_i. Then, since $U(X,Y) = 1 + (U(X,Y) - 1)$, by expansion,

$$\prod_{i<j} U(X_i,X_j) = \sum_{H \subset G} \prod_{ij \in H} (U(X_i,X_j) - 1) = \sum_{H \subset G} (-1)^{|E(H)|}. \tag{4.28}$$

Thus

$$K(X_1) \cdots K(X_n) \prod_{i<j} U(X_i,X_j) = \sum_{H \subset G} (-1)^{|E(H)|} K(X_1) \cdots K(X_n). \tag{4.29}$$

The sum over the subgraphs H of G can be performed by first summing over partitions of the vertices into connected components V_s and then over the connected subgraphs H_s with vertex sets V_s:

$$\begin{aligned} \sum_{H_1,\dots,H_r} \prod_{s=1}^{r} (-1)^{|E(H_s)|} \prod_{i \in V_s} K(X_i) &= \prod_{s=1}^{r} \sum_{H_s \subset G|_{V_s}} (-1)^{|E(H_s)|} \prod_{i \in V_s} K(X_i) \\ &= \prod_{s=1}^{r} I(G|_{V_s}) \prod_{i \in V_s} K(X_i). \end{aligned} \tag{4.30}$$

Summing over the choices of $X_1,\dots,X_n$ such that G has connected components $V_1,\dots,V_r$, writing $V_s = \{v_{s,1},\dots,v_{s,n_s}\}$ with $|V_s| = n_s$, this becomes

$$\prod_{s=1}^{r} \Bigg[\sum_{\substack{X_{v_{s,1}},\dots,X_{v_{s,n_s}} \in \mathbf{C} \\ G(X_{v_{s,1}},\dots,X_{v_{s,n_s}}) = G|_{V_s}}} I(G|_{V_s}) \prod_{i \in V_s} K(X_i) \Bigg]. \tag{4.31}$$

The term in the bracket only depends on $G|_{V_s}$ up to relabelling of its vertices. We may therefore replace V_s by $\{1,\dots,n_s\}$ and the constraint $G(X_{v_{s,1}},\dots,X_{v_{s,n_s}}) = G|_{V_s}$ becomes that $X_1,\dots,X_{n_s}$ is connected. Thus the term is the bracket equals:

$$\sum_{\substack{X_1,\dots,X_{n_s} \in \mathbf{C} \\ \text{connected}}} K(X_1) \cdots K(X_{n_s}) I(G(X_1,\dots,X_{n_s})). \tag{4.32}$$

For each choice of the sizes $n_1,\dots,n_r$ of the connected components, the number of possibilities to choose the sets $V_1,\dots,V_r$ is

$$\frac{n!}{n_1! \dots n_r!}. \tag{4.33}$$

This gives

$$
\begin{aligned}
Z &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{X_1,\dots,X_n \in \mathbf{C}} K(X_1)\cdots K(X_n) \prod_{i<j} U(X_i, X_j) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r} \frac{1}{r!} \sum_{n_1,\dots,n_r} \frac{n!}{n_1!\dots n_r!} \prod_{s=1}^{r} \Bigg[\sum_{\substack{X_1,\dots,X_{n_s} \in \mathbf{C} \\ \text{connected}}} K(X_1)\cdots K(X_{n_s}) I(G(X_1,\dots,X_{n_s})) \Bigg] \\
&= \sum_{r} \frac{1}{r!} \Bigg[\sum_{n} \frac{1}{n!} \sum_{\substack{X_1,\dots,X_n \in \mathbf{C} \\ \text{connected}}} K(X_1)\cdots K(X_n) I(G(X_1,\dots,X_n)) \Bigg]^r \\
&= \exp\Bigg[\sum_{n} \frac{1}{n!} \sum_{\substack{X_1,\dots,X_n \in \mathbf{C} \\ \text{connected}}} K(X_1)\cdots K(X_n) I(G(X_1,\dots,X_n)) \Bigg], \qquad (4.34)
\end{aligned}
$$

which was the claim. □

Example 4.5. Consider $\mathbf{B} = \{\gamma\}$ for a single element γ and $K(\gamma) = z$. Then

$$
Z = 1 + z, \qquad \log Z = \log(1+z) = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n-1} z^k. \qquad (4.35)
$$

The only connected $X_1, \dots, X_n$ are $\gamma, \dots, \gamma$, and the associated connectivity graph is the complete graph K_n, whose index is

$$
I(K_n) = (-1)^{n-1}(n-1)!, \qquad (4.36)
$$

consistent with (4.27).

Next, we seek a criterion that guarantees convergence of the right-hand side of (4.27). For this we need the following two preparatory lemmas.

Lemma 4.6 (Penrose resummation)**.** *Let $v_{ij} \in [0, \infty]$ for all $ij \in K_n$. Then*

$$
\Big| \sum_{H} \prod_{ij \in H} (\mathrm{e}^{-v_{ij}} - 1) \Big| \leqslant \sum_{T} \prod_{ij \in T} |\mathrm{e}^{-v_{ij}} - 1|, \qquad (4.37)
$$

where the sum over H is over all connected spanning subgraphs of K_n, and the sum over T is over spanning trees of K_n. In particular, we have (Rota's Theorem)

$$
\Big| \sum_{H \subset G} (-1)^{|E(H)|} \Big| \leqslant \sum_{T \subset G} 1, \qquad (4.38)
$$

where the sum over H is over connected spanning subgraphs in G, and that over T is over spanning trees in G.

Proof. Fix an order on the edges of K_n. Given a connected spanning subgraph $H \subset K_n$, select a spanning tree T of H by selecting edges from H in the given order discarding any edge that completes a loop among prior edges. This algorithm (Kruskal's algorithm) defines a function $H \mapsto T(H)$, and we can write

$$
\sum_{H} \prod_{ij \in H} (\mathrm{e}^{-v_{ij}} - 1) = \sum_{T} \sum_{H: T(H) = T} \prod_{ij \in H} (\mathrm{e}^{-v_{ij}} - 1). \qquad (4.39)
$$

From the construction of the algorithm it follows that for any T there is a maximal graph H_T such that $T(H_T) = T$ and the sum over H such that $T(H) = T$ is the sum $T \subset H \subset H_T$. Therefore the inner sum on the right-hand side becomes

$$\sum_{T\subset H\subset H_T} \prod_{ij\in H} (\mathrm{e}^{-v_{ij}} - 1) = \prod_{ij\in T} (\mathrm{e}^{-v_{ij}} - 1) \sum_{X\subset H_T\setminus T} \prod_{ij\in X} (\mathrm{e}^{-v_{ij}} - 1) = \prod_{ij\in T} (\mathrm{e}^{-v_{ij}} - 1) \prod_{ij\in H_T\setminus T} \mathrm{e}^{-v_{ij}}. \quad (4.40)$$

Since the second factor is bounded by 1, the claim follows. □

Lemma 4.7 (Cayley's Theorem)**.** *The number of trees on $[n]$ with degree sequence $(d_1, \dots, d_n)$ is*

$$\frac{(n-2)!}{\prod_i (d_i - 1)!}. \quad (4.41)$$

Exercise 4.8. Prove Cayley's Theorem using the following steps. Associate to each edge $ij \in K_n$ a weight w_{ij} and define the weighted Laplacian matrix

$$L_{ij} = -1_{i\neq j} w_{ij} + 1_{i=j} \sum_{k\neq i} w_{ik}. \quad (4.42)$$

Look up Kirchhoff's Matrix Tree Theorem which states that (for any k, l)

$$\sum_T \prod_{ij\in T} w_{ij} = (-1)^{k+l} \det L^{kl}, \quad (4.43)$$

where the matrix L^{kl} is obtained by deleting the k-th row and l-th column. Set $w_{ij} = x_i x_j$. Then

$$\prod_{ij\in T} w_{ij} = \prod_i x_i^{\deg_T(i)}, \quad (4.44)$$

and this gives the generating function

$$\sum_T \prod_i x_i^{\deg_T(i)} = \det(L^{11}) = x_1 \dots x_n (x_1 + \dots + x_n)^{n-2}. \quad (4.45)$$

The number of trees on $[n]$ with degree sequence $(d_1, \dots, d_n)$ is

$$\prod_{i=1}^n \frac{1}{d_i!} \frac{\partial^{d_i}}{\partial x_i^{d_i}} \sum_T \prod_i x_i^{\deg_T(i)} \Bigg|_{x_1=\dots=x_n=0} = \prod_{i=1}^n \frac{1}{d_i!} \frac{\partial^{d_i}}{\partial x_i^{d_i}} x_1 \dots x_n (x_1 + \dots + x_n)^{n-2} \Bigg|_{x_1=\dots=x_n=0}. \quad (4.46)$$

Since a spanning tree on $[n]$ has $n-1$ edges and each edge contributes to the degrees of two vertices, we have $\sum d_i = 2n - 2$, and the claim follows.

The following criterion guarantees convergence of the right-hand side of (4.27). Its assumption (4.47) is not optimal, but it gives a fairly simple proof (along the lines of [27]). Sharper conditions and references are given in [38].

Theorem 4.9 (Convergence of polymer expansion)**.** *Let $A > \mathrm{e}$ and $\alpha = (A\mathrm{e})/(A-\mathrm{e})^2$, and assume that*

$$\|K\| := \sup_{\gamma\in\mathbf{B}} \sum_{\substack{X\in\mathbf{C}\\ X\ni\gamma}} |K(X)| A^{|X|} \leqslant 1/\alpha. \quad (4.47)$$

Then the right-hand side of (4.27) *converges absolutely and*

$$\log Z \leqslant |\mathbf{B}| \left(\|K\| + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \|K\|^n \alpha^n \right). \tag{4.48}$$

Moreover, there is $s > 0$ such that for any $a \geqslant 1$ and any $Y \subset \mathbf{B}$, we have the improvement

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{X_1,\dots,X_n \in \mathbf{C} \\ \text{connected}}} |K(X_1)\cdots K(X_n) I(G(X_1,\dots,X_n))| 1_{|X_1|+\cdots+|X_n| \geqslant a} 1_{(X_1 \cup \cdots \cup X_n) \cap Y \neq \emptyset}$$
$$\leqslant |Y| \mathrm{e}^{-sa} \left(\|K\| + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \|K\|^n \alpha^n \right). \tag{4.49}$$

Proof of Theorem 4.9. First consider the term $n = 1$. Fix k and $\gamma \in \mathbf{B}$ and sum over $X \in \mathbf{C}$ with $|X| = k$ and $X \ni \gamma$, and then over the choices of γ and $k \geqslant 1$:

$$\sum_{X \in \mathbf{C}} |K(X)| \leqslant \sum_{k} \sum_{\gamma} \sum_{|X|=k, X \ni \gamma} |K(X)| \leqslant \|K\| |\mathbf{B}| \sum_{k} A^{-k} \leqslant \|K\| |\mathbf{B}| (A-1)^{-1}. \tag{4.50}$$

Now consider the $n > 1$ terms. As for $n = 1$, fix $\gamma_1, \dots, \gamma_n \in \mathbf{B}$, $k_1, \dots, k_n \geqslant 1$, and sum over $X_1, \dots X_n \in \mathbf{C}$ with $X_1 \ni \gamma_1, \dots, X_n \ni \gamma_n$ and $|X_1| = k_1, \dots, |X_n| = k_n$, and then sum over the possible choices of the γ and k. If X_r intersects X_s, then the sum over γ_r can be restricted to X_s; further if X_r intersects multiple X_s we can choose which X_s the sum over γ_r should be restricted to. To specify the choice, use

$$|I(G)| \leqslant \sum_{T \subset G} 1, \tag{4.51}$$

where the sum runs over the set of spanning trees of G, by (4.38). Given a spanning tree T of G, choose the summations of the γ_r as follows:

- For any leaf in T, choose the sum over the polymers associated to its unique neighbour.
- Remove the leaf and iterate the procedure.
- Once the tree only contains one vertex, sum the associated γ_r over all of $\mathbf{B}$.

For T with degree sequence $(d_1, \dots, d_n)$, this bounds the sum over the γ by

$$|\mathbf{B}| \prod_{i=1}^{n} k_i^{d_i}, \tag{4.52}$$

and thus gives the bound

$$\sum_{\substack{X_1,\dots,X_n \in \mathbf{C} \\ \text{connected}}} |K(X_1)\cdots K(X_n)| \leqslant |\mathbf{B}| \|K\|^n \sum_{k_1,\dots,k_n} \sum_{T \subset G} \prod_{i} k_i^{d_i} A^{-k_i}. \tag{4.53}$$

Lemma 4.7 implies that

$$\sum_{T \subset G} \prod_{i} k_i^{d_i} \leqslant (n-2)! \sum_{d_1,\dots,d_n} \prod_{i} \frac{k_i^{d_i}}{(d_i-1)!} \leqslant (n-2)! \prod_{i=1}^{n} \sum_{d=1}^{\infty} \frac{k_i^d}{(d-1)!} \leqslant (n-2)! \prod_{i=1}^{n} k_i \mathrm{e}^{k_i}. \tag{4.54}$$

Since (exercise)

$$\sum_k k(\mathrm{e}/A)^k = \frac{\mathrm{e}/A}{(1-e/A)^2} = \alpha, \tag{4.55}$$

we obtain the claim

$$\log Z \leqslant |\mathbf{B}| \left(\|K\| + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \|K\|^n \alpha^n \right). \tag{4.56}$$

For the improvement if $|X_1| + \cdots + |X_n| \geqslant a$ and $(X_1 \cup \cdots X_n) \cap Y = \emptyset$, use that

$$\sum_{k_1+\cdots+k_n \geqslant a} \prod_{i=1}^{n} k_i (\mathrm{e}/A)^{k_i} \leqslant (C\mathrm{e}/A)^a \alpha^n \leqslant \mathrm{e}^{-sa} \alpha^n, \tag{4.57}$$

and that the factor $|\mathbf{B}|$ can be replaced by $|Y|$. This completes the proof. □

4.3.1. Application of polymer expansion to the Ising model.

Lemma 4.10. *For any A, the polymer activity of the high temperature expansion (4.7) obeys $\|K\| = O(\beta)$ as $\beta \to 0$, and that the Peierls expansion obeys $\|K\| = O(\mathrm{e}^{-2\beta})$ as $\beta \to \infty$.*

Proof. Since $|\mathrm{e}^{\beta\sigma_x\sigma_y} - 1| \leqslant \beta \mathrm{e}^{\beta}$, from (4.7) we obtain

$$|K(X)| \leqslant (\beta \mathrm{e}^{\beta})^{|X|} \sum_{\sigma \in \{\pm 1\}^{V(X)}} \prod_{x \in V(X)} \mathrm{e}^{h_x \sigma_x} (2\cosh(h_x))^{-1} = (\beta \mathrm{e}^{\beta})^{|X|}. \tag{4.58}$$

Given any edge $e \in \mathbf{B}$, the number of connected polymers X of size k containing e is bounded by $(2d)^k$. Therefore, for any A,

$$\|K\| \leqslant \sum_{k=1}^{\infty} (2dA\beta \mathrm{e}^{\beta})^k = O(\beta) \quad \text{as } \beta \to 0, \tag{4.59}$$

as claimed. The claim for the Peierls expansion is an exercise. □

Exercise 4.11. Complete the proof for the Peierls expansion. Use that by symmetry $Z_+ = Z_-$ for $h = 0$. For $h \geqslant 0$ show that $Z_- \leqslant Z_+$ by differentiating $\log Z_- - \log Z_+$ in h and using Griffith's first inequality.

For a cube $\Lambda \subset \mathbb{Z}^d$ denote by $\mathbf{C}(\Lambda)$ the connected polymers contained in Λ, and define analogously $\mathbf{C}(\mathbb{Z}^d)$ such that $\mathbf{C}(\mathbb{Z}^d) = \cup_\Lambda \mathbf{C}(\Lambda)$.

Exercise 4.12. Use Theorem 4.9 to show that, for $\beta \ll 1$ and for $\beta \gg 1$ with $+$ or $-$ boundary conditions, the pressure exists and is given by

$$|\Lambda|^{-1} \log Z \longrightarrow \sum_{n=1} \frac{1}{n!} \sum_{\substack{X_1,\dots,X_n \in \mathbf{C}(\mathbb{Z}^d) \\ \text{connected}}} \frac{K(X_1)}{|X_1|} \cdots \frac{K(X_n)}{|X_n|} I(G(X_1,\dots,X_n)) 1_{0 \in (X_1 \cup \cdots \cup X_n)}. \tag{4.60}$$

Show that the truncated correlation functions have *tree graph decay*:

$$\langle \sigma_{x_1}; \cdots ; \sigma_{x_k} \rangle \leqslant O(\mathrm{e}^{-s\tau(x_1,\dots,x_k)}), \tag{4.61}$$

where $\tau(x_1,\dots,x_k)$ is the length of the shortest tree connecting $x_1,\dots,x_k$.

Sketch. Existence of the limit involves bounding the difference of the right-hand side for $\Lambda' \supset \Lambda$. This is done using the exponential decay in the size of the clusters in (4.49). For the second claim, use that for $X_1,\dots,X_n \in \mathbf{C}$ connected and containing points $x_1,\dots,x_k$, it is necessary that

$$|X_1| + \cdots + |X_n| \geqslant \tau(x_1,\dots,x_k). \tag{4.62}$$

More details can be found in [27, Section 20.5], for example. □

5 Abelian spin systems: currents, charges, and spin waves

5.1 Currents and charges

A very insightful approach to the study of spin systems with *abelian* symmetry (Ising and XY model) are their representations by *currents* and *charges*, which follow from Fourier analysis on the abelian symmetry groups $O(1) = \mathbb{Z}_2$ and $O(2)$. For the XY model, such representations were extensively used in [23, 25] (and other references), for the Ising model in [1]. For the Ising model, they provide a percolation picture, while they relate the XY model to a Coulomb gas.

Ising model. The partition function of the Ising model is zero external field is

$$Z = \frac{1}{2^{|V|}} \sum_{\sigma} \prod_{xy} \mathrm{e}^{\beta \sigma_x \sigma_y} \tag{5.1}$$

where we added the convenient normalization $1/2^{|V|}$. Expanding the exponential as

$$\mathrm{e}^{\beta \sigma_x \sigma_y} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (\sigma_x \sigma_y)^n = \sum_{n=0}^{\infty} w_\beta(n) (\sigma_x \sigma_y)^n, \quad w_\beta(n) = \frac{\beta^n}{n!}, \tag{5.2}$$

the partition function can be rewritten as

$$Z = \sum_{n:E \to \mathbb{N}_0} W_\beta(n) \frac{1}{2^{|V|}} \sum_{\sigma} \prod_{xy} (\sigma_x \sigma_y)^{n_{xy}}, \quad W_\beta(n) = \prod_{xy} w_\beta(n_{xy}). \tag{5.3}$$

The map $n : \mathbb{E} \to \mathbb{N}_0$ is called a *current*. The *divergence* of a current n is the map $\nabla \cdot n : V \to \mathbb{Z}$ given by

$$(\nabla \cdot n)_x = \sum_{y \sim x} n_{xy}. \tag{5.4}$$

Lemma 5.1. *For any current $n : E \to \mathbb{N}_0$ and $m : V \to \{0, 1\}$,*

$$\frac{1}{2^{|V|}} \sum_{\sigma} \sigma^m \prod_{xy} (\sigma_x \sigma_y)^{n_{xy}} = 1_{\nabla \cdot n = m}, \tag{5.5}$$

where the equality $\nabla \cdot n = m$ is interpreted mod 2*.*

Proof. Note that

$$\prod_{xy} (\sigma_x \sigma_y)^{n_{xy}} = \prod_{x} \prod_{y \sim x} \sigma_x^{n_{xy}} = \prod_{x} \sigma_x^{(\nabla \cdot n)_x} \tag{5.6}$$

since in the product $\prod_x \prod_{y \sim x}$ each edge appears twice. The claim follows from

$$\frac{1}{2} \sum_{\sigma_x \in \{\pm 1\}} \sigma_x^{(\nabla \cdot n)_x + m_x} = 1_{(\nabla \cdot n)_x + m_x \text{ is even}}. \tag{5.7}$$

This completes the proof. □

This represents the partition function and correlation functions as those of systems of currents:

$$Z = \sum_{\nabla \cdot n = 0} W_\beta(n), \qquad \langle \sigma^m \rangle = \frac{\sum_{\nabla \cdot n = m} W_\beta(n)}{\sum_{\nabla \cdot n = 0} W_\beta(n)}. \tag{5.8}$$

Instead of (5.2), one can also write

$$\mathrm{e}^{\beta \sigma_x \sigma_y} = \cosh(\beta) + \sinh(\beta) \sigma_x \sigma_y = \sum_{n=0}^{1} w_\beta(n) (\sigma_x \sigma_y)^n, \quad w_\beta(n) = \begin{cases} \cosh(\beta) & (n = 0) \\ \sinh(\beta) & (n = 1). \end{cases} \tag{5.9}$$

Then the partition function becomes a sum over currents $n : E \to \{0, 1\}$.

XY model. The expansion (5.2) should be viewed as that of a function on the abelian group $\mathbb{Z}_2$ into its characters. In particular, analogous considerations apply to the XY model, with the abelian group now $SO(2)$. Indeed, the partition function of the XY model is

$$Z = \frac{1}{(2\pi)^{|V|}} \int e^{\beta \sum_{xy} \cos(\theta_x - \theta_y)} \, d\theta. \tag{5.10}$$

By Fourier expansion, there are $w_\beta(n)$ such that

$$e^{\beta \cos t} = \sum_{n \in \mathbb{Z}} w_\beta(n) e^{int}. \tag{5.11}$$

It is convenient to select a standard orientation xy for every edge $\{x, y\}$. Then when writing $xy \in E$ we mean that xy is in the standard orientation. Integrating over θ, the partition function becomes

$$\begin{aligned} Z = \int \prod_{xy} e^{\beta \cos(\theta_x - \theta_y)} \, d\theta &= \sum_n \int \prod_{xy} w_\beta(n_{xy}) e^{i n_{xy}(\theta_x - \theta_y)} \, d\theta, \\ &= \sum_n W_\beta(n) \int \prod_{xy} e^{i n_{xy}(\theta_x - \theta_y)} \, d\theta, \qquad W_\beta(n) = \prod_{xy} w_\beta(n_{xy}), \end{aligned} \tag{5.12}$$

where the sum is over all currents $n : E \to \mathbb{Z}$. Extend n to directed edges by $n_{yx} = -n_{xy}$. The divergence is then defined by (on the right-hand side the edges xy are not necessarily in the standard orientation)

$$(\nabla \cdot n)_x = \sum_{y \sim x} n_{xy}. \tag{5.13}$$

Lemma 5.2. *For any current $n : E \to \mathbb{Z}$ and $m : V \to \mathbb{Z}$,*

$$\frac{1}{(2\pi)^{|V|}} \int e^{-i\theta \cdot m} \prod_{xy} e^{i n_{xy}(\theta_x - \theta_y)} \, d\theta = 1_{\nabla \cdot n = m}. \tag{5.14}$$

Proof. The proof is analogous to that of Lemma 5.1 replacing σ_x by $e^{i\theta_x}$ and using

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta_x((\nabla \cdot n)_x - m_x)} \, d\theta_x = 1_{(\nabla \cdot n)_x = m_x}, \tag{5.15}$$

which shows the claim. □

Thus the partition function and correlations take the analogous form

$$Z = \sum_{\nabla \cdot n = 0} W_\beta(n), \qquad \langle \cos(\theta \cdot m) \rangle = \frac{\sum_{\nabla \cdot n = m} W_\beta(n)}{\sum_{\nabla \cdot n = 0} W_\beta(n)}. \tag{5.16}$$

Interpretation. Correlations of the Ising and XY models are equivalent to current correlations. For the Ising model, there sources and sinks are the same since $+1 = -1$ in $\mathbb{Z}_2$. On the other hand, for the XY model, sources and sinks are distinct. They are *vortices* which which can turn in either direction. These representations have many consequences, which we will discuss later.

Villain XY model. For the usual XY model, the function $w_\beta(n)$ in (5.11) can be computed explicitly in terms of Bessel functions. A model that is somewhat easier to handle but has the same features as the standard XY model is the *Villain XY model*, obtained by the replacement

$$f_{XY}(t) = \mathrm{e}^{\beta(\cos t - 1)} \rightsquigarrow f_V(t) = \sum_{n \in \mathbb{Z}} w^\beta(t + 2\pi n), \quad w^\beta(t) = \mathrm{e}^{-\frac{1}{2}\beta t^2}. \tag{5.17}$$

The right-hand side is a periodized Gaussian. It preserves the important features of the function $f_{XY}(t)$: periodicity in 2π and the its behaviour for small t.

Proposition 5.3 (Poisson summation)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ have sufficient decay. Then*

$$\sum_{n \in \mathbb{Z}} f(t + 2\pi n) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}(n) \mathrm{e}^{\mathrm{i}nt}, \quad \hat{f}(k) = \int f(x) \mathrm{e}^{-\mathrm{i}kx} \, dx. \tag{5.18}$$

Proof. The left-hand side defines a 2π-periodic function $g(t)$ and its Fourier coefficients are

$$\hat{g}_n = \int_0^{2\pi} g(t) \mathrm{e}^{-\mathrm{i}nt} \, dt = \sum_{n \in \mathbb{Z}} \int_0^{2\pi} f(t + 2\pi n) \mathrm{e}^{-\mathrm{i}nt} \, dt = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}nt} \, dt = \hat{f}(n). \tag{5.19}$$

The same holds for the right-hand side. □

By the Poisson summation formula and the fact that the Fourier transform of a Gaussian is Gaussian, the Villain interaction has the two dual representations

$$f_V(t) = \sum_{n \in \mathbb{Z}} w^\beta(t + 2\pi n) = \sum_{n \in \mathbb{Z}} w_\beta(n) \mathrm{e}^{\mathrm{i}nt}, \quad w^\beta(t) = \mathrm{e}^{-\frac{1}{2}\beta t^2}, \quad w_\beta(n) = \frac{1}{2\pi} \mathrm{e}^{-\frac{1}{2}n^2/\beta}. \tag{5.20}$$

For the standard XY model, $f_{XY}(t)$ can also be represented as the periodization of a function w^β defined on $\mathbb{R}$; see [23, Appendix B].

5.2 Discrete calculus

Let $G = (V, E)$ be graph that can locally be embedded into $\mathbb{Z}^d$, and denote its faces by F for $d \geqslant 2$, and its cubes by Q if $d \geqslant 3$. Define oriented versions of V, E, F, Q (see Figure 1). Each vertex x corresponds to two oriented vertices x (its standard orientation) and x^{-1} (its reversal). An edge has two boundary vertices $e = \{x, y\}$ and an orientation selects an order of these, written $e = \{x, y^{-1}\}$ with $e^{-1} = \{x^{-1}, y\}$. The faces have four consecutive boundary edges $a = \{e_1, e_2, e_3, e_4\}$ and an orientation determines the direction in which these are traversed; so, e.g., if $e \in a$ then $e^{-1} \in a^{-1}$. Denote the oriented vertices, edges, faces, cubes by $\vec{V}$, $\vec{E}$, $\vec{F}$, $\vec{Q}$, and fix a standard orientation on V, E, F, Q. Define

- 0-form: $s : V \to \mathbb{R}$, extended to $s : \vec{V} \to \mathbb{R}$ by $s_x = -s_x$ if $x^{-1} \in V$;
- 1-form: $v : E \to \mathbb{R}$, extended to $v : \vec{E} \to \mathbb{R}$ by $v_e = -v_e$ if $e^{-1} \in E$;
- 2-form: $\phi : F \to \mathbb{R}$, extended to $\phi : \vec{F} \to \mathbb{R}$ by $\phi_a = -\phi_a$ if $a^{-1} \in F$;
- 3-form: $\phi : Q \to \mathbb{R}$, extended to $\phi : \vec{Q} \to \mathbb{R}$ by $\phi_q = -\phi_q$ if $q^{-1} \in Q$;

and k-forms of higher degree can be defined similarly. For $s : V \to \mathbb{R}$, $v : E \to \mathbb{R}$, $\phi : F \to \mathbb{R}$, set

- Exterior derivative:

$$(ds)_{xy} = (\nabla s)_{xy} = \sum_{x \in e} s_x = s_x - s_y, \qquad (dv)_a = \sum_{e \in a} v_e, \qquad (d\phi)_q = \sum_{a \in q} \phi_q. \tag{5.21}$$

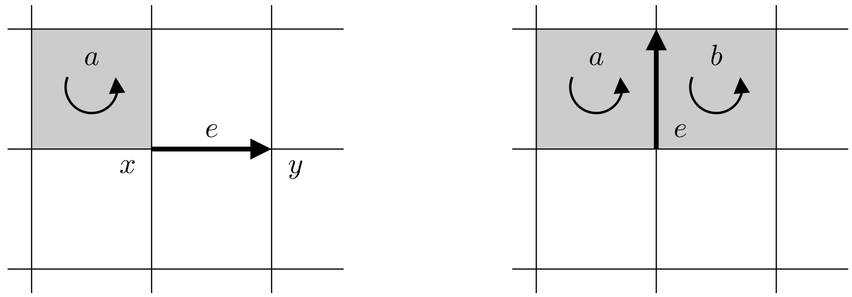

Figure 1: The vertices, edges, faces are drawn with an assigned standard orientation (for which we drop the arrow). Left: $e = xy = \{x, y^{-1}\}$. Right: $e \in a$ but $e^{-1} \in b$, and in $d = 2$ one has $(\nabla \times \phi)_e = (d^*\phi)_e = \phi_a + \phi_{b^{-1}} = \phi_a - \phi_b$.

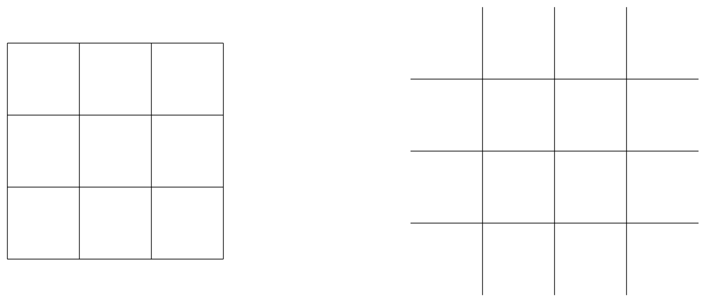

Figure 2: Free (left) and Dirichlet (right) boundary conditions.

- Codifferential:

$$d^* s = 0, \qquad (d^* v)_x = (\nabla \cdot v)_x = \sum_{e \ni x} v_{\vec{e}}, \qquad (d^*\phi)_e = (\nabla \times \phi)_e = \sum_{a \ni e} \phi_{\vec{a}}. \tag{5.22}$$

Denote the standard Euclidean inner product on the spaces $\mathbb{R}^V$, $\mathbb{R}^E$, $\mathbb{R}^F, \mathbb{R}^Q$ of 0-, 1-, 2-, 3-forms by $(\cdot, \cdot)$.

Exercise 5.4. Verify that $dd = 0$ and $d^*d^* = 0$ and that d and d^* are adjoint. Then $(d^*\phi, ds) = (\phi, d^2 s) = 0$ implies that $\operatorname{im} d^* \perp \operatorname{im} d$ in $\mathbb{R}^E$. The 1-forms that are orthogonal both to $\operatorname{im} d$ and $\operatorname{im} d^*$ obey $d^*v = 0$ and $dv = 0$. Thus they are harmonic, i.e., they lie in $\ker -\Delta$ where $-\Delta = dd^* + d^*d$. Check that d and d^* both commute with Δ.

The following boundary conditions are most common:

- Periodic boundary conditions: $G = (V, E)$ is a d-dimensional discrete torus.
- Dirichlet boundary conditions: $G = (\bar{V}, E)$ is the subgraph of $\mathbb{Z}^d$ shown in Figure 2 (right). Denote the vertices completely contained by V and $\partial V = \bar{V} \setminus V$, and extend 0-forms $s \in \mathbb{Z}^V$ by 0 to ∂V.
- Free boundary conditions: $G = (V, E)$ is a hypercube in $\mathbb{Z}^d$ and F is the set of its internal faces as in Figure 2 (left). The boundary faces are denoted by ∂F, and 2-forms $\phi \in \mathbb{Z}^F$ are extended by 0 to ∂F.

Lemma 5.5. *(Dirichlet boundary conditions) For $s : V \to \mathbb{R}$ (with $s|_{\partial V} = 0$) and $ds = 0$, it follows that $s = 0$. For any $n : E \to \mathbb{Z}$ with $dn = 0$, there exists $s : V \to \mathbb{Z}$ (with $s|_{\partial V} = 0$) such that $ds = n$. For any $q \in \mathbb{Z}^F$ with $dq = 0$, there exists $n \in \mathbb{Z}^E$ such that $dn = q$. Thus:*

$$\ker d_0 = \emptyset, \quad \mathbb{Z}^E = \operatorname{im} d_0, \quad \mathbb{Z}^F = \operatorname{im} d_1. \tag{5.23}$$

The same holds with $\mathbb{Z}$ replaced by $\mathbb{R}$, and Δ is invertible on $\mathbb{R}^E$.

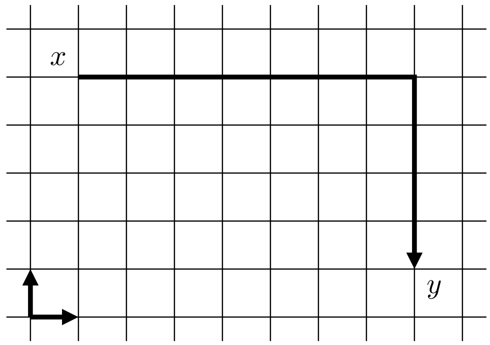

Figure 3: For $g = 1_x - 1_y$, an integer-valued solution to $d^*h = g$ is given by a string from x to y. Orienting the edges as shown in the bottom-left corner, h defined by $h_e = +1$ for the horizontal edges on the path, $h_e = -1$ for the vertical edges, and $h_e = 0$ for all other edges, is indeed a solution to $d^*h = g$.

Sketch. For example, let $n \in \mathbb{Z}^E$ with $dn = 0$. Then, for some $x \sim y$ with $y \in \partial V$, define $s_x = s_x - s_y = n_e$ with e the edge between x and y. This is consistent since if also $x \sim y'$ then $n_e - n_{e'} = (dn)_a = 0$ some a face a and so also $\phi_x = n_{e'}$. Now one can proceed by induction (now regarding x as part of the boundary) analogously. □

Example 5.6. Let $g \in \mathbb{Z}^V$ with $\sum g = 0$. By linearity, it suffices to consider $g = 1_x - 1_y$, for $x, y \in V$. Define $h \in \mathbb{Z}^E$ by choosing a path $\gamma \in (x_0 = x, \dots, x_n = y)$ and set $h_e = \pm 1$ if $e = x_i x_{i+1}$ for some $i < n$, with the sign determined by the orientation (see Figure 3). Then $d^*h = g$.

Two dimensions. In $d = 2$ the faces F form the *dual graph*. The dual graph the same same edges in the sense that two faces are neighbours if they share an edge of the original graph. Free boundary conditions are mapped to Dirichlet boundary conditions and vice versa (see Figure 2), and clearly periodic boundary conditions are invariant. For example, let (V, E) and (V^*, E^*) be the graphs in Figure 2 with free boundary conditions and Dirichlet boundary conditions respectively. Then there are natural maps $* : \mathbb{Z}^V \to \mathbb{Z}^{F^*}$, $* : \mathbb{Z}^{F^*} \to \mathbb{Z}^V$, and so on, and $d^* = *d*$. In particular, Lemma 5.5 implies that $\ker d_2^* = 0$ if (V, E) has free boundary conditions and $\ker d_2^* = \{\text{constants}\}$ if (V, E) has Dirichlet boundary conditions.

Example 5.7. $(d^*\phi)_e = (\nabla \times \phi)_e = \phi_a - \phi_b$ where a and b are the two faces containing e.

Closed 2-forms. For $d = 2$ (with Dirichlet boundary conditions), the condition $dq = 0$ is $\sum q = 0$ and such q are neutral configurations of charges. As in Example 5.6 there is $n \in \mathbb{Z}^E$ such that $dn = q$, where n can be interpreted as paths between the charges of opposite sign. In $d \geqslant 3$, the $q \in \mathbb{Z}^F$ with $dq = 0$ form closed loops.

5.3 Dual models

To be concrete, assume that (V, E) is a (hyper)cube with free boundary conditions. Then, by Lemma 5.5, for any $n \in \mathbb{Z}^E$ with $d^*n = 0$ there exists $\phi : F \to \mathbb{Z}$ such that $n = d^*\phi$. The partition function of the Ising and XY model then take the form

$$Z = \sum_{d^*n=0} W_\beta(n) = \sum_\phi W_\beta(d^*\phi), \tag{5.24}$$

where the sum runs over $\phi : F \to \mathbb{Z}$ respectively $\phi : F \to \mathbb{Z}_2$, in both cases with $\phi|_{\partial F} = 0$.

5.3.1. Ising model and Kramers–Wannier duality. By (5.24), the partition function of the Ising model on (V, E) is

$$Z_{V,\beta} = \sum_{\phi \in \mathbb{Z}_2^F} W_\beta(d^*\phi) = (\cosh \beta)^{|E|} \sum_{\phi \in \mathbb{Z}_2^F} \prod_{xy} (\tanh \beta)^{|(d^*\phi)_{xy}|}. \tag{5.25}$$

Now assume that $d = 2$. Then $(d^*\phi)_{xy} = \phi_a - \phi_b$ and

$$Z_{V,\beta} = (\cosh \beta)^{|E|} \sum_{\phi \in \mathbb{Z}_2^F} \prod_{ab} (\tanh \beta)^{|\phi_a - \phi_b|} \tag{5.26}$$

Identifying $\phi \in \{0, 1\}$ with $\tau \in \{+1, -1\}$, we have $|\phi_a - \phi_b| = \frac{1}{2}(1 - \tau_a \tau_b)$. Defining β^* by $\mathrm{e}^{-2\beta^*} = \tanh \beta$, the partition function can thus be written as

$$\begin{aligned} Z_{V,\beta} &= (\cosh \beta)^{|E|} (\tanh \beta)^{|E|/2} \sum_{\tau \in \{\pm 1\}^F} \prod_{ab} \mathrm{e}^{\beta^* \tau_a \tau_b} \\ &= (\cosh \beta)^{|E|} (\tanh \beta)^{|E|/2} Z_{F,\beta^*,0} \end{aligned} \tag{5.27}$$

Exercise 5.8. Assume that V is a (hyper)cube. Show that then $Z_{V,\beta,\omega} = Z_{V,\beta,0} + o(N)$, and moreover that the limit $p(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_{V,\beta,\omega}$ exists (for any boundary condition ω).

Since $|V| = N$, $|F|/N \to 1$ and $|E|/N \to 2$, it follows that (*Kramers–Wannier duality*)

$$p(\beta) = p(\beta^*) + 2 \log \cosh \beta + \log \tanh \beta = p(\beta^*) - \log(2 \sinh 2\beta) \tag{5.28}$$

The function $\log \sinh$ is real-analytic on $(0, \infty)$. Assuming that p is analytic in β except at a single point β_c (this is *not* clear), it follows that β_c must be fixed point of the function $g(\beta) = \operatorname{artanh}(\mathrm{e}^{-2\beta})$.

Exercise 5.9. The function g as exactly one fixed point $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ and it maps $(0, \beta_c)$ to (β_c, ∞) and vice-versa.

5.3.2. Villain model and discrete Gaussian. By (5.24), the partition function of the XY model is

$$Z = \sum_{n \in \mathbb{Z}^E : d^* n = 0} W_\beta(n) = \sum_{\phi \in \mathbb{Z}^F / \ker d^*} W_\beta(d^*\phi). \tag{5.29}$$

In particular, the partition function of the Villain model is equal to

$$Z \propto \sum_{\phi \in \mathbb{Z}^F / \ker d^*} \mathrm{e}^{-\frac{1}{2\beta}(d^*\phi, d^*\phi)}. \tag{5.30}$$

Let $d = 2$. Then this is the partition function of *discrete Gaussian model* on the dual lattice at inverse temperature $\beta^* = 1/\beta$. The discrete Gaussian model is a discrete model for an interface. By yet another Poisson summation, the partition function can be further rewritten as

$$Z \propto \sum_{q \in \mathbb{Z}^F : \sum q = 0} \mathrm{e}^{-\frac{1}{2}(2\pi)^2 \beta (d^*(dd^*)^{-1} q, d^*(dd^*)^{-1} q)} = \sum_{q \in \mathbb{Z}^F : \sum q = 0} \mathrm{e}^{-\frac{1}{2}(2\pi)^2 \beta (q, (-\Delta)^{-1} q)}. \tag{5.31}$$

Here we assumed Dirichlet boundary conditions in the original graph so that $\ker d_2^* = \{\text{constants}\}$. This is the partition function of a *Coulomb gas*.

5.3.3. Dual models via periodization. Now we use the representation

$$f(t) = \sum_{m \in \mathbb{Z}} w^\beta(t + 2\pi m), \tag{5.32}$$

and assume Dirichlet boundary conditions (on the original graph). (In particular, the Villain XY model is the case $w^\beta(n) = \mathrm{e}^{-\frac{1}{2}\beta n^2}$.) The partition function then is

$$Z = \sum_{m \in \mathbb{Z}^E} \int_{\mathbb{T}^V} W^\beta(d\theta + 2\pi m)\, d\theta. \tag{5.33}$$

The sum over $m \in \mathbb{Z}^E$ can written as a double sum over $q \in \mathbb{Z}^F$ and then over $m \in \mathbb{Z}^E$ with $dm = q$. The nontrivial terms obey $dq = 0$ (which in $d = 2$ with Dirichlet boundary conditions on the original graph is to be interpreted as $\sum q = 0$):

$$Z = \sum_{q \in \mathbb{Z}^F : dq = 0} \sum_{m \in \mathbb{Z}^E : dm = q} \int_{\mathbb{T}^V} W^\beta(d\theta + 2\pi m)\, d\theta. \tag{5.34}$$

By Lemma 5.5, for any $q \in \mathbb{Z}^F$ with $dq = 0$ there exists $m \in \mathbb{Z}^E$ with $dm = q$, and for any $n \in \mathbb{Z}^E$ with $dn = 0$ there is $\psi \in \mathbb{Z}^V$ such that $d\psi = n$, and ψ is unique for Dirichlet boundary conditions. It follows that

$$Z = \sum_{q \in \mathbb{Z}^F : dq = 0} \sum_{\psi \in \mathbb{Z}^V} \int_{\mathbb{T}^V} W^\beta(d\theta + d\psi + 2\pi n_q)\, d\theta = \sum_{q \in \mathbb{Z}^F : dq = 0} \int_{\mathbb{R}^V} W^\beta(d\tilde{\varphi} + 2\pi n_q)\, d\tilde{\varphi} \tag{5.35}$$

By Lemma 5.5, and using that d commutes with Δ,

$$n_q = -(dd^* + d^*d)\Delta^{-1} n = -dd^*\Delta^{-1} n_q - d^*\Delta^{-1} q. \tag{5.36}$$

Translating $\tilde{\varphi}$ to $\varphi = \tilde{\varphi} - 2\pi d^*\Delta^{-1} n_q$, it follows that

$$Z = \sum_{q \in \mathbb{Z}^F : dq = 0} \int_{\mathbb{R}^V} W^\beta(d\varphi - 2\pi d^*\Delta^{-1} q)\, d\varphi. \tag{5.37}$$

This gives a decomposition of the angle field θ into *spin waves* φ and *vortex charges* q. For the Villain model, these contributions decouple:

$$W^\beta(d\varphi - 2\pi d^*\Delta^{-1} q) = \mathrm{e}^{-\frac{1}{2}\beta(d\varphi, d\varphi)} \times \mathrm{e}^{-\frac{1}{2}\beta(2\pi)^2(q, -\Delta^{-1} q)} \tag{5.38}$$

using that the images of d and d^* are orthogonal and (using $dq = 0$ in the last equality)

$$(d^*\Delta^{-1} q, d^*\Delta^{-1} q) = (dd^*\Delta^{-1} q, \Delta^{-1} q) = (q, -\Delta^{-1} q) - (d^*d\Delta^{-1} q, \Delta^{-1} q) = (q, -\Delta^{-1} q). \tag{5.39}$$

5.4 Sine–Gordon representation

Let ϕ be a Gaussian field on the faces F (to match the set-up above) with covariance matrix $\beta(dd^* + m^2)^{-1}$ (a massive free field), and denote its expectation by $\mathbb{E}_{\beta,m^2}$. Its Fourier transform is

$$\mathbb{E}_{\beta,m^2}(\mathrm{e}^{\mathrm{i}(q,\phi)}) = \mathrm{e}^{-\frac{1}{2}\beta(q,(dd^* + m^2)^{-1} q)} \quad \text{for any } q : \mathbb{R}^F \to \mathbb{R}. \tag{5.40}$$

Lemma 5.10. *As* $m^2 \downarrow 0$*,*

$$\lim_{m^2\downarrow 0} \mathbb{E}_{\beta,m^2}(\mathrm{e}^{\mathrm{i}(q,\phi)}) = \begin{cases} \mathrm{e}^{-\frac{1}{2}\beta(q,(dd^*)^{-1}q)} & (dq=0) \\ 0 & (dq \neq 0). \end{cases} \tag{5.41}$$

Proof. Write $q = q_1 + q_2$ with $dq_1 = 0$ and $d^*q_2 = 0$. Then

$$(q,(dd^*+m^2)^{-1}q) = \frac{(q_2,q_2)}{m^2} + (q_1,(dd^*+m^2)^{-1}q_1). \tag{5.42}$$

The second term is continuous as $m^2 \downarrow 0$, while the first term diverges if $q_2 \neq 0$. □

In the following, we will simply write $\mathbb{E}_\beta$ instead of $\lim_{m^2\downarrow 0}\mathbb{E}_{\beta,m^2}$. It follows that the partition function of the Coulomb gas can be written as

$$Z = \sum_{q\in\mathbb{Z}^F:\sum q=0} \mathrm{e}^{-\frac{1}{2}\beta(2\pi)^2(q,(-\Delta)^{-1}q)} = \sum_{q\in\mathbb{Z}^F:\sum q=0} \mathbb{E}_\beta(\mathrm{e}^{2\pi\mathrm{i}(q,\phi)}) = \mathbb{E}_\beta\left[\sum_{q\in\mathbb{Z}^F} \mathrm{e}^{2\pi\mathrm{i}(q,\phi)}\right]. \tag{5.43}$$

The sum can be rearranged to

$$\prod_a \sum_{q\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}q\phi_a} = \prod_a \left[1 + 2\sum_{q=1}^\infty \cos(2\pi q\phi_a)\right], \tag{5.44}$$

and in summary, we obtain the Sine–Gordon representation of the Coulomb gas:

$$Z = \mathbb{E}_\beta\left[\prod_a \left[1 + 2\sum_{q=1}^\infty \cos(2\pi q\phi_a)\right]\right]. \tag{5.45}$$

Remark 5.11. This is formally the same as

$$(\text{normalization})\int_{\mathbb{R}^F} \mathrm{e}^{-\frac{1}{2\beta}(d^*\phi,d^*\phi)} \prod_a \left[1 + 2\sum_{q=1}^\infty \cos(2\pi q\phi_a)\right] = \sum_{\phi\in\mathbb{Z}^F} \mathrm{e}^{-\frac{1}{2\beta}(d^*\phi,d^*\phi)} \tag{5.46}$$

since

$$\sum_{q\in\mathbb{Z}} \mathrm{e}^{2\pi iqx} = \sum_{m\in\mathbb{Z}} \delta_m(dx). \tag{5.47}$$

5.5 Long-range order and Kosterlitz–Thouless transition of the XY model

The same derivation as (5.35) can be carried out for the expectation of a function $F(d\theta)$ that is 2π-periodic in every edge. For such a function, the above shows that

$$\langle F(d\theta)\rangle = \langle F(d\tilde{\varphi} + 2\pi n_q)\rangle = \langle F(d\varphi - 2\pi\Delta^{-1}d^*q)\rangle, \tag{5.48}$$

where the first equality follows from 2π-periodicity of F. To study the two-point function between 0 and x, set $g = 1_x - 1_0$, and define $h \in \mathbb{Z}^E$ as in Example 5.6 such that $d^*h = g$ and thus

$$\theta_0 - \theta_x = -(g,\theta) = -(h,d\theta). \tag{5.49}$$

It follows that

$$\langle e^{i(\theta_0-\theta_x)}\rangle = \langle e^{i(\varphi_0-\varphi_x)} \times e^{-2\pi i(\sigma,q)}\rangle, \qquad \text{where } \sigma = d\Delta^{-1}h, \tag{5.50}$$

and we used that

$$(h, \Delta^{-1}d^*q) = (d\Delta^{-1}h, q) = (\sigma, q). \tag{5.51}$$

The first factor in the expectation on the right-hand side is the *spin wave* contribution, the second factor the *vortex* contribution. For the Villain model, they are independent:

$$\langle e^{i(\theta_0-\theta_x)}\rangle = \langle e^{i(\varphi_0-\varphi_x)}\rangle\langle e^{-2\pi i(\sigma,q)}\rangle. \tag{5.52}$$

5.5.1. Spin wave contribution. The spin wave contribution to the two-point function is (in the Villain XY model):

$$\langle e^{i(\varphi_0-\varphi_x)}\rangle = e^{-\frac{1}{2\beta}(g,-\Delta^{-1}g)}. \tag{5.53}$$

Since $(-\Delta)^{-1}_{0x} \sim -\frac{1}{2\pi}\log|x|$ for $d=2$ by (3.59) and similarly $(-\Delta)^{-1}_{0x} \sim C_d|x|^{-(d-2)}$ for $d \geqslant 3$, as $|x| \to \infty$,

$$\langle e^{i(\varphi_0-\varphi_x)}\rangle \sim \begin{cases} C|x|^{-1/(2\pi\beta)} & (d=2) \\ C_1 + C_2|x|^{-(d-2)} & (d \geqslant 3). \end{cases} \tag{5.54}$$

Spin wave theory states that for sufficiently low temperatures the spin wave part should give the correct behaviour of the system. It thus predicts that there is long-range order in $d \geqslant 3$, with power-law correction (Goldstone mode), and there is power-law decay in $d = 2$.

5.5.2. Vortex gas contribution. The contribution of the vortex part to the two-point function is (again in the Villain XY model)

$$\langle e^{-2\pi i(\sigma,q)}\rangle. \tag{5.55}$$

Clearly, it is bounded from above by 1, and one immediately obtains that the spin wave part is an upper bound on the two-point function. In particular, for the Villain XY model in $d = 2$,

$$\langle e^{i(\theta_0-\theta_x)}\rangle \leqslant e^{\frac{1}{2\beta}(g,-\Delta^{-1}g)} = O(|x|^{-1/(2\pi\beta)}). \tag{5.56}$$

(Compare with Theorem 3.11.) The first bound in (5.56) of course also holds in $d \geqslant 3$, but does not provide strong information since the Green's function $-\Delta^{-1}$ is bounded. The main difficulty is to obtain a lower bound.

Proposition 5.12 (Fröhlich–Spencer)**.** *For the Villain XY model in $d \geqslant 2$, there is $c(\beta)$ with $c(\beta) \to 0$ as $\beta \to \infty$ such that*

$$1 \geqslant \langle e^{-2\pi i(\sigma,q)}\rangle \geqslant e^{c(\beta)(g,-\Delta^{-1}g)}. \tag{5.57}$$

Together with the spin wave contribution (5.54), it follows that, for β sufficiently large,

$$e^{\frac{1}{2\beta}(g,-\Delta^{-1}g)} \geqslant \langle \sigma_x \cdot \sigma_y\rangle \geqslant e^{(\frac{1}{2\beta}+c(\beta))(g,-\Delta^{-1}g)}. \tag{5.58}$$

An analogous estimate holds for the standard XY model, but its proof is more involved. Moreover, the proof of (5.57) for $d = 2$ is significantly more difficult [23], and we will only consider $d \geqslant 3$ as in [25].

For $d \geqslant 3$, (5.58) implies that there is *long-range order* for sufficiently small temperatures (the right-hand side is bounded below by a strictly positive constant). Using reflection positivity and the infrared bound, in Corollary 3.5, we have already seen that the $O(n)$ model exhibits long-range order for any $d \geqslant 3$ and $n \geqslant 1$, for small enough temperatures (and also seen in Theorems 3.10–3.11 that in $d = 2$ they do *not* if $n \geqslant 2$). For the Ising model ($n = 1$), we have seen that long-range order can be proved by the Peierls argument (and expansion). For $d = 2$, (5.58) implies the existence of the *Kosterlitz–Thouless transition*, a phase transition without long-range order: for high temperatures correlations decay exponentially, while for low temperatures they have power-law decay.

5.5.3. Proof of Proposition 5.12 for $d \geqslant 3$. Define

$$Z(\sigma) = \sum_{dq=0} \mathrm{e}^{-\frac{1}{2}\beta(2\pi)^2(q,-\Delta^{-1}q)}\mathrm{e}^{-2\pi i(\sigma,q)} \tag{5.59}$$

where the sum runs over 2-forms q with $dq = 0$. Then

$$\langle \mathrm{e}^{-2\pi \mathrm{i}(\sigma,q)}\rangle = \frac{Z(\sigma)}{Z(0)}. \tag{5.60}$$

In the Sine–Gordon representation, it follows that

$$Z(\sigma) = \mathbb{E}_\beta\left(\sum_{dq=0} \mathrm{e}^{2\pi i(\phi+\sigma,q)}\right) \tag{5.61}$$

Exercise 5.13. Let ζ_1 and ζ_2 be two independent centered Gaussian field with covariances C_1 and C_2. Then $\phi = \zeta_1 + \zeta_2$ is a centered Gaussian field with covariance $C = C_1 + C_2$.

Since dd^* is bounded on $\ell^2(F)$, there is a constant $c > 0$ such that $(dd^*)^{-1} \geqslant c\mathrm{id}$. By Exercise 5.13, the field ϕ can therefore be decomposed in distribution as $\phi = \zeta + \phi'$, where ζ is a Gaussian field with covariance $\beta c\mathrm{id}$ and ϕ' has covariance $\beta((dd^*)^{-1} - c\mathrm{id})$. Taking the conditional expectation over ζ then gives

$$Z(\sigma) = \mathbb{E}_\beta\left(\sum_{dq=0} \mathrm{e}^{-\frac{1}{2}c\beta(q,q)}\mathrm{e}^{2\pi i(\phi'+\sigma,q)}\right). \tag{5.62}$$

This is one step of renormalization that integrates out fluctuations on the smallest scale, and makes the coefficients small for large β. Now write the partition function as

$$Z(\sigma) = \mathbb{E}_\beta\left(\sum_{X\subset F}\sum_{dq=0,\mathrm{supp}\, q=X} \mathrm{e}^{-\frac{1}{2}c\beta(q,q)}\mathrm{e}^{2\pi i(\phi'+\sigma,q)}\right) = \mathbb{E}_\beta\left(\sum_X K(X,\phi'+\sigma)\right) \tag{5.63}$$

where

$$K(X,\phi'+\sigma) = \sum_{dq=0,\mathrm{supp}\, q=X} \mathrm{e}^{-\frac{1}{2}c\beta(q,q)}\mathrm{e}^{2\pi i(\phi'+\sigma,q)}. \tag{5.64}$$

Note that $K(X_1 \cup X_2) = K(X_1)K(X_2)$ if $X_1, X_2 \subset F$ have distance at least 2. This factorization into *locally neutral* contributions is where the condition $d \geqslant 3$ enters crucially. (In $d = 2$, the condition $dq = 0$ is the non-local condition $\sum q = 0$, and a significantly more complicated argument is required.)

The next step is a cluster expansion that writes the polymer partition function as an exponential. In order to do so, the next lemma verifies that the polymer activity K is small.

Lemma 5.14. *For any A, the norm (4.47) satisfies*

$$\|K\| \to 0 \quad (\beta \to \infty). \tag{5.65}$$

Proof. Since q can only take integer values $q_a \neq 0$ for any face a in its support X,

$$|K(X)| \leqslant \left(2\sum_{q=1}^{\infty} \mathrm{e}^{-\frac{1}{2}c\beta q^2}\right)^{|X|} = \left(\frac{2\mathrm{e}^{-\frac{1}{2}c\beta}}{1-\mathrm{e}^{-\frac{1}{2}c\beta}}\right)^{|X|}. \tag{5.66}$$

Given any $a \in F$, the number of connected sets $X \subset F$ containing a are bounded by $\mathrm{e}^{C|X|}$. It follows that

$$\|K\| = \sup_a \sum_{X \ni a} |K(X)| A^{|X|} \to 0 \quad (\beta \to \infty), \tag{5.67}$$

as needed. □

By the polymer expansion, Theorems 4.4, 4.9, it follows that

$$Z(\sigma) = \mathbb{E}_\beta \exp\left(\sum_{n=1}^\infty \sum_{X_1,\dots,X_n \in \mathbf{C}} \frac{1}{n!} I(G(X_1,\dots,X_n)) K(X_1)\cdots K(X_n) \right) \tag{5.68}$$

It is convenient to rewrite the term in the exponential as sum over charge configurations. The product $K(X_1)\cdots K(X_n)$ is

$$K(X_1)\cdots K(X_n) = \sum_{q_1} \cdots \sum_{q_n} \mathrm{e}^{-\frac{1}{2}c\beta \sum_i (q_i,q_i)} \mathrm{e}^{2\pi \mathrm{i}(\phi'+\sigma, q_1+\cdots+q_n)}, \tag{5.69}$$

where the sum over q_i is over q_i with $dq_i = 0$ and $\operatorname{supp} q_i = X_i$. Thus

$$\sum_{n=1}^\infty \sum_{X_1,\dots,X_n \in \mathbf{C}} \frac{1}{n!} I(G(X_1,\dots,X_n)) K(X_1)\cdots K(X_n) = \sum_q z(\beta,q) \mathrm{e}^{2\pi \mathrm{i}(q,\phi'+\sigma)} \tag{5.70}$$

where the sum over q with connected support, and

$$z(\beta,q) = \sum_{n=1}^\infty \frac{1}{n!} \sum_{q_1+\cdots+q_n = q} I(G(X_1,\dots,X_n)) \mathrm{e}^{-\frac{1}{2}c\beta \sum_i (q_i,q_i)}, \tag{5.71}$$

where the sum is again over q_i with $dq_i = 0$ and $X_i = \operatorname{supp} q_i$.

Lemma 5.15. *There is $c > 0$ such that for β large enough, $|z(\beta,q)| \leqslant \mathrm{e}^{-c\beta\|q\|_1}$.*

Sketch. Since q is integer valued,

$$\|\sum_i q_i\|_1 \leqslant \sum_i \|q_i\|_1 \leqslant \sum_i \|q_i^2\|_1 = \sum_i (q_i,q_i). \tag{5.72}$$

The combinatorial factor can also be bounded. □

In summary, and using that for every charge configuration q, the configuration $-q$ also appears in the sum, we have obtained

$$Z(\sigma) = \mathbb{E}_\beta \exp\left(\sum_{dq=0} z(\beta,q) \cos(2\pi(\phi'+\sigma,q)) \right). \tag{5.73}$$

Next, write

$$\begin{aligned} \cos(2\pi(\phi'+\sigma,q)) &= \cos(2\pi(\phi',q))\cos(2\pi(\sigma,q)) - \sin(2\pi(\phi',q))\sin(2\pi(\sigma,q)) \\ &= \cos(2\pi(\phi',q)) + \underbrace{\cos(2\pi(\phi',q))(\cos(2\pi(\sigma,q))-1)}_{R(q,\sigma,\phi')} - \underbrace{\sin(2\pi(\phi',q))\sin(2\pi(\sigma,q))}_{O(q,\sigma,\phi')}, \end{aligned} \tag{5.74}$$

and note that O is odd in ϕ'. It remains to estimate R. Since $dq = 0$, by Lemma 5.5, there exists an integer valued solution $n = n_q$ to $dn = q$. Thus

$$(\phi',q) = (d^*\phi', n_q), \tag{5.75}$$

is gradient perturbation to the free field. The 1-form n_q can be chosen with good properties.

Lemma 5.16. *Let $d \geqslant 3$ and q be 2-form with $dq = 0$. Then there exists a 1-form n_q with $dn_q = q$ such that $\|n_q\|_\infty \leqslant \|q\|_1$.*

Sketch. Let T be a (well-chosen) spanning tree of G. Given $x \in V$ then there is a unique path Γ_x from 0 to x. Given $xy \in E$ choose S_{xy} be a (nonunique) surface with boundary $\{xy\} \cup \gamma_x \cup \gamma_y^{-1}$, and

$$n_{xy} = \sum_{a \in S_{xy}} q_a. \tag{5.76}$$

Since $dq = 0$, this definition is independent of the choice of S_{xy} and the bound $\|n\|_\infty \leqslant \|q\|_1$ is obvious. See e.g. [31, Appendix B] for a detailed proof. □

Lemma 5.17. *There is $c(\beta)$ with $c(\beta) \to 0$ as $\beta \to \infty$ such that*

$$\sum_q |R(q, \sigma, \phi')| \leqslant c(\beta)(g, -\Delta^{-1} g). \tag{5.77}$$

Proof. Since h and n_q are integer valued, it follows that

$$(\sigma, q) = (d^*\sigma, n_q) = (d^* d\Delta^{-1} h, n_q) = (dd^*\Delta^{-1} h, n_q) = (d\Delta^{-1} g, n_q) \qquad \text{mod } 1. \tag{5.78}$$

Thus

$$\cos(2\pi(\sigma, q)) = \cos(2\pi(d\Delta^{-1} g, n_q)), \tag{5.79}$$

where

$$|\cos(2\pi(\sigma, q)) - 1| \leqslant 2\pi^2 (d\Delta^{-1} g, n_q)^2 \leqslant 2\pi^2 (n_q, n_q)(d\Delta^{-1} g, d\Delta^{-1} g) = 2\pi^2 (n_q, n_q)(g, -\Delta^{-1} g). \tag{5.80}$$

It follows that

$$\sum_q |R(q, \sigma, \phi')| \leqslant \left(2\pi^2 \sum_q z(\beta, q) \|n_q\|_2^2 \right) (g, -\Delta^{-1} g) = c(\beta)(g, -\Delta^{-1} g) \tag{5.81}$$

and using $\|n_q\|_2^2 \leqslant \|q\|_1$, we obtain

$$c(\beta) \leqslant C \sum_q \mathrm{e}^{-c\beta \|q\|_1} \|q\|_1 \tag{5.82}$$

from which the claim follows. □

Proof of Proposition 5.12. Applying Jensen's inequality, using that $O(q, \sigma, \phi')$ is odd in ϕ', it follows that

$$Z(\sigma) = \mathbb{E}_\beta \exp\left(\sum_q \Big(z(\beta, q) \cos(2\pi(\phi', q)) - O(q, \sigma, \phi') + R(q, \sigma, \phi') \Big) \right) \geqslant Z(0) \mathrm{e}^{c(\beta)(g, -\Delta^{-1} g)}. \tag{5.83}$$

This completes the proof of Proposition 5.12. □

6 Random geometric representations

6.1 Random currents as percolation

For the Ising model, the representation (5.8) provides and interpretation in terms of percolation, a point of view developed in [2]. A current $n : E \to \mathbb{N}_0$ can be seen as a subgraph of G with n_{xy} edges between xy. For vertices $x, y \in V$, we write $x \leftrightarrow y$ in n if x and y are connected in n, i.e., if there exists a path γ from x to y such that $n_e > 0$ for $e \in \gamma$. We also write

$$\partial n = \{x \in V : (d^* n)_x = 1 \mod 2\}. \tag{6.1}$$

Proposition 6.1 (Switching Lemma). *Let $H \subset G$ be a subgraph, $x, y \in V(H)$, $A \subset V(G)$. Then, for any function $F : \{\pm 1\}^{E(G)} \to \mathbb{R}$,*

$$\sum_{\partial n_1 = \{x,y\}, \partial n_2 = A} F(n^1 + n^2) W(n^1) W(n^2) = \sum_{\partial n_1 = \emptyset, \partial n_2 = A \Delta \{x,y\}} F(n^1 + n^2) W(n^1) W(n^2) 1_{\{x \leftrightarrow y \text{ in } n^1 + n^2\}}, \tag{6.2}$$

where the sums run over currents n_1 on H (thus $n_e = 0$ for $e \in G \setminus H$) and n_2 on G.

Proof. Set $n = n^1 + n^2$ and first sum over n. Since $\partial(n_1 + n_2) = \partial n_1 \Delta \partial n_2$, and since

$$W(n^1) W(n^2) = \prod_e \left(\frac{J_e^{n_e^1}}{n_e^1!} \right) \prod_e \left(\frac{J_e^{n_e^2}}{n_e^2!} \right) = W(n^1 + n^2) \binom{n^1 + n^2}{n^1}, \tag{6.3}$$

the left-hand side of (6.2) is equal to

$$\sum_{\partial n = A \Delta \{x,y\}} F(n) W(n) \sum_{\partial n_1 = \{x,y\}, n_1 \subseteq n} \binom{n}{n_1}, \tag{6.4}$$

while the right-hand side of (6.2) is

$$\sum_{\partial n = A \Delta \{x,y\}} F(n) W(n) 1_{\{x \leftrightarrow y \text{ in } n\}} \sum_{\partial n_1 = \emptyset, n_1 \subseteq n} \binom{n}{n_1}. \tag{6.5}$$

Thus to prove the claim, it suffices to show

$$\sum_{\partial n_1 = \{x,y\}, n_1 \subseteq n} \binom{n}{n_1} = 1_{\{x \leftrightarrow y \text{ in } n\}} \sum_{\partial n_1 = \emptyset, n_1 \subseteq n} \binom{n}{n_1}. \tag{6.6}$$

For n such that $x \not\leftrightarrow y$ in n both sides vanish. Therefore we assume $x \leftrightarrow y$ in n. View n as the multigraph with n_{xy} edges between vertices x and y. For any graph m, denote by ∂m the set of vertices with an odd number of edges. Since x and y are connected in n, there exists a subgraph $k \subset n$ such that $\partial k = \{x, y\}$. Note that $m \mapsto m \Delta k$ is an involution and maps graphs with $\partial m = \emptyset$ to graphs with $\partial m = \{x, y\}$. In particular, both sets of graphs have equal cardinality, i.e., (6.6) holds. ☐

Corollary 6.2.

$$\langle \sigma_x \sigma_y \rangle^2 = \frac{\sum_{\partial n^1 = \emptyset, \partial n^2 = \emptyset} W(n^1) W(n^2) 1_{\{x \leftrightarrow y \text{ in } n^1 + n^2\}}}{\sum_{\partial n^1 = \emptyset, \partial n^2 = \emptyset} W(n^1) W(n^2)} \tag{6.7}$$

Proof. By the switching lemma, the right-hand side of (6.7) is equal to

$$\frac{\sum_{\partial n^1=\partial n^2=\{x,y\}} W(n^1)W(n^2)}{\sum_{\partial n^1=\emptyset,\partial n^2=\emptyset} W(n^1)W(n^2)} = \left(\frac{\sum_{\partial n=\{x,y\}} W(n)}{\sum_{\partial n=\emptyset} W(n^1)}\right)^2 = \langle\sigma_x\sigma_y\rangle^2, \tag{6.8}$$

as claimed. □

Thus the square of the two-point function of the Ising model $\langle\sigma_x\sigma_y\rangle$ equals the two-point function of the percolation configuration n^1+n^2:

$$\langle\sigma_x\sigma_y\rangle^2 = \mathbb{E}\left(x \leftrightarrow y \text{ in } n^1+n^2\middle|\emptyset,\emptyset\right), \tag{6.9}$$

where $\mathbb{E}(\cdot|A_1,A_2)$ denotes the expectation of two current configurations n^1 and n^2 with sources $\partial n_1 = A_1$ and $\partial n_2 = A_2$. Let $C_{n^1+n^2}(x) = \{y \in V : x \leftrightarrow y \text{ in } n_1+n_2\}$ denote the connected cluster containing x. Its expected size is

$$\mathbb{E}\left(|C_{n_1+n_2}(x)|\middle|\emptyset,\emptyset\right) = \sum_y \langle\sigma_x\sigma_y\rangle^2. \tag{6.10}$$

Thus long-range order corresponds to percolation of the currents (the expected cluster size is infinite), while exponential decay implies a finite expected cluster size. One can see many consequences from the percolation point of view, see [2], as exemplified in the following proof of Simon's inequality [42].

Corollary 6.3 (Simon's inequality)**.** *Let B be a set such that $V\setminus B$ has at least two components, and let $x,y\in V$ be in distinct components. Then*

$$\langle\sigma_x\sigma_z\rangle \leqslant \sum_{y\in B}\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle. \tag{6.11}$$

Proof. For any $B\subset V$,

$$\begin{aligned}\sum_{y\in B}\frac{\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle}{\langle\sigma_x\sigma_z\rangle} &= \sum_{y\in B}\frac{\sum_{\partial n^1=\{x,y\}\partial n^2=\{y,z\}} W(n^1)W(n^2)}{\sum_{\partial n^1=\{x,z\},\partial n^2=\emptyset} W(n^1)W(n^2)}\\ &= \sum_{y\in B}\frac{\sum_{\partial n^1=\{x,z\}\partial n^2=\emptyset} 1_{\{x\,\leftrightarrow\, z \text{ in } n^1+n^2\}} W(n^1)W(n^2)}{\sum_{\partial n^1=\{x,z\},\partial n^2=\emptyset} W(n^1)W(n^2)}\\ &= \mathbb{E}\left(|B\cap C_{n^1+n^2}(x)|\middle|\{x,z\},\emptyset\right).\end{aligned} \tag{6.12}$$

The assumption that B separates x and z implies $|B\cap C_{n^1+n^2}(x)|\geqslant 1$, and (6.11) follows. □

Many results discussed in the previous sections apply mostly to high or low temperatures. To understand the transition between these regimes is a fundamental question (of which many aspects are widely open). For high temperatures, we have seen by various methods that

$$\chi(\beta)<\infty, \quad \text{where } \chi = \lim_{N\to\infty}\sum_{x\in V}\langle\sigma_0\sigma_x\rangle_{\beta,0}, \tag{6.13}$$

providing that the infinite volume limit is taken appropriately (e.g., along a sequence of finite tori). For low temperatures, for the Ising model on $\mathbb{Z}^d$, we have seen that

$$M_+(\beta)>0, \quad \text{where } M_+(\beta) = \lim_{h\downarrow 0}\lim_{N\to\infty}\langle\sigma_x\rangle_{\beta,h}. \tag{6.14}$$

Thus a natural charactization of the boundary of the high temperature phase is the temperature at which (6.13) fails and that of the boundary of the low temperature phase at which (6.14) fails. Let

$$\beta_c^\chi = \sup\{\beta : \chi(\beta)<\infty\}, \quad \beta_c^M = \inf\{\beta : M_+(\beta)>0\}. \tag{6.15}$$

Theorem 6.4. *For the Ising model (and more general* 1*-component models that can be approximated by Ising models such that the* φ^4 *model),* $\beta_c^\chi = \beta_c^M$.

This result is sometimes called the *sharpness* of the phase transition of the Ising model. Its proof [3] makes use of inequalities between correlation function, derived using the random current representation; we will not discuss it in detail. Recently, a new and shorter proof was given in [18]. For more general n-component spin models, the result is certainly expected to be true, but in general this is not proved.

6.2 Random walks and local time

Spin systems also have representations by random walks (instead of random currents), for *any* number of components $n \geqslant 1$. This point of view was developed in [10, 15]. It goes back to [48].

Continuous-time random walk. Let J be a symmetric matrix with no entries on the diagonal and nonnegative entries elsewhere. The continuous-time simple random walk generated by J can be defined as follows. Let $Y = (Y_n)$ be a discrete-time simple random with jump probability from x to y given by J_{xy}/J_x where $J_x = \sum_y J_{xy}$, and given Y, let τ_n be a sequence of independent $\mathrm{Exp}(J_{Y_n})$ distributed random variables. Here an $\mathrm{Exp}(a)$ random variable has density $a e^{-at} 1_{t>0}$ and thus mean $1/a$. Then $(X_T)_{T \geqslant 0}$ is defined by

$$X_T = Y_{N_T}, \quad N_T = \min \left\{ n : \sum_{i=1}^n \tau_i < T \right\}. \tag{6.16}$$

Thus, at every step, from position x say, after waiting an exponential time with mean given by the inverse of the total weight of the current vertex $1/J_x$, the walk takes a step to one of its neighbors. Equivalently, by properties of the exponential distribution, each edge has an exponential clock with mean given by its inverse weight, and the walk then jumps to the neighbor whose clock rings first. We write $\mathbb{E}_x$ for the expectation of X with initial condition $X_0 = x$.

Exercise 6.5. X is a continuous-time Markov process with right-continuous sample paths and for any $f : V \to \mathbb{R}$,

$$\frac{\partial}{\partial t} \mathbb{E}(f_{X_t}) = -\mathbb{E}(Qf(X_t)), \quad (Qf)_x = \sum_y J_{xy}(f_x - f_y). \tag{6.17}$$

Sketch. That X is a Markov process follows from the property that the exponential distribution has no memory (if τ has exponential distribution then $\mathbb{P}(\tau > s + t | \tau > s) = \mathbb{P}(\tau > t)$), and that Y is a discrete-time Markov process. Thus conditioned on $X_0 = x$, the next jump occurs at an $\mathrm{Exp}(J_x)$ distributed time, and

$$\frac{1}{t}\mathbb{E}_x(f_{X_t} - f_{X_0}) = \frac{1}{t}\mathbb{E}_x((f_{X_t} - f_{X_0}) 1_{N_t \leqslant 1}) + O(t) = \underbrace{\mathbb{E}_x(f_{Y_1} - f_x)}_{\sum_y \frac{J_{xy}}{J_x}(f_y - f_x)} \underbrace{\frac{1}{t}\mathbb{E}(\tau_1 < t)}_{\frac{1}{t}(1 - e^{-J_x t})} + O(t) = -(Qf)_x + O(t), \tag{6.18}$$

where we used that

$$\mathbb{P}(N_t \geqslant 2) = \mathbb{P}(\tau_1 + \tau_2 \leqslant t) = O(t^2). \tag{6.19}$$

This completes the sketch of the proof. □

The *local time* of X up to time T at vertex $x \in V$ is defined by

$$L_x^T = \int_0^T 1_{X_t = x}\, dt. \tag{6.20}$$

Thus L_x^T is the time spent by the walk at that vertex x up to T.

Proposition 6.6 (Feynman–Kac formula)**.** *Let $b : V \to \mathbb{R}$. Then, for any $T > 0$ and $f : V \to \mathbb{R}$,*

$$(\mathrm{e}^{-T(Q+b)}f)_x = \mathbb{E}_x\left(\mathrm{e}^{-(b,L^T)}f_{X_T}\right) = \mathbb{E}_x\left(\mathrm{e}^{-\int_0^T b_{X_t}\,dt}f_{X_T}\right). \tag{6.21}$$

Proof. Define $P_T f$ by the right-hand side of (6.21). Since, by the Markov property, $X_{[0,t]}$ and $X_{[t,t+s]}$ are independent given X_t, one can check that

$$(P_{t+s}f)_x = (P_t(P_s f))_x. \tag{6.22}$$

Thus P is a semigroup. Its generator is

$$\frac{P_T - P_0}{T} = \frac{1}{T}\mathbb{E}_x(f_{X_T} - f_x) - \mathbb{E}_x\left(\frac{1}{T}\int_0^T b_{X_t}\mathrm{e}^{-\int_0^t b_{X_s}\,ds}\,dt\right)f_x \to -(Qf)_x - (bf)_x. \tag{6.23}$$

The claim follows from this (see, e.g., [49] for more details). ☐

In the following, actually only use the integrated Feynman–Kac formula for the resolvent will be used:

$$(Q+b)^{-1}_{xy} = \int_0^\infty \mathbb{E}_x\left(\mathrm{e}^{-(b,L^T)}1_{X_T=y}\right)\,dT. \tag{6.24}$$

Gaussian field. Let D be a diagonal $V \times V$ matrix with $\operatorname{Re} D_{xx} > J_x$, and set $A = D - J$ so that A has positive definite real part. Let $\varphi = (\varphi_x)_{x\in V}$ be the n-component Gaussian field whose density is proportional to

$$\mathrm{e}^{-\frac{1}{2}(\varphi,A\varphi)}\,d\varphi, \tag{6.25}$$

where $(\varphi, A\varphi) = \sum_{i=1}^n(\varphi^i, A\varphi^i)$, and $d\varphi$ is the Lebesgue measure on $(\mathbb{R}^n)^V$. We write $\mathbb{E}$ for the expectation of the Gaussian field and include it in the expectation for the random walk $\mathbb{E}_x$. Abbreviate

$$\tau_x = \frac{1}{2}|\varphi_x|^2. \tag{6.26}$$

Proposition 6.7 (Brydges–Fröhlich–Spencer representation)**.** *For $x, y \in V$ and nice enough $g : \mathbb{R}_+^V \to \mathbb{R}$,*

$$\mathbb{E}(g(\tau)\varphi_x^1\varphi_y^1) = \int_0^\infty \mathbb{E}_x\left(g(L^T+\tau)\mathrm{e}^{-(A-Q,L^T)}1_{X_T=y}\right)\,dT. \tag{6.27}$$

In our applications, g will usually have sufficient decay to eventually take $A \to Q$, which we will then do implicitly and then simply write $A = Q$.

Remark 6.8. The integral over the final time T of the random walk on the right-hand side of (6.27) can be written in terms of a random walk with *killing rate* $A_{xx} - Q_{xx}$ at vertex x. This is a random walk on state space $V \cup \{\partial\}$, where ∂ is an additional vertex called cemetery state. At each step, the killed walk jumps to ∂ with rate $A_{xx} - Q_{xx}$. Denoting the death time of such a walk X by T_∂, the right-hand side of (6.27) is equal to

$$\mathbb{E}_x(g(L^\infty+\tau)1_{X_{T_\partial}=y}). \tag{6.28}$$

Thus (6.27) has an interpretation that τ under the signed measure $\varphi_x^1\varphi_y^1 \times$ (Gaussian measure on φ) has the same distribution as $\tau + L^\infty$ under the product measure of the Gaussian field and a measure described in terms of the random walk with transition rate Q and killing $A-Q$. See e.g. [49] for further development of this point of view. The formula (6.27) is sometimes referred to as “Dynkin isomorphism.”

Proof. It suffices to prove (6.27) for $g(t) = \mathrm{e}^{-(b,t)}$ and arbitrary $b \in \mathbb{R}^V$. Indeed, replacing b by zb with $z \in \mathbb{C}$, one can check that both sides of(6.27) are analytic in $z \in I = (-\varepsilon, \varepsilon) + \mathrm{i}\mathbb{R}$, for $\varepsilon > 0$ small enough that $\operatorname{Re} A > Q + \varepsilon|b|$. Since both sides agree for $z \in (-\varepsilon, \varepsilon)$ and are analytic in I, they must also agree in all of I. Hence the Fourier transforms of both sides agree, which implies the equality.

Define

$$J(b) = \int \mathrm{e}^{-\frac{1}{2}(\varphi,(A+b)\varphi)}\, d\varphi. \tag{6.29}$$

Then the left-hand side of (6.27) is

$$\mathbb{E}(g(\tau)\varphi_x^1\varphi_y^1) = \mathbb{E}(\mathrm{e}^{-\frac{1}{2}(\varphi,b\varphi)}\varphi_x^1\varphi_y^1) = \frac{J(b)}{J(0)}(A+b)^{-1}_{xy}. \tag{6.30}$$

The right-hand side of (6.27) factorizes into the product of the following two terms:

$$\int_0^\infty \mathbb{E}_x(\mathrm{e}^{-(L^T,b)}1_{X_T=y})\, dT = (A+b)^{-1}_{xy}, \qquad \mathbb{E}(\mathrm{e}^{-\frac{1}{2}(\varphi,b\varphi)}) = \frac{J(b)}{J(0)}, \tag{6.31}$$

where the first equality follows from the integrated Feynman–Kac formula (6.24). □

Given a nice function $g : \mathbb{R}_+^V \to \mathbb{R}_+$, define for $t \in \mathbb{R}_+^V$,

$$Z(t) = \mathbb{E}(g(\tau+t)), \quad \langle F(\varphi)\rangle_t = \frac{1}{Z(t)}\mathbb{E}(F(\varphi)g(\tau+t)), \tag{6.32}$$

and abbreviate $\langle\cdot\rangle = \langle\cdot\rangle_0$.

Example 6.9. Let $g(t) = \mathrm{e}^{-\sum_x(gt_x^2+\nu t_x)}$. Then $\langle\cdot\rangle$ is the expectation of the $|\varphi|^4$ model.

For $g(t) = \mathrm{e}^{-\sum_x v_x(t_x)}$, we have

$$\frac{\partial}{\partial t_x}\langle F\rangle_t = -\left\langle F; v_x'(\tau_x+t_x)\right\rangle_t. \tag{6.33}$$

For $n = 1, 2$, under certain assumptions of F and v_x, the second Griffith inequality implies that right-hand side is nonpositive, and thus $\langle F\rangle_t$ is monotone decreasing in t:

$$\frac{\partial}{\partial t}\langle F\rangle_t \leqslant 0. \tag{6.34}$$

This holds in particular for the $|\varphi|^4$ model and appropriate F and we will assume (6.34) from now.

Lemma 6.10 (Gaussian integration by parts formula)**.** *Let $C_{xy} = \mathbb{E}(\varphi_x^1\varphi_y^1)$. Then*

$$\mathbb{E}(\varphi_x^1 F(\varphi)) = \sum_y C_{xy}\mathbb{E}\left(\frac{\partial F}{\partial\varphi_y^1}\right) = \sum_y \int_0^\infty \mathbb{E}_x\left(\frac{\partial F}{\partial\varphi_y^1}1_{X_T=y}\right) dT. \tag{6.35}$$

Proof. The last equality follows from integrating the Feynman–Kac formula. By integration by parts, the right-hand side is of (6.35) is proportional to

$$\begin{aligned}\sum_y C_{xy}\int \frac{\partial F}{\partial\varphi_y^1}\mathrm{e}^{-\frac{1}{2}(\varphi,A\varphi)}\, d\varphi &= -\sum_y C_{xy}\int F(\varphi)\frac{\partial}{\partial\varphi_y^1}\mathrm{e}^{-\frac{1}{2}(\varphi,A\varphi)}\, d\varphi \\ &= \sum_y C_{xy}\sum_z A_{yz}\int F(\varphi)\varphi_z^1\mathrm{e}^{-\frac{1}{2}(\varphi,A\varphi)}\, d\varphi. \end{aligned} \tag{6.36}$$

This is equal to the left-hand side of (6.35) since $C = A^{-1}$. □

The following non-Gaussian integration by parts formula generalizes (6.27) (and follows from essentially the same proof).

Proposition 6.11 (BFS integration by parts)**.**

$$\langle \varphi_x^1 F(\varphi)\rangle = \sum_y \int_0^\infty \mathbb{E}_x \left(\mathcal{Z}(L) \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_L 1_{X_T=y} \right) dT, \qquad \text{where } \mathcal{Z}(t) = \frac{Z(t)}{Z(0)}. \tag{6.37}$$

Proof. As in the proof of (6.27), first consider $g(t) = \mathrm{e}^{-(b,t)}$, and recall $J(b)$ from (6.29). Then by Gaussian integration by parts and the integrated Feynman–Kac formula,

$$\begin{aligned} Z(0)\langle \varphi_x^1 F(\varphi)\rangle = \frac{1}{J(0)} \int \mathrm{e}^{-\frac12(\varphi,(A+b)\varphi)} \varphi_x^1 F(\varphi)\, d\varphi &= \frac{1}{J(0)} \int \mathrm{e}^{-\frac12(\varphi,(A+b)\varphi)} \sum_y (A+b)^{-1}_{xy} \frac{\partial F}{\partial \varphi_y^1}\, d\varphi \\ &= \sum_y \int_0^\infty \mathbb{E}_x \left(\mathrm{e}^{-(\tau+L^T,b)} \frac{\partial F}{\partial \varphi_y^1} 1_{X_T=y} \right) dT \\ &= \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L) \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_L 1_{X_T=y} \right) dT. \end{aligned} \tag{6.38}$$

Since both sides are linear in g, as in the proof of (6.27), this identity extends to all nice g by analyticity in b and Fourier transform. □

Corollary 6.12 (Gaussian upper bounds)**.** *Assume* (6.34). *Then*

$$\langle \varphi_x^1 F(\varphi)\rangle \leqslant \sum_y \langle \varphi_x^1 \varphi_y^1\rangle \left\langle \frac{\partial F(\varphi)}{\partial \varphi_y^1} \right\rangle. \tag{6.39}$$

Proof. By (6.37) and monotonicity of $\langle\cdot\rangle_t$ in t,

$$\langle \varphi_x^1 F(\varphi)\rangle = \sum_y \int_0^\infty \mathbb{E}_x (\mathcal{Z}(L) \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_L 1_{X_T=y})\, dT \leqslant \sum_y \int_0^\infty \mathbb{E}_x (\mathcal{Z}(L) \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_0 1_{X_T=y})\, dT. \tag{6.40}$$

The right-hand side is equal to the right-hand side of (6.39). □

In particular, (6.39) implies for $n = 1, 2$ and reasonable g that

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2p}} \rangle \leqslant \sum_\pi \langle \varphi_{\pi(1)} \varphi_{\pi(2)}\rangle \cdots \langle \varphi_{\pi(2p-1)} \varphi_{\pi(2p)}\rangle \tag{6.41}$$

where the sum runs over all pairings (matchings) of $\{1, \dots, 2p\}$. The case $p = 2$ is the Lebowitz inequality.

Corollary 6.13 (Lebowitz inequality)**.**

$$\langle \varphi_{x_1}\varphi_{x_2}\varphi_{x_3}\varphi_{x_4}\rangle - \sum_\pi \langle \varphi_{x_{\pi(1)}}\varphi_{x_{\pi(2)}}\rangle\langle \varphi_{x_{\pi(3)}}\varphi_{x_{\pi(4)}}\rangle \leqslant 0 \tag{6.42}$$

6.2.1. Multiple walks. Let $X^1, \dots, X^p$ be independent simple random walks with local times $L_1, \dots L_p$. We write $L^T = \sum_{i=1}^n L_i^{T_i}$ for the total local time up to times $T = (T_1, \dots, T_p)$, and denote by $\mathbb{E}_{x_1,\dots,x_p}$ the expectation with initial conditions $X_0^1 = x_1, \dots, X_0^p = x_p$.

Exercise 6.14. Iterate (6.37) to show that, for $x_1, \dots, x_{2p} \in V$,

$$\mathbb{E}(g(\tau)\varphi^1_{x_1}\cdots\varphi^1_{x_{2p}}) = \sum_\pi \int_0^\infty \cdots \int_0^\infty \mathbb{E}_{x_{\pi(1)},\dots,x_{\pi(p)}}(g(L^T+\tau)1_{X^1_{T_1}=x_{\pi(p+1)}}\cdots 1_{X^n_{T_p}=x_{\pi(2p)}})\, dT_1\cdots dT_p. \quad (6.43)$$

Lemma 6.15. *Let $A \subset V$ and T_A be the first time that X exits A. Then, for $x \in A$,*

$$\int_0^\infty \mathbb{E}_x\left(\mathcal{Z}(L^T)1_{X_T=y}1_{T_A<T}\right) dT = \sum_{z\in A}\sum_{z'\notin A} J_{zz'}\int_0^\infty\int_0^\infty \mathbb{E}_{x,z'}\left(\mathcal{Z}(L^T)1_{X^1_{T_1}=z}1_{X^1_{[0,T_1]}\subset A}1_{X^2_{T_2}=y}\right) dT_1\, dT_2. \quad (6.44)$$

The left-hand side sums over walks from x to y going through z, while the right-hand side involves a walk from x to z, then a step from z to z' and then a walk from z' to y.

Proof. We will estimate the integrand on the left-hand side for fixed T. Set $I_{n,i} = [T2^{-n}i, T2^{-n}(i+1))$, for integers $0 \leqslant i < 2^n - 1$. Then

$$\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{z\in X_{[0,T)}}) = \sum_{i=0}^{2^n-1}\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{T_A\in I_{n,i}}). \quad (6.45)$$

Defining the interval-valued functions $I_n : [0,T) \to 2^{[0,T)}$ by $I_n(S) = I_{n,i}$ if $S \in I_{n,i}$, the right-hand side is

$$\int_0^T |I_n(S)|^{-1}\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{T_A\in I_n(S)})\, dS. \quad (6.46)$$

It suffices to show that, for every $S \in [0,T)$,

$$\lim_{n\to\infty}|I_n(S)|^{-1}\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=x}1_{T_A\in I_n(S)}) = \sum_{z\in A}\sum_{z'\notin A} J_{zz'}\mathbb{E}_{x,z'}(\mathcal{Z}(L_1^S + L_2^{T-S})1_{X^1_S=z}1_{X^1_{[0,S]}\subset A}1_{X^2_{T-S}=y}). \quad (6.47)$$

Since we also have the bound (exercise),

$$|I|^{-1}\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=x}1_{T_z\in I}) \leqslant |I|^{-1}P(T_z \in I) = O(1), \quad (6.48)$$

the claim then follows by integrating (6.47) and taking $n \to \infty$ (using dominated convergence) and that

$$\int_0^\infty dT \int_0^T dS\, f(S)g(T-S) = \int_0^\infty\int_0^\infty f(T_1)g(T_2)\, dT_1\, dT_2. \quad (6.49)$$

To verify (6.47), consider any interval $I = [a,b)$. Then

$$|\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{T_A\in I}) - \mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{T_A\geqslant a}1_{X_a\in A}1_{X_b\notin A})| \leqslant \mathbb{E}_x(|1_{T_z\in I} - 1_{T_z\geqslant a}1_{X_a\in A}1_{X_b\notin A}|) \leqslant \mathbb{P}(N_I \geqslant 2) = O(|I|^2), \quad (6.50)$$

where N_I is the number of jumps in the time interval I. Thus

$$\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{T_A\in I}) = \mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{X_{[0,a]}\subset A}1_{X_b\notin A}) + O(|I|^2). \quad (6.51)$$

Summing over the possibilities for X_a and X_b, write

$$\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_T=y}1_{X_{[0,a]}\subset A}1_{X_b\notin A}) = \sum_{z\in A}\sum_{z'\notin A}\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_{[0,a]}\subset A}1_{X_a=z}1_{X_b=z'}1_{X_T=y}). \tag{6.52}$$

Using the Markov property, the right-hand side has contributions from three independent walks: a walk $x\to z$ in time $[0,a]$, a step $z\to z'$ in time (a,b), and a walk $z'\to y$ in time $[b,T]$:

$$\begin{aligned}\mathbb{E}_x(\mathcal{Z}(L^T)1_{X_a=z}1_{X_b=z'}1_{X_T=y}) &= \mathbb{E}_{x,z'}(\mathcal{Z}(L_1^a+L_2^{T-b})1_{X_{[0,a]}\subset A}1_{X_a=z}1_{X_{T-b}=y})\mathbb{P}(X_b=z'|X_a=z)+O(|I|^2)\\ &= \mathbb{E}_{x,z'}(\mathcal{Z}(L_1^a+L_2^{T-b})1_{X_{[0,a]}\subset A}1_{X_a=z}1_{X_{T-b}=y})J_{zz'}|I|+O(|I|^2),\end{aligned} \tag{6.53}$$

where the $O(|I|^2)$ errors accounts for omitting the contribution of the walk from a to b from $\mathcal{Z}$ and the error from the replacement (see Exercise 6.5)

$$P(X_b=z'|X_a=z) = J_{zz'}|I|+O(|I|^2). \tag{6.54}$$

Taking $I=I_n(S)$, by right-continuity of the sample paths, as $n\to\infty$, left-hand side of (6.47) is bounded by

$$\sum_{z\in A}\sum_{z'\notin A}J_{zz'}\mathbb{E}_{x,z'}(\mathcal{Z}(L_1^S+L_2^{T-S})1_{X_S=z}1_{X_{[0,S]}\subset A}1_{X_{T-S}=y}), \tag{6.55}$$

as claimed. □

Corollary 6.16 (Lieb–Simon inequality)**.** *Let $n=1,2$ and $A\subset V$. Then*

$$0\leqslant\langle\varphi_x\varphi_z\rangle-\langle\varphi_x\varphi_z\rangle_A\leqslant\sum_{y\in A,y'\notin A}J_{y'y}\langle\varphi_x\varphi_y\rangle_A\langle\varphi_{y'}\varphi_z\rangle, \tag{6.56}$$

where $\langle\cdot\rangle_A$ denotes the expectation with J_{xy} set to 0 unless $x,y\in A$.

Proof. The lower bound is the second Griffith inequality. The left-hand side of the second inequality is

$$\begin{aligned}\langle\varphi_x\varphi_z\rangle-\langle\varphi_x\varphi_z\rangle_A &= \int_0^\infty\mathbb{E}_x\left(\mathcal{Z}(L^T)1_{X_T=z}1_{T_A<T}\right)dT\\ &= \sum_{y\in A}\sum_{y'\notin A}J_{yy'}\int_0^\infty\int_0^\infty\mathbb{E}_{x,y'}\left(\mathcal{Z}(L^T)1_{X^1_{T_1}=y}1_{X^1_{[0,T_1]}\subset A}1_{X^2_{T_2}=z}\right)dT_1\,dT_2\\ &\leqslant\sum_{y\in A}\sum_{y'\notin A}J_{yy'}\langle\varphi_x\varphi_y\rangle_A\langle\varphi_{y'}\varphi_z\rangle,\end{aligned} \tag{6.57}$$

where in the last step we used that

$$\begin{aligned}&\int_0^\infty\int_0^\infty\mathbb{E}_{x,y'}\left(\left(\mathcal{Z}(L^T)-\mathcal{Z}(L_1^{T_1})\mathcal{Z}(L_2^{T_2})\right)1_{X^1_{T_1}=y}1_{X^1_{[0,T_1]}\subset A}1_{X^2_{T_2}=z}\right)dT_1\,dT_2\\ &=\int_0^\infty\mathbb{E}_x\left(\mathcal{Z}(L^T)1_{X_T=y}1_{X_{[0,T]}\subset A}\left(\langle\varphi_{y'}\varphi_z\rangle_{L^T}-\langle\varphi_{y'}\varphi_z\rangle_0\right)\right)dT\leqslant 0\end{aligned} \tag{6.58}$$

by monotonicity of $\langle\cdot\rangle_t$, i.e., (6.34). □

6.3 Triviality above dimension four

Both random current and random walk representations can be used to show the behaviour near the critical point must be the same as that of mean field theory if $d \geqslant 5$, provided that $n = 1, 2$ (see [20, 2]). Let

$$\chi = \lim_{m\to\infty} \sum_{x\in\mathbb{T}_m^d} \langle \varphi_0 \varphi_x \rangle_{\mathbb{T}_m^d} \tag{6.59}$$

be the susceptibility, and define the critical point β_c by

$$\beta_c = \sup\{\beta : \chi(\beta) < \infty\}. \tag{6.60}$$

From Sections 2–3, χ is finite for high temperatures and infinite for low temperatures, and thus $\beta_c \in (0, \infty)$. By the second Griffith inequality, χ is monotone in β, assuming that $n = 1, 2$, which we do in this section. Moreover, we assume homogenous nearest-neighbour spin-spin coupling, i.e.,

$$J_{xy} = \beta 1_{x\sim y}, \tag{6.61}$$

so that in particular the infrared bound (3.13) holds.

Comment: Should move a discussion of the infinite volume limit here.

Theorem 6.17 (Aizenman, Fröhlich)**.** *Let $d \geqslant 5$. For the Ising and XY model, and also the* 1- *and* 2-*component $|\varphi|^4$ model, as $\varepsilon = \beta_c - \beta \downarrow 0$, the susceptibility obeys*

$$\frac{c}{\varepsilon} \leqslant \chi \leqslant \frac{C}{\varepsilon}. \tag{6.62}$$

(In fact, the lower bound holds in any dimension $d \geqslant 3$

Remark 6.18. (6.62) shows that for $d \geqslant 5$ the critical exponent for the susceptibility is 1 (in the sense of upper and lower bounds), as in mean field theory (Proposition 1.10 (b)). Similar bounds can also be established for other observables (in $d \geqslant 5$). For $d = 4$, the susceptibility (as well as other observables) are expected to have logarithmic corrections to mean field theory. Using similar techniques as here, it is known that corrections can be at most logarithmic. Full asymptotics are known under under stronger assumptions (the $|\varphi|^4$ model with small coupling constant $g > 0$), using the renormalization group. Then, for example, as $\varepsilon \downarrow 0$,

$$\chi \sim C\frac{1}{\varepsilon}(-\log \varepsilon)^{(n+2)/(n+8)}. \tag{6.63}$$

For $d = 2$, the divergence of the susceptibility is known for the Ising model, for which it diverges as

$$\chi \sim C\frac{1}{\varepsilon^{7/4}}. \tag{6.64}$$

The most interesting case $d = 3$ remains a great challenge for statistical physics.

The proof of Theorem 6.17 relies on the analysis of the truncated four-point function, defined by

$$U_4(x_1, x_2, x_3, x_4) = \langle \varphi_{x_1}\varphi_{x_2}\varphi_{x_3}\varphi_{x_4} \rangle - \sum_\pi \langle \varphi_{x_{\pi(1)}}\varphi_{x_{\pi(2)}} \rangle \langle \varphi_{x_{\pi(3)}}\varphi_{x_{\pi(4)}} \rangle. \tag{6.65}$$

The Lebowitz inequality (6.42) provides the upper bound $U_4(x_1, x_2, x_3, x_4) \leqslant 0$. Theorem 6.17 is proved by proving a lower bound on $U_4(x_1, x_2, x_3, x_4)$. Such lower bounds can be obtained both from the random current representation [2], and from the BFS random walk representation [20].

To illustrate it, we use the random walk representation and show the Aizenman–Fröhlich inequality for the $|\varphi|^4$ model with $n = 1$. In preparation of Fröhlich's proof of the Aizenman–Fröhlich inequality, we note the following inequality for the weight $\mathcal{Z}$ in the random walk representation.

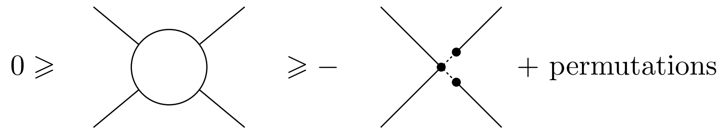

Figure 4: Diagrammatic representation of the Lebowitz and Aizenman–Fröhlich inequalities. On the right-hand side, the dotted line symbolizes J_{ab}, and the solid line a two-point function $\langle \varphi_a \varphi_b \rangle$.

Lemma 6.19. *For the $|\varphi|^4$ model, the weight $\mathcal{Z}$ obeys*

$$\mathcal{Z}(t+s) \geqslant \mathcal{Z}(t)\mathcal{Z}(s)\mathrm{e}^{-2g(t,s)}. \tag{6.66}$$

Proof. The logarithm of $\mathcal{Z}(t+s)$ equals

$$\log \mathcal{Z}(t+s) = \log \mathcal{Z}(t) + \int_0^1 \frac{\partial}{\partial u} \log \mathcal{Z}(t+us)\, du, \tag{6.67}$$

and, using the remark below (6.33),

$$\begin{aligned} \frac{\partial}{\partial u} \log \mathcal{Z}(t+us) &= -\sum_x s_x \langle 2g(\tau_x + t_x + us_x) + \nu \rangle_{t+us} \\ &\geqslant -\sum_x s_x \langle 2g(\tau_x + us_x) + \nu \rangle_{us} - 2\sum_x g s_x t_x \\ &= \frac{\partial}{\partial u} \log \mathcal{Z}(us) - 2g(s,t). \end{aligned} \tag{6.68}$$

The inequality (6.66) follows by reversing the previous steps. □

Proposition 6.20 (Aizenman–Fröhlich inequality)**.** *For the 1-component $|\varphi|^4$ model,*

$$\begin{aligned} &U_4(x_1,x_2,x_3,x_4) \\ &\quad \geqslant -\sum_z \langle \varphi_{x_{\pi(1)}} \varphi_z \rangle \langle \varphi_{x_{\pi(2)}} \varphi_z \rangle \left[\delta_{z x_{\pi(3)}} + \sum_{z'} J_{zz'} \langle \varphi_{z'} \varphi_{x_{\pi(3)}} \rangle \right] \left[\delta_{z x_{\pi(4)}} + \sum_{z''} J_{zz''} \langle \varphi_{z''} \varphi_{x_{\pi(4)}} \rangle \right]. \end{aligned} \tag{6.69}$$

Proof of Proposition 6.20. By (6.43), the truncated four-point function equals

$$\sum_\pi \int_0^\infty \int_0^\infty \mathbb{E}_{x_{\pi(1)},x_{\pi(2)}} \Big(\Big(\mathcal{Z}(L_1^{T_1} + L_2^{T_2}) - \mathcal{Z}(L_1^{T_1})\mathcal{Z}(L_2^{T_2}) \Big) 1_{X^1_{T_1} = x_{\pi(3)}} 1_{X^2_{T_2} = x_{\pi(4)}} \Big)\, dT_1\, dT_2. \tag{6.70}$$

By (6.66), this is bounded below by

$$\sum_\pi \int_0^\infty \int_0^\infty \mathbb{E}_{x_{\pi(1)},x_{\pi(2)}} \Big(\mathcal{Z}(L_1^{T_1})\mathcal{Z}(L_2^{T_2}) 1_{X^1_{T_1} = x_{\pi(3)}} 1_{X^2_{T_2} = x_{\pi(4)}} \Big(\mathrm{e}^{-2g(L^1,L^2)} - 1 \Big) \Big)\, dT_1\, dT_2. \tag{6.71}$$

The right-hand side vanishes unless X_1 and X_2 intersect, say at $z \in V$. Thus it is bounded below by

$$-\sum_z \sum_\pi \int_0^\infty \int_0^\infty \mathbb{E}_{x_{\pi(1)},x_{\pi(2)}} \Big(\mathcal{Z}(L_1^{T_1})\mathcal{Z}(L_2^{T_2}) 1_{X^1_{T_1} = x_{\pi(3)}} 1_{X^2_{T_2} = x_{\pi(4)}} 1_{z \in X^1_{[0,T_1]}, z \in X^2_{[0,T_2]}} \Big)\, dT_1\, dT_2. \tag{6.72}$$

The double integral on the right-hand side factorizes into two integrals of the type

$$\int_0^\infty \mathbb{E}_x \left(\mathcal{Z}(L^T) 1_{X_T=y} 1_{z\in X_{[0,T)}} \right) dT \leqslant \langle \varphi_x \varphi_y \rangle \delta_{zx} + \sum_{z'} J_{zz'} \langle \varphi_x \varphi_{z'} \rangle \langle \varphi_z \varphi_y \rangle, \tag{6.73}$$

where the inequality follows from the Simon inequality (6.56) with $A = V \setminus \{z\}$. Taking the sum over z, this then shows the lower bound on the four-point function

$$-\sum_z \langle \varphi_{x_{\pi(1)}} \varphi_z \rangle \langle \varphi_{x_{\pi(2)}} \varphi_z \rangle \left[\delta_{zx_{\pi(3)}} + \sum_{z'} J_{zz'} \langle \varphi_{z'} \varphi_{x_{\pi(3)}} \rangle \right] \left[\delta_{zx_{\pi(4)}} + \sum_{z''} J_{zz''} \langle \varphi_{z''} \varphi_{x_{\pi(4)}} \rangle \right], \tag{6.74}$$

as claimed. □

To bound the right-hand side of (6.69), a real-space version of the infrared bound is useful. It proof uses the following delicate monotonicity of the two-point function.

Lemma 6.21 (Schrader–Messager–Miracle-Solé inequality)**.** *Let $n = 1, 2$. For general single spin measure and homogeneous nearest-neighbour coupling on $\mathbb{Z}^d$, it holds that*

$$\langle \varphi_0 \varphi_x \rangle \leqslant \langle \varphi_0 \varphi_y \rangle \quad \textit{whenever } |y|_1 \leqslant |x|_\infty. \tag{6.75}$$

Proof. □

Lemma 6.22 (Infrared bound, real-space version)**.** *Let $n = 1, 2$ and $d \geqslant 3$, and assume $\lim_{|x|\to\infty} \langle \varphi_0 \varphi_x \rangle = 0$. Then*

$$\langle \varphi_0 \varphi_x \rangle \leqslant \frac{C}{\beta (1+|x|)^{d-2}}. \tag{6.76}$$

Comment from RB: Be more precise about infinite volume limit.

Proof [44, Appendix A]. Let $\chi(x) = 1_{|x|_\infty \leqslant L}$ and $S(x,y) = S(x-y) = \langle \varphi_x \varphi_y \rangle$. By the infrared bound,

$$(\chi, S\chi) \leqslant \frac{1}{\beta} (\chi, G\chi) = \frac{1}{\beta} O(L^{d+2}), \tag{6.77}$$

where in the last bound we used that

$$\sum_{|x|\leqslant L} \sum_{|y|\leqslant L} \frac{1}{1 \vee |x-y|^{-(d-2)}} \leqslant \sum_{j=0}^{1+\log_2 L} O(2^{-(d-2)j}) \sum_{|x|\leqslant L} \underbrace{\sum_{y} 1_{2^j \leqslant |x-y| \leqslant 2^{j+1}}}_{O(2^{dj})} = O(L^{d+2}). \tag{6.78}$$

Thus

$$\min_{|x|\leqslant 2L} S(x) \leqslant (2L+1)^{-2d} \sum_{|x|_\infty \leqslant L} \sum_{|y|_\infty \leqslant L} S(x-y) \leqslant \frac{1}{\beta} O(L^{-(d-2)}). \tag{6.79}$$

The claim then follows from (6.75) and choosing $L \approx c|x|$. □

Lemma 6.23. *Let $d > 4$. Then*

$$\sum_{u\in\mathbb{Z}^d} \frac{1}{1 \vee |u|^{d-2}} \frac{1}{1 \vee |u-v|^{d-2}} = O\left(\frac{1}{1 \vee |v|^{d-4}} \right) \tag{6.80}$$

Proof. Let $v \neq 0$. Then

$$\sum_{|u| \leqslant |v|/2} \frac{1}{|u|^{d-2}} \frac{1}{|u-v|^{d-2}} \leqslant \frac{1}{(|v|/2)^{d-2}} \sum_{|u| \leqslant |v|/2} \frac{1}{|u|^{d-2}} = O\left(\frac{1}{|v|^{d-4}}\right) \tag{6.81}$$

$$\sum_{|u| \geqslant 2|v|} \frac{1}{|u|^{d-2}} \frac{1}{|u-v|^{d-2}} \leqslant \sum_{|u| \geqslant 2|v|} \frac{1}{|u|^{d-2}} \frac{1}{(|u|/2)^{d-2}} = O\left(\frac{1}{|v|^{d-4}}\right) \tag{6.82}$$

$$\sum_{|v|/2 \leqslant |u| \leqslant 2|v|} \frac{1}{|u|^{d-2}} \frac{1}{1 \vee |u-v|^{d-2}} \leqslant \frac{1}{(|v|/2)^{d-2}} \sum_{|u| \leqslant 2|v|} \frac{1}{1 \vee |u-v|^{d-2}} = O\left(\frac{1}{|v|^{d-4}}\right), \tag{6.83}$$

where in the last inequality we used that, given $v \in \mathbb{Z}^d$, there are $O(2^{dj})$ points $u \in \mathbb{Z}^d$ with $2^j \leqslant |u-v| \leqslant 2^{j+1}$, from which it follows that

$$\sum_{|u| \leqslant 2|v|} \frac{1}{1 \vee |u-v|^{d-2}} \leqslant 1 + \sum_{j=0}^{1+\log_2 |v|} O(2^{dj} 2^{-(d-2)j}) = O(|v|^2). \tag{6.84}$$

This completes the proof. □

An immediate consequence of (6.69) is the inequality

$$\begin{aligned} -\sum_{u,v,w} J_{uv} U_4(0,u,v,w) &\leqslant \sum_{u,v,w} J_{uv} \sum_z \langle \varphi_0 \varphi_z \rangle \langle \varphi_u \varphi_z \rangle \left[\delta_{zv} + \sum_{z'} J_{zz'} \langle \varphi_{z'} \varphi_v \rangle\right] \left[\delta_{zw} + \sum_{z''} J_{zz''} \langle \varphi_{z''} \varphi_w \rangle\right] \\ &\leqslant |J| \sum_z \langle \varphi_0 \varphi_z \rangle^2 (1 + |J|\chi)^2 = |J| \mathsf{B} (1 + |J|\chi)^2 \end{aligned} \tag{6.85}$$

where B is the *bubble diagram*

$$\mathsf{B} = \sum_z \langle \varphi_0 \varphi_z \rangle^2 = \int |\hat{S}(k)|^2 \, dk. \tag{6.86}$$

By the infrared bound, in $d > 4$, the bubble diagram is uniform bounded for $\beta \leqslant \beta_c$. For the Ising and 1-component φ^4 model, the inequality (6.85) can be improved to (Aizenman–Graham [4])

$$-\sum_{u,v,w} J_{uv} U_4(0,u,v,w) \leqslant |J| \mathsf{B} \left(\beta \frac{\partial \chi}{\partial \beta} + |J|\chi\right). \tag{6.87}$$

(This is indeed an improvement, as seen by (6.88) below, together with the Lebowitz inequality $U_4 \leqslant 0$.)

> **Comment: Can the proof of (6.62) also be completed using the weaker inequality (6.69), as claimed in [20]?**

Proof of Theorem 6.17 using (6.87). Differentiating the susceptibility gives

$$\begin{aligned} \beta \frac{\partial}{\partial \beta} \chi &= \sum_z \langle \varphi_0 \varphi_z ; -\beta H(\varphi) \rangle \\ &= \frac{1}{2} \sum_{x,y,z} J_{xy} \left(\langle \varphi_0 \varphi_z \varphi_x \varphi_y \rangle - \langle \varphi_0 \varphi_z \rangle \langle \varphi_x \varphi_y \rangle \right) \\ &= \frac{1}{2} \sum_{x,y,z} J_{xy} \left(\langle \varphi_0 \varphi_x \rangle \langle \varphi_z \varphi_y \rangle + \langle \varphi_0 \varphi_y \rangle \langle \varphi_x \varphi_z \rangle + U_4(0,x,y,z) \right). \\ &= |J|\chi^2 + \frac{1}{2} \sum_{x,y,z} J_{xy} U_4(0,x,y,z) \end{aligned} \tag{6.88}$$

The last term on the right-hand side is bounded above by Lebowitz' inequality (6.42) (for all β) and from below by the summed Aizenman–Graham inequality (6.87) (for $\beta < \beta_c$):

$$\frac{\chi^2}{1+CB} - C\chi \leqslant \frac{\partial}{\partial\beta}\chi \leqslant C\chi^2. \tag{6.89}$$

Let $f(t) = 1/\chi(\beta_c - t)$. Then f is decreasing by Griffith inequality (for all t), $f'(t) = \chi'(\beta_c - t)/\chi(\beta_c - t)^2 \leqslant C$ also for all t, and $f(t) = 0$ for $t < 0$ by (6.60), it follows that $f(t) \downarrow 0$ as $t \downarrow 0$, and

$$f(\varepsilon) = \int_0^\varepsilon f'(t)\,dt \leqslant C\varepsilon \tag{6.90}$$

i.e., $\chi \geqslant 1/(C\varepsilon)$. Moreover, the lower bound implies

$$f(\varepsilon) = \int_0^\varepsilon f'(t)\,dt \geqslant c\varepsilon, \tag{6.91}$$

and thus also $\chi \leqslant 1/(c\varepsilon)$. □

6.4 Continuum and scaling limits

Now consider the $|\varphi|^4$ model on the scaled lattice $(a\mathbb{Z})^d$ with $a > 0$, and mostly $n = 1$. More precisely, it is defined as follows. Let φ be distributed according to the probability measure proportional to $e^{-S_{a,R}(\varphi)}\,d\varphi$ on $\Lambda_{a,R} = [-R, R]^d \cap (a\mathbb{Z})^d$, where

$$S_{a,R}(\varphi) = -\frac{1}{2}\sum_{x,y}\varphi_x J_{xy}\varphi_y + \sum_x\left(\frac{1}{4}g\varphi_x^4 + \frac{1}{2}\nu\varphi_x^2\right), \tag{6.92}$$

where the sum only extends over x, y in $\Lambda_{a,R}$. Let a be fixed. Then, for any $x_1, \dots, x_p \in (a\mathbb{Z})^d$, the limit

$$S_p^a(x_1, \dots, x_p) = \lim_{R\to\infty}\langle\varphi_{x_1}\cdots\varphi_{x_p}\rangle \tag{6.93}$$

exists by the second Griffith inequality (assuming $n = 1$). (Alternatively, one could consider a discrete torus or a discretized domain $(a\mathbb{Z}^d) \cap D$.) The correlation functions are also called *Schwinger functions*, and can be interpreted as Schwartz distributions in $S'(\mathbb{R}^{pd})$. The truncated correlation functions are denoted by U_p^a. In particular,

$$\begin{aligned} U_4^a(x_1, x_2, x_3, x_4) = \langle\varphi_{x_1}\varphi_{x_2}\varphi_{x_3}\varphi_{x_4}\rangle^a - \langle\varphi_{x_1}\varphi_{x_2}\rangle^a\langle\varphi_{x_3}\varphi_{x_4}\rangle^a - \langle\varphi_{x_1}\varphi_{x_3}\rangle^a\langle\varphi_{x_2}\varphi_{x_4}\rangle^a \\ - \langle\varphi_{x_1}\varphi_{x_4}\rangle^a\langle\varphi_{x_2}\varphi_{x_3}\rangle^a. \end{aligned} \tag{6.94}$$

There are different ways to take continuum limits $a \downarrow 0$ of the (truncated) Schwinger functions. Under certain technical conditions (which we do not discuss), continuum limits of the correlation functions correspond to random fields on $\mathbb{R}^d$ (in the sense that they are the correlation functions of a random Schwartz distribution on $\mathbb{R}^d$). Under further conditions (the Osterwalder–Schrader Axioms [36], involving in particular *reflection positivity*), they are further in correspondence with relativistic Quantum Field Theories in $d-1$ space dimensions.

Scaling limit (infrared problem). For *fixed* coupling constants J, g, ν, does there exist a scaling $s = s(a)$ such that $s\varphi$ converges to a nonvanishing random field as $a \to 0$?

This is the scaling of interest in statistical mechanics (describing the system's large scale behaviour). Above the critical temperature, any nontrivial scaling limit is (expected to be) white noise. (In particular, for one-component spin systems and more general FKG systems, see [35, 34].) The most interesting case is the critical point, at which the field is strongly correlated. The limiting random fields should be scale invariant, and actually conformally invariant, at least in $d = 2$ (for the Ising model, which is believed to have the same scaling limit as the critical one-component φ^4 model, see [**?**]). The Wilson renormalization group [50], for which Wilson received the Nobel Prize in 1982, provides a largely conjectural picture to explain many of the associated heuristics.

Continuum limit (ultraviolet problem). For which *a-dependent* choices of *bare* coupling constants $J(a)$, $g(a), \nu(a)$ does φ converge to a nonvanishing random field as $a \to 0$?

The choice of coupling constants typically corresponds to much weaker interaction than for the scaling limit (which in principle is a special case). Such scaling is of interest for Quantum Field Theory.

There are different natural topologies for convergence of the Schwinger functions. In particular, one of them is *weak convergence at non-coinciding arguments*, in which $S_p^a(x_1, \dots, x_p)$ is tested against function in the Schwartz space $S(\mathbb{R}^{pd})$ that vanish whenever $x_i = x_j$ for some $i \neq j$. (It turns out that coinciding arguments are irrelevant for QFT [36]; see also [19] for further discussion of topologies.)

Theorem 6.24 (Glimm–Jaffe)**.** *Let $d = 2, 3$ and $n = 1$. Then nontrivial (ultraviolet) continuuum limits exist for which the Osterwalder–Schrader Axioms hold.*

A simple proof of this result (except for rotation invariance), based on the random walk representation, was given by Brydges–Fröhlich–Sokal [14].

On the other hand, non-Gaussian continuum limits of the $|\varphi|^4$ model (for $n = 1, 2$) are known not to exist for $d > 4$. In fact, the Aizenman–Fröhlich inequality (6.69) implies the following theorem.

Theorem 6.25 (Aizenman, Fröhlich)**.** *Let $d \geqslant 5$ and $n = 1, 2$. Assume that $S_2^a(0, x) \to 0$ as $x \to 0$ (no long range order) for every $a > 0$, and that $S_2^a \to S_2$ weakly in $S'(\mathbb{R}^{2d})$ as $a \to 0$. Then*

$$0 \geqslant U_4^a(x_1, \dots, x_4) \geqslant -Ca^{d-4} \tag{6.95}$$

if $|x_i - x_j| \geqslant \delta > 0$ (with δ-dependent constant). This implies $U_4^a \to 0$ weakly at non-coinciding arguments.

As a consequence of $U_4^a \to 0$, it is known that in fact all truncated correlation functions converge to 0 *(weakly at non-coinciding arguments), i.e., the continuum limit must be Gaussian.*

For $d = 4$, it is conjectured that no non-Gaussian continuums limits exist either. (Results that prove existence of Gaussian scaling limits exist for small coupling constants, but the ultraviolet problem remains unsolved.)

Proof of Theorem 6.25. Write $J_{xy} = \beta 1_{x \sim y}$, $\beta = a^{d-2}\zeta$ with $\zeta = \zeta(a)$ possibly depending on a. The real-space infrared bound (6.76) implies that (assuming there is no long-range order)

$$S_2^a(0, x) \leqslant \frac{C}{\zeta(a + |x|)^{d-2}}. \tag{6.96}$$

Therefore we can assume that $\lim_{a \to 0} \zeta < \infty$, since otherwise $\lim_{a \to 0} \lim S_2^a = 0$ is trivial. Furthermore, the inequality (6.75) implies that

$$\sup_a \sup_{|x| \geqslant \delta/2} S_2^a(0, x) < \infty. \tag{6.97}$$

Indeed, otherwise (6.75) implies that there is a subsequence such that (along this subsequence)

$$\inf_{|x|\leqslant c\delta} S_2^a(0,x) \to \infty, \tag{6.98}$$

which is a contradiction to the assumption that $S_2^a \to S_2$.

The main contribution to (6.69) is essentially given by

$$|J|^2 \sum_{z\in(a\mathbb{Z})^d} S(x_1,z)S(x_2,z)S(x_3,z)S(x_4,z), \tag{6.99}$$

with $|J|^2 \propto \zeta^2 a^{2d-4}$. (Indeed, the difference between z and its neighbours z' and z'' is unimportant and the contributions involving δ are not difficult to see to be of lower order as $a \to 0$.) Using (6.96) for the term involving x_1, and (6.97) for the terms involving x_2, x_3, x_4,

$$\begin{aligned}\zeta a^d \sum_{z\in(a\mathbb{Z})^d:|z-x_1|\leqslant\delta/2} S(x_1,z)S(x_2,z)S(x_3,z)S(x_4,z) &\leqslant Ca^d \sum_{z\in(a\mathbb{Z})^d:|z|\leqslant\delta_2} \frac{1}{(a+|z|)^{d-2}} \\ &\leqslant Ca^d \int_{|z|\leqslant\delta/2} \frac{1}{|z|^{d-2}}\, dz \leqslant C, \end{aligned}\tag{6.100}$$

and of course the same bound holds with x_1 replaced by any of x_2, x_3, x_4. On the other hand, again using (6.96) and (6.97),

$$\begin{aligned}\zeta^2 a^d &\sum_{z\in(a\mathbb{Z})^d:\min_i|z-x_i|\geqslant\delta/2} S(x_1,z)S(x_2,z)S(x_3,z)S(x_4,z) \\ &\leqslant Ca^d \sum_{z\in(a\mathbb{Z})^d:\min_i|z-x_i|\geqslant\delta/2} \frac{1}{|z-x_1|^{d-2}}\frac{1}{|z-x_2|^{d-2}} \leqslant C.\end{aligned}\tag{6.101}$$

Therefore

$$|J|^2 \sum_{z\in(a\mathbb{Z})^d} S(x_1,z)S(x_2,z)S(x_3,z)S(x_4,z) = O(a^{d-4}). \tag{6.102}$$

Since the other contributions to (6.69) are smaller (as remarked above), this shows (6.95). □

6.5 Supersymmetry and the self-avoiding walk

k-forms. Consider a two-component field $(u_x, v_x)_{x\in V}$ and denote by du_x and dv_x the associated 1-forms, with the usual exterior algebra generator by the wedge product $\wedge$. Thus

$$du_x \wedge du_y = -du_y \wedge du_x, \quad dv_x \wedge dv_y = -dv_y \wedge dv_x, \quad dv_x \wedge du_y = -dv_y \wedge du_x. \tag{6.103}$$

Then k-forms are sums of forms $f(u,v)\, du_{x_1} \wedge \cdots \wedge du_{x_l} \wedge dv_{x_{l+1}} \wedge \cdots \wedge dv_{x_k}$, with $f(u,v)$ an ordinary function of (u,v), say smooth. They define a real vector space (in fact an algebra) denoted Ω_k. Differential forms are formal linear combinations of k-forms:

$$\Omega = \bigoplus_{k=0}^{2N} \Omega_k. \tag{6.104}$$

Notice that k-forms only exist for $k \leqslant 2N$, where $N = |V|$ and $2N$ is the dimension of $\mathbb{R}^V \oplus \mathbb{R}^V$. Moreover, any $2N$-form ω can be uniquely written as

$$\omega = f\, du_{x_1} \wedge dv_{x_1} \wedge \cdots \wedge du_{x_N} \wedge dv_{x_N}, \tag{6.105}$$

where f is a function and $x_1, \dots, x_N$ is an arbitrary enumeration of V. As usual in differential geometry, the integral of $\omega \in \Omega_{2N}$ is then defined as

$$\int \omega = \int f \, du_{x_1} dv_{x_1} \cdots du_{x_N} dv_{x_N}, \tag{6.106}$$

with the integral on the right-hand side being the Lebesgue integral. For $\omega \in \Omega_k$, $k < 2N$, it is useful for our purposes to set

$$\int \omega = 0. \tag{6.107}$$

The integral is then extended to Ω by linearity.

Complex field. We identify (u, v) with the complex field ϕ defined by

$$\phi_x = u_x + \mathrm{i}v_x, \quad \bar{\phi} = u_x - \mathrm{i}v_x. \tag{6.108}$$

Moreover, for an arbitrary fixed choice of square root, we define the 1-forms

$$\psi_x = \frac{1}{\sqrt{2\pi\mathrm{i}}} d\phi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi\mathrm{i}}} d\bar{\phi}_x. \tag{6.109}$$

Then, for example,

$$\bar{\psi}_x \wedge \psi_x = \frac{1}{2\pi\mathrm{i}} d\bar{\phi}_x \wedge d\phi_x = \frac{1}{\pi} du_x \wedge dv_x. \tag{6.110}$$

From now on, we will almost always omit $\wedge$ from our notation, and write, e.g., $\bar{\psi}_x \psi_x = \bar{\psi}_x \wedge \psi_x$.

Definition 6.26. *For $A = (A_{xy})_{x,y \in V}$ with positive definite real part, define the differential form*

$$S = S_A = \sum_{x,y} \left(\phi_x A_{xy} \bar{\phi}_x + \psi_x A_{xy} \bar{\psi}_x \right). \tag{6.111}$$

The super-Gaussian measure *is the differential form (so not really a measure)*

$$\mathrm{e}^{-S} = \mathrm{e}^{-(\phi, A\bar{\phi})} \sum_{k=0}^{2N} \frac{(-1)^k}{k!} \left(\sum_{x,y} \psi_x A_{xy} \bar{\psi}_y \right)^k. \tag{6.112}$$

Given any differential form ω, we denote by $\omega|_k$ its degree k part.

Proposition 6.27.

$$\mathrm{e}^{-S}\big|_{2N} = (\det A) \mathrm{e}^{-(\phi, A\bar{\phi})} \prod_{i=1}^{N} \frac{d\bar{\phi}_x d\phi_x}{2\pi\mathrm{i}} = (\det A) \mathrm{e}^{-(\phi, A\bar{\phi})} \prod_{i=1}^{N} \frac{du_x dv_x}{\pi} \tag{6.113}$$

The right-hand side is the (normalized!) complex Gaussian measure, with covariance $C = A^{-1}$.

Proof. The right-hand side is

$$(\det A)\mathrm{e}^{-(\phi,A\bar{\phi})}\prod_{i=1}^{N}\frac{d\bar{\phi}_{x_i}d\phi_{x_i}}{2\pi\mathrm{i}} = \sum_{\sigma}(-1)^{\sigma}A_{x_1\sigma(x_1)}\cdots A_{x_N\sigma(x_N)}\bar{\psi}_{x_1}\psi_{x_1}\cdots\bar{\psi}_{x_N}\psi_{x_N}$$
$$= \sum_{\sigma}A_{x_1\sigma(x_1)}\cdots A_{x_N\sigma(x_N)}\bar{\psi}_{x_1}\psi_{\sigma(x_1)}\cdots\bar{\psi}_{x_N}\psi_{\sigma(x_N)}$$
$$= \sum_{y_1,\dots,y_N}A_{x_1y_1}\cdots A_{x_Ny_N}\bar{\psi}_{x_1}\psi_{y_1}\cdots\bar{\psi}_{x_N}\psi_{y_N}$$
$$= \frac{1}{N!}\sum_{x_1,\dots,x_N}\sum_{y_1,\dots,y_N}A_{x_1y_1}\cdots A_{x_Ny_N}\bar{\psi}_{x_1}\psi_{y_1}\cdots\bar{\psi}_{x_N}\psi_{y_N}$$
$$= \frac{1}{N!}\left(-\sum_{x,y}A_{xy}\psi_{x_1}\bar{\psi}_{y_1}\right)^N. \tag{6.114}$$

This is the same as the left-hand side. □

Exercise 6.28.

$$\int \phi_{x_1}\bar{\phi}_{y_1}\cdots\phi_{x_k}\bar{\phi}_{x_k} = \mathrm{per}(C_{x_iy_i})_{i=1}^k \tag{6.115}$$
$$\int \psi_{x_1}\bar{\psi}_{y_1}\cdots\psi_{x_k}\bar{\psi}_{x_k} = \det(C_{x_iy_i})_{i=1}^k \tag{6.116}$$

where the determinant and permanent of a $k\times k$ matrix M are

$$\det(M) = \sum_{\sigma\in S_k}(-1)^{\sigma}M_{i\sigma(i)}, \quad \mathrm{per}(M) = \sum_{\sigma\in S_k}M_{i\sigma(i)}. \tag{6.117}$$

Exercise 6.29. Understand in which sense ϕ has the interpretation of a field of free *bosons*, while ψ has the interpretation of the field of free *fermions*. To this end, let $\mathcal{H}$ be the finite-dimensional complex Hilbert space with inner product C, i.e.,

$$(f,g)_{\mathcal{H}} = \sum_{x,y}C_{xy}f_x\bar{g}_y. \tag{6.118}$$

Let $\mathcal{F} = \bigoplus_{k=0}^{\infty}\mathcal{H}^{\otimes k}$ be the Fock space generated by $\mathcal{H}$, where $\mathcal{H}^{\otimes 0}\cong\mathbb{C}$ is spanned by a vector Ω called the vacuum. Let $\mathcal{F}_s$ and $\mathcal{F}_a$ be the symmetric and antisymmetric subspaces of $\mathcal{F}$. Define the creation and annihilation operators a_x^* and a_x on $\mathcal{F}_s$ respectively $\mathcal{F}_a$ by

$$a_x\psi = \dots, \quad a_x^*\psi = (\dots). \tag{6.119}$$

Define the *field operators* by

$$\phi_x = a_x + \mathrm{i}a_x^*, \quad \dots \tag{6.120}$$

Show that

$$(\Omega, \phi_{x_1}\phi_{x_1}^*\cdots\phi_{x_k}\phi_{x_k}^*\Omega) = (\dots) \tag{6.121}$$

the relation of (6.115)–(6.116).

Next, we require some more notation.

Exterior derivative and interior product. For $\omega = f du_{x_1} \wedge \dots \in \Omega_k$, the *exterior derivative* is

$$d\omega = \sum_x \frac{\partial f}{\partial u_x} du_x \wedge (du_{x_1} \wedge \dots) + \sum_x \frac{\partial f}{\partial v_x} dv_x \wedge (du_{x_1} \wedge \dots). \tag{6.122}$$

It extends to Ω by linearity and has the properties

$$d^2 = 0, \quad d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta) \quad \text{if } \eta \in \Omega_k. \tag{6.123}$$

Given a vector field $X : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ and $\omega \in \Omega_k$, the interior product $\iota_X \omega \in \Omega_{k-1}$ is defined by (...). In particular, for the vector field (...), we have

$$\iota_X d\phi_x = -2\pi \mathrm{i} \phi_x, \quad \iota d\bar{\phi}_x = 2\pi \mathrm{i} \bar{\phi}_x. \tag{6.124}$$

Definition 6.30. *The supersymmetry generator is*

$$Q = d + \iota, \tag{6.125}$$

and a form $\omega \in \Omega$ is supersymmetric *if $Q\omega = 0$.*

Proposition 6.31. *The forms $\phi_x \bar{\phi}_y + \psi_x \bar{\psi}_y$ are supersymmetric and*

$$\phi_x \bar{\phi}_y + \psi_x \bar{\psi}_y \in \operatorname{im} Q. \tag{6.126}$$

Proof. Indeed,

$$Q(\phi_x \bar{\phi}_y + \psi_x \bar{\psi}_y) = d(\phi_x \bar{\phi}_y) + \iota(\psi_x \bar{\psi}_y) = (d\phi_x)\bar{\phi}_y + \phi_x (d\bar{\phi}_y) - \phi_x (d\bar{\phi}_y) - (d\phi_x)\bar{\phi}_y = 0, \tag{6.127}$$

showing the supersymmetry, and

$$Q(\phi_x d\bar{\phi}_y) = (d + \iota)(\phi_x + d\bar{\phi}_y) = d\phi_x d\phi_y + (2\pi \mathrm{i})\phi_x \bar{\phi}_y \tag{6.128}$$

implies that $\phi_x \bar{\phi}_y + \psi_x \bar{\psi}_y \in \operatorname{im} Q$. □

Definition 6.32. *For even forms $\omega_1, \dots, \omega_p$ and $F \in C^\infty(\mathbb{R}^p)$, define $F(\omega_1, \dots, \omega_p)$ by Taylor expansion about the 0-degree parts of the forms ω_i. More precisely, set*

$$F(\omega_1, \dots, \omega_p) = \sum_{k_1, \dots, k_p} \frac{\partial^{k_1}}{\partial \omega_1^{k_1}} \cdots \frac{\partial^{k_1}}{\partial \omega_1^{k_1}} F(\omega_1|_0, \dots, \omega_p|_0) \frac{(\omega_1 - \omega_1|_0)^{k_1}}{k_1!} \cdots \frac{(\omega_p - \omega_p|_0)^{k_p}}{k_p!} \in \Omega, \tag{6.129}$$

where the sum can be restricted to $k_1 + \dots + k_p \leqslant N$.

Exercise 6.33.

$$QF(\omega_1, \dots \omega_p) = \sum_{i=1}^p \frac{\partial}{\partial \omega_i} F(\omega_1, \dots, \omega_p) Q\omega_i. \tag{6.130}$$

Thus supersymmetric fields can be generated as functions of supersymmetric forms. We apply this in particular applied to the forms (the "square" of (ϕ, ψ))

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x. \tag{6.131}$$

It is not a coincidence that we use the same letter τ as in (6.26). The apparently missing factor $\frac{1}{2}$ is due to normalization conventions for complex Gaussian measures. For example, in terms of the two-component field (u, v), we have $\bar{\phi}\phi = \frac{1}{2}(u^2 + v^2)$.

Proposition 6.34 (Localization Lemma)**.** *Let S be supersymmetric. Then for any $F \in C^\infty(\mathbb{R}^N)$,*

$$\int F(\tau)\mathrm{e}^{-S} = F(0) \int \mathrm{e}^{-S}. \tag{6.132}$$

Thus, for functions of the square of the super-Gaussian field, the saddle point approximation is *exact.*

Proof. It suffices to show that the following derivative vanishes:

$$\begin{aligned}\frac{\partial}{\partial\lambda} \int F(\lambda\tau)\mathrm{e}^{-S} &= \sum_x \int F_x(\lambda\tau)\tau_x \mathrm{e}^{-S} \\ &= \sum_x \int Q(F_x(\lambda\tau)\omega_x \mathrm{e}^{-S}), \quad \text{where } \tau_x = Q\omega_x,\end{aligned} \tag{6.133}$$

and the last equality follows from $Q(\mathrm{e}^{-S}) = 0$ and $Q(F_x(\lambda\tau)) = 0$. For any form ω (with sufficient decay), we have by the definition of $\int$ and Stokes' Theorem

$$\int Q\omega = \int d\omega = 0, \tag{6.134}$$

and the claim follows. □

Example 6.35. Let S be supersymmetric with $\int \mathrm{e}^{-S} = 1$. Then

$$0 = \int (\phi_x\bar\phi_y + \psi_x\bar\psi_y)\mathrm{e}^{-S} \quad \text{implies} \quad \int \bar\psi_y\psi_x\mathrm{e}^{-S} = \int \bar\phi_y\phi_x\mathrm{e}^{-S}. \tag{6.135}$$

The following is a supersymmetric version of the BFS representation (6.27).

Proposition 6.36. *For $g \in C^\infty(\mathbb{R}^N)$ with at most exponential growth (say),*

$$\int g(\tau)\phi_x\bar\phi_y\mathrm{e}^{-S} = \int_0^\infty \mathbb{E}_x\left(g(L^T)\mathrm{e}^{-(A-Q,L^T)}1_{X_T=y}\right) dT. \tag{6.136}$$

The crucial difference to (6.27) is that the right-hand side only involves L^T not $L^T+\tau$. The proposition has the interpretation that the distribution of the local time L^∞ of the killed walk is the same as that of the square of the supersymmetric Gaussian field associated to the walk.

Proof. The proof is analogous to that of (6.27). It again suffices to check the claim for $g(t) = \mathrm{e}^{-(b,t)}$. Then the left- and right-hand sides of (6.136) both become

$$(A+b)^{-1}_{xy}, \tag{6.137}$$

now without further normalization (in contrast to the proof of (6.27)). For the left-hand side, this follows from (6.115), and for the right-hand directly from the integrated Feynman-Kac formula. □

The supersymmetric version of the $|\varphi|^4$ model is the weakly self-avoiding walk (or random polymer). Indeed, application of (6.136) with $g(t) = \mathrm{e}^{-\sum_x(gt_x^2+\nu t_x)}$ yields the following relation. Let

$$I(T) = \int_0^T \int_0^T 1_{X_{S_1}=X_{S_2}}\, dS_1\, dS_2 = \sum_x (L_x^T)^2 \tag{6.138}$$

be the *self-intersection local time* of the random walk X (up to time T).

Corollary 6.37 (Parisi–Sourlas [37], McKane [33], Brydges–Evans–Imbrie [9])**.**

$$\int \phi_x \bar{\phi}_y \mathrm{e}^{-S_{g,\nu}} = \int_0^\infty \mathbb{E}_x \left(\mathrm{e}^{-gI(T)} 1_{X_T = y} \right) \mathrm{e}^{-\nu T} \, dT, \tag{6.139}$$

where

$$S_{g,\nu} = \frac{1}{2} \sum_x (\phi_x(-\Delta\bar{\phi})_x + \psi_x(-\Delta\bar{\psi})_x) + \sum_x \left(\frac{1}{4} g(\phi_x\bar{\phi}_x + \psi_x\bar{\psi}_x)^2 + \frac{1}{2}\nu(\phi_x\bar{\phi}_x + \psi_x\bar{\psi}_x) \right). \tag{6.140}$$

Analogous to the correspondence between $|\varphi|^4$ and $O(n)$ model, the right-hand side is up to constants the $|\varphi|^4$ version of the generating function for self-avoiding walks

$$\sum_{n=0}^\infty |\{\text{self-avoiding walks from } x \text{ to } y \text{ of length } n\}| z^n. \tag{6.141}$$

Indeed, the generating function for self-avoiding walks is a limit of that of weakly self-avoiding walks with $g \to \infty$; see [13]. (This is also called the "point-to-point" partition function.)

There are other, perhaps more transparent, ways to understand that supersymmetry is intimately connected to self-avoiding walks. To explain this, note that the Gaussian integration by parts formula extends to the supersymmetric case.

Exercise 6.38. Let $S = S_A$ be as in (6.111). Then

$$\int \bar{\phi}_x F \mathrm{e}^{-S} = \sum_y C_{xy} \int \frac{\partial F}{\partial \phi_y} \mathrm{e}^{-S}. \tag{6.142}$$

Example 6.39 (SUSY and self-avoiding walk)**.** Write $\mathbb{E}(\cdot)$ for the $\int \mathrm{e}^{-S}(\cdot)$. Then

$$\mathbb{E}\left(\bar{\phi}_x\phi_y g(\bar{\phi}\phi)\right) = C_{xy}\mathbb{E}\left(g(\bar{\phi}\phi)\right) + \sum_{x_1} C_{xx_1}\mathbb{E}\left(\bar{\phi}_{x_1}\phi_y g^{(n)}(\bar{\phi}\phi)\right) \tag{6.143}$$

where $n_x = 1$ if $x = x_1$ and $n_x = 0$ otherwise and $g^{(n)}$ has n_x derivatives in the t_x variable. Iterating this identity, one obtains

$$\mathbb{E}\left(\bar{\phi}_x\phi_y g(\bar{\phi}\phi)\right) = \sum_{w:x\to y} C_{xx_1}\cdots C_{x_k y}\mathbb{E}\left(g^{(n)}(\bar{\phi}\phi)\right), \tag{6.144}$$

where n_x is the number of times the walk w visits site x. Thus the right-hand side is sum over walks with weight in terms of the vertices it visits. (Note that the weight involves $C = A^{-1}$ rather than A, as in (6.27); this has to do with the fact that it is now discrete rather than continuous.) The same can be done with $\tau_x = \bar{\phi}_x\phi_x$ replaced by $\tau_x = \bar{\phi}_x\phi_x + \bar{\psi}_x\psi_x$ inside g. Then, by (6.132),

$$\mathbb{E}\left(g^{(n)}(\tau_x)\right) = g^{(n)}(0). \tag{6.145}$$

In particular, for $g(t) = \prod_x (1 + t_x)$ the right-hand side becomes the indicator function that the walk w is self-avoiding:

$$\mathbb{E}\left(\bar{\phi}_x\phi_y g(\bar{\phi}\phi + \bar{\psi}\psi)\right) = \sum_{w:x\to y} C_w 1_{w \text{ is self-avoiding}}, \tag{6.146}$$

where C_w is the product $C_{xx_1}\cdots C_{x_k y}$ over the steps of the walk.

More details concerning the connection between supersymmetry and self-avoiding walks can be found in [16] and [9].

Example 6.40 (SUSY and random matrix theory)**.** Let H be a symmetric matrix and $z = E + \mathrm{i}\eta \in \mathbb{C}_+$. Then the matrix $\mathrm{i}(H-z) = \mathrm{i}(H-E)+\eta$ has positive definite real part. Let $G = (H-z)^{-1}$ be the Green's function. It has the supersymmetric representation

$$G_{ij} = \mathrm{i}\int \bar\phi_i \phi_j \mathrm{e}^{-\mathrm{i}(\phi,(H-z)\bar\phi)-\mathrm{i}(\psi,(H-z)\bar\psi)}. \tag{6.147}$$

Let's say that H has independent entries. Then the expectation of the Green's function is

$$\mathbb{E}G_{ij} = \mathrm{i}\int \bar\phi_i\phi_j \prod_{i<j}\mathbb{E}\left(\mathrm{e}^{-\mathrm{i}H_{ij}(2\phi_i\bar\phi_j+\psi_i\bar\psi_j+\psi_j\bar\psi_i)}\right)\prod_i \mathbb{E}\left(\mathrm{e}^{-\mathrm{i}H_{ii}(\phi_i\bar\phi_i+\psi_i\bar\psi_i)}\right)\mathrm{e}^{(\mathrm{i}E-\eta)(\phi_i\bar\phi_i+\psi_i\bar\psi_i)} \tag{6.148}$$

The expectations are particularly easy to calculate if the entries are centered Gaussians, say with $\mathrm{Var}(H_{ij}) = 1/N$ and $\mathrm{Var}(H_{ii}) = 2/N$. Then

$$\mathbb{E}G_{ij} = \mathrm{i}\int \bar\phi_i\phi_j \mathrm{e}^{-\frac{1}{2N}\left((\phi,\bar\phi)^2-(\psi,\bar\psi)^2-2(\phi,\bar\psi)(\bar\phi,\psi)\right)}\mathrm{e}^{(\mathrm{i}E-\eta)((\phi,\bar\phi)+(\psi,\bar\psi))}. \tag{6.149}$$

Formally dropping ψ (this is called the boson–boson contribution) and setting $E=0$ for the moment, the previous line resembles a correlation function of a spin model. For $i=j$, it is really the one-point function of a mean-field spin model, where $\sigma = |\phi|^2$ is the spin field and the action is (recall (1.21)):

$$\frac{N}{2}m^2 + N\eta m, \quad m = \frac{1}{N}(\phi,\bar\phi) = \frac{1}{N}\sum_i \sigma_i. \tag{6.150}$$

The single spin measure induced on σ is the radial marginal of the two-dimensional Lebesgue measure: $\mu(d\sigma) = \delta(|\phi|^2-\sigma)\, d\bar\phi\, d\phi \propto 1_{\sigma\geqslant 0}\,\sigma\, d\sigma$, the imaginary part η has the interpretation of magnetic field, and i times the average magnetization is the expected Stieltjes transform of the empirical measure. However, differently from ordinary spin models, since the single spin measure is not symmetric about 0, there is only one phase.

Still ignoring ψ, as a mean-field model, the asymptotic magnetization can be studied through a saddle point approximation. To obtain a suitable form, one can either do a Hubbard–Stratonovich transform, as in Section 1.4, or a direct change variables to m. Then there is a Jacobi factor proportional to m^{N-1}. Including m^N in the action, it becomes

$$N\left(\frac{1}{2}m^2 - \mathrm{i}zm - \log m\right). \tag{6.151}$$

Thus the stationary points obey

$$(\mathrm{i}m)^2 + z(\mathrm{i}m) + 1 = 0, \tag{6.152}$$

which is the self-consistent equation for the semicircle law. This only provides a glimpse of the connection of random matrices and supersymmetry. Ultimately, the fermionic variables ψ cannot be ignored, and provide crucial cancellations. For random matrix theory, the expected Green's function $\mathbb{E}G_{ij}$ only provides quite limited information. The analysis of $\mathbb{E}|G_{ij}|^2$ becomes significantly more difficult as the state space of the model becomes noncompact with hyperbolic symmetry. This is explained very well in [45] (see also [46]).

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