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1. **Wick formula.** Let $\langle\cdot\rangle$ be the expectation of a Gaussian field $(\varphi_x)_{x\in\Lambda}$. Show the Wick formula:

$$\langle \varphi_{x_1}\cdots\varphi_{x_{2k}}\rangle = \sum_\pi \prod_{i=1}^k \langle \varphi_{x_{\pi(2i-1)}}\varphi_{x_{\pi(2i)}}\rangle$$

where the sum runs over pairings π of $1,\dots,2k$.

2. **Newman inequality.** For a 1-component φ^4 model and the Ising model (on a finite set Λ with $h=0$) show

$$\langle \varphi_{x_1}\cdots\varphi_{x_{2k}}\rangle \leqslant \sum_\pi \prod_{i=1}^k \langle \varphi_{x_{\pi(2i-1)}}\varphi_{x_{\pi(2i)}}\rangle,$$

where π is as in the previous question.

3. **Intersections of simple random walks.** Let X^1 and X^2 be independent simple random walks on $\mathbb{Z}^d$ with initial condition $X_0^1 = X_0^2 = 0$. Show that the expected time that the two walks intersect each other satisfies

$$\mathbb{E}\left[\int_0^\infty\int_0^\infty 1_{X^1_{t_1}=X^2_{t_2}}\,dt_1\,dt_2\right]\begin{cases} =\infty & (d\leqslant 4)\\ <\infty & (d>4).\end{cases}$$

Show further that if $d>4$ and the two walks start at x and y, respectively, then the expected time the two walks intersect each other tends to 0 as $|x-y|\to\infty$.

4. **Edwards–Sokal coupling.** Complete the parts omitted in the proof of the Edwards–Sokal coupling in class and in the lecture notes.

5. **Percolation, $d=1$.** For percolation $\mathbb{P}_p$ on $\mathbb{Z}$ compute $\mathbb{P}_p(0\leftrightarrow x)$. Compute the analogous probability for the random cluster model with $q>0$.

6. **Percolation, high temperature.** For percolation $\mathbb{P}_p$ on $\mathbb{Z}^d$, $d\geqslant 1$, show that there is $p_0>0$ such that $\mathbb{P}_p(0\leftrightarrow x)\leqslant Ce^{-c|x|}$ for $p<p_0$.

7. **Percolation, low temperature.** For percolation $\mathbb{P}_p$ on $\mathbb{Z}^d$, $d\geqslant 2$, use a Peierls' argument to show that there is $p_1<1$ such that $\mathbb{P}_p(0\leftrightarrow x)\geqslant c>0$ for $p>p_1$.

8. **Random current representation.** Let $A\subset\Lambda$ and $n: E\to\mathbb{N}_0$. Show that

$$2^{-|\Lambda|}\sum_{\sigma\in\{\pm1\}^\Lambda}\prod_{x\in A}\sigma_x\prod_{xy\in E}(\sigma_x\sigma_y)^{n_{xy}} = 1_{\partial n=A}$$

where $(d^*n)_x = \sum_{y:y\sim x} n_{xy}$ and $\partial n\subset\Lambda$ is the set of vertices such that $(d^*n)_x$ is odd. The $n: E\to\mathbb{N}_0$ are called *currents* and ∂n its sources. Show that the Ising partition function (with $h=0$) can be written as

$$Z_\beta = 2^{-|\Lambda|}\sum_{\sigma\in\{\pm1\}^\Lambda} e^{\beta\sum_{xy}\sigma_x\sigma_y} = \sum_{n:E\to\mathbb{N}_0:\partial n=\emptyset} W_\beta(n),\qquad W_\beta(n)=\prod_{xy\in E}\frac{\beta^{n_{xy}}}{n_{xy}!}.$$

9. **Switching lemma.** Let $x,y\in\Lambda$ and $A\subset\Lambda$. For any $F:\{\pm1\}^E\to\mathbb{R}$,

$$\sum_{\partial n^1=\{x,y\}}\sum_{\partial n^2=A} F(n^1+n^2)W_\beta(n^1)W_\beta(n^2) = \sum_{\partial n^1=\emptyset}\sum_{\partial n^2=A\Delta\{x,y\}} F(n^1+n^2)W_\beta(n^1)W_\beta(n^2)1_{x\leftrightarrow y \text{ in } n^1+n^2},$$

where the sums run over currents n^1 and n^2, Δ denotes the symmetric difference of sets, and $x\leftrightarrow y$ in n^1+n^2 means that there is a sequence of edges e connecting x and y with $n^1_e+n^2_e>0$.

10. **Two-point function and random currents.** By applying the switching lemma, show that the square of the two-point function of the Ising model can be expressed in terms of currents as

$$\langle \sigma_x \sigma_y \rangle^2 = \frac{\sum_{\partial n^1 = \emptyset} \sum_{\partial n^2 = \emptyset} W_\beta(n^1) W_\beta(n^2) 1_{x \leftrightarrow y \text{ in } n^1 + n^2}}{\sum_{\partial n^1 = \emptyset} \sum_{\partial n^2 = \emptyset} W_\beta(n^1) W_\beta(n^2)}.$$

11. **Simon–Lieb inequality from random currents.** Prove the Simon–Lieb inequality for the Ising model by using the random current representation.