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1. **Continuous-time simple random walk.** Given Λ finite and rates $J_{xy} = J_{yx} \geqslant 0$, define the continuous-time simple $X = (X_t)_{t\geqslant 0}$ random walk with initial condition $X_0 = x \in \Lambda$ as in class. Show that X has generator Δ_J, i.e., for any $f : \Lambda \to \mathbb{R}$,

$$\frac{\partial}{\partial t} f_t = \Delta_J f_t, \qquad f_t(x) = \mathbb{E}_x(f(X_t)). \tag{0.1}$$

2. **Local time of simple random walk.** Let $(L_x(t))_{x\in\Lambda}$ be the local time (or occupation time) of the continuous-time simple random walk X defined by $L_x(t) = \int_0^t 1_{X_s=x}\, ds$. Show that for any sufficiently nice function $f : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\frac{\partial}{\partial t} f_t(x,\ell) = \mathcal{L} f_t(x,\ell), \qquad f_t(x,\ell) = \mathbb{E}_x(f(X_t, \ell + L_t)) \tag{0.2}$$

where $\mathcal{L} f(x,\ell) = \Delta_J f(x,\ell) + \frac{\partial}{\partial \ell_x} f(x,\ell)$ and the discrete Laplacian Δ_J applies in the first argument of f.

3. **No transversal magnetisation.** Consider the $O(n)$ model (with free or periodic boundary conditions) on $\Lambda \subset \mathbb{Z}^d$. Let $e = (1,0,\dots,0)$ denote the direction of the external field h. Show that

$$\langle e' \cdot \sigma \rangle^\Lambda_{\beta,h} = 0, \qquad \text{for any } e' \in \mathbb{R}^n \text{ with } e \cdot e' = 0. \tag{0.3}$$

4. **Ward identity.** For the $O(n)$ model as in the previous question, show the Ward identity

$$\sum_{y\in\Lambda} \langle \sigma_x^2 \sigma_y^2 \rangle_{\beta,h} = \frac{\langle \sigma_x^1 \rangle_{\beta,h}}{h}. \tag{0.4}$$

[Hint: Only consider $n = 2$. The general case is the same but notationally more cumbersome. It is helpful to integrate by parts.]

5. **Rotationally invariant random vectors.** Let M be an $\mathbb{R}^n$-valued random variable whose distribution is rotationally invariant, i.e., for any $R \in SO(n)$, the distribution of RM is the same as that of M. Show that the distributions of $M/|M|$ and $|M|$ are independent. Here $|M|$ is the Euclidean norm of M.

6. **Tetrahedral representation of the Potts model.** The q-state Potts models is an analogue of the Ising model in which spins can take q values (with $q = 2$ corresponding to the Ising model). This definition amounts to the following definition of the measure of the Potts model: For $\theta \in \{1,\dots,q\}^\Lambda$,

$$\mathbb{P}_\beta(\theta) \propto e^{\beta \sum_{xy} 1_{\theta_x=\theta_y}}.$$

Show that there are q vectors $v_1 \in \mathbb{R}^{q-1}, \dots, v_q \in \mathbb{R}^{q-1}$ with the property that

$$v_i \cdot v_j = \begin{cases} q-1 & (i = j) \\ -1 & (i \neq j). \end{cases}$$

The set of these q points forms a tretrahedron T_q. *[Hint: Use induction in q.]*

The configurations $\theta \in \{1\dots,q\}^\Lambda$ can thus be identified with spin configurations $\sigma \in (T_q)^\Lambda \subset (\mathbb{R}^n)^\Lambda$. Deduce that the q-state Potts model is reflection positive.

7. **Reflection positivity through sites.** Let Λ be a discrete torus with an odd number of vertices along every coordinate direction, and let $P \subset \Lambda$ be a plane of vertices (as opposed to edges considered in class) so that $\Lambda = \Lambda_+ \cup P \cup \Lambda_-$. The corresponding reflection $\theta : \Lambda_\pm \to \Lambda_\mp$ now leaves P invariant. Show that any product measure $\mu^{\otimes\Lambda}$ is reflection positive for this reflection.

As a consequence, show that the Ising model is reflection positive also with respect to planes of vertices.

8. **Reflection positivity of the hard-core model.** Let Λ be a discrete torus as in the previous question. Configurations of the hard-core model are $n = (n_x)_{x\in\Lambda}$ with $n_x \in \{0, 1\}$ for each $x \in \Lambda$ with the interpretation that there is a particle at $x \in \Lambda$ if $n_x = 1$ and $n_x = 0$ otherwise. In the hard-core model, two particles are not permitted to occupy neighbouring sites, i.e., the admissible configuration n obey the constraint $n_x n_y = 0$ if $xy \in E$. For $z > 0$ (called the activity), the probability of such a configuration n is

$$\mathbb{P}_z(n) = \frac{1}{Z_z^\Lambda} z^N, \qquad N = \sum_{x\in\Lambda} n_x. \tag{0.5}$$

Show that the hard-core model is reflection positive through planes of vertices (and analogously it is also reflection positive through planes of edges).

[Hint: one can approximate the hard-core model as the limit of an Ising model with inverse temperature going to $-\infty$ (antiferromagnetic).]