Roland Bauerschmidt ⟨rb812@cam.ac.uk⟩ *Michaelmas 2022*

1. **Connected graphs.** Given a finite connected graph G and a vertex x in G, show that there exists a path in G starting from x that crosses every edge exactly twice. *[Hint: Induction on the number of vertices of the graph.]* Deduce that the number of connected subgraphs of G with n edges that contain x is bounded by D^{2n} where D is the maximal degree of G.

2. **1d Ising model.** Use the high temperature expansion to show that when $\Lambda = \{1, \dots, L\} \subset \mathbb{Z}$ then $\langle \sigma_x \sigma_y \rangle_{\beta,0}^{\Lambda} = (\tanh \beta)^{|x-y|}$ for $x, y \in \Lambda$. Thus the two-point function decays exponentially for all $\beta > 0$.

3. **Exponential lower bound.** Show that the two-point function of the Ising model on $\Lambda \subset \mathbb{Z}^d$ cannot decay faster than exponentially.

4. **Boundary conditions.** For $\Lambda \subset \mathbb{Z}^d$, let $\bar{\Lambda} \subset \mathbb{Z}^d$ consist of all vertices in Λ or neighbouring a vertex in Λ. Spin configurations with boundary conditions ξ are $\sigma : \bar{\Lambda} \to \{\pm 1\}$ with $\sigma_x = \xi_x$ for $x \in \bar{\Lambda} \setminus \Lambda$. Let $\bar{E}$ consist of the neighest-neighbour edges of $\mathbb{Z}^d$ with both vertices contained in $\bar{\Lambda}$. The Ising model on Λ with boundary condition ξ assigns the following probability to such configurations:

$$\mathbb{P}_{\beta,h}^{\Lambda,\xi}(\sigma) = \frac{1}{Z_{\beta,h}^{\Lambda,\xi}} e^{\beta \sum_{xy \in \bar{E}} \sigma_x \sigma_y + h \sum_{x \in \Lambda} \sigma_x}. \tag{0.1}$$

Plus boundary conditions are the choice $\xi_x = +1$ for all x and we then write + instead of ξ (and analogously for minus boundary conditions).

By adapting the high temperature expansion to + boundary conditions, show $\langle \sigma_0 \rangle_{\beta,0}^{\Lambda_L,+} \to 0$ as $L \to \infty$ for β small enough.

By adapting the Peierls argument to + boundary conditions, show that $\limsup_{L \to \infty} \langle \sigma_0 \rangle_{\beta,0}^{\Lambda_L,+} > 0$ for β large enough.

5. (*) **Peierls argument for integer-valued height functions.** Let $\Lambda = \Lambda_L \subset \mathbb{Z}^d$ be a hypercube of side length L centered at 0, and consider integer-valued spin configurations $\sigma : \Lambda_L \to \mathbb{Z}$ with $\sigma_0 = 0$ and $\sigma_x - \sigma_y \in \{-1, 0, 1\}$ for all $x \sim y$. The probability of a spin configuration is given by

$$\mathbb{P}_{\beta}^{\Lambda}(\sigma) = \frac{1}{Z_{\beta}^{\Lambda}} e^{-\frac{\beta}{2} \sum_{xy \in E} |\sigma_x - \sigma_y|}. \tag{0.2}$$

Show that there is $\beta_1 > 0$ such that if $\beta > \beta_1$, then for all $x \in \mathbb{Z}^d$ fixed and L sufficiently large, $\langle \sigma_x^2 \rangle_{\beta}^{\Lambda_L} \leqslant C$.

6. **Gaussian measure and Hubbard–Stratonovich transform.** Let H be a real symmetric (strictly) positive definite $n \times n$ matrix. Show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(\varphi, H\varphi)} \, d\varphi = \sqrt{\frac{(2\pi)^n}{\det H}}, \tag{G}$$

where $(\varphi, \varphi') = \sum_{x=1}^n \varphi_x \varphi'_x$ is the standard inner product on $\mathbb{R}^n$. Derive the Laplace transform of the corresponding normalised measure, also known as Hubbard–Stratonovich transform in statistical physics:

$$\sqrt{\frac{\det H}{(2\pi)^n}} \int_{\mathbb{R}^n} e^{(f,\varphi)} e^{-\frac{1}{2}(\varphi, H\varphi)} \, d\varphi = e^{+\frac{1}{2}(f, H^{-1} f)}. \tag{0.3}$$

7. **Laplace's Principle.** Let $S : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded below, assume that $\{\varphi \in \mathbb{R}^n : S(\varphi) \leqslant \min S + 1\}$ is compact, and that $\int_{\mathbb{R}^n} e^{-S(\varphi)} \, d\varphi < \infty$.

Let φ_0 be a minimum of S, i.e., $S(\varphi_0) = \min S$. Show that for any $f : \mathbb{R}^n \to \mathbb{R}$ bounded continuous with $f(\varphi_0) \neq 0$,

$$\int f(\varphi) e^{-tS(\varphi)} \, d\varphi = e^{-t \min S + o(t)} \qquad (t \to \infty). \tag{0.4}$$

Now assume S is twice continuously differenable and has a unique minimum φ_0. Show that then for any $f : \mathbb{R}^n \to \mathbb{R}$ bounded continuous with $f(\varphi_0) \neq 0$,

$$\int f(\varphi) e^{-tS(\varphi)}\, d\varphi = \sqrt{\frac{(2\pi)^n}{\det(\text{Hess}\, tS(\varphi_0))}} f(\varphi_0) e^{-tS(\varphi_0)} (1 + o(1)) \qquad (t \to \infty), \tag{0.5}$$

where $\text{Hess}\, S = (\frac{\partial S}{\partial \varphi_i \partial \varphi_j})_{i,j=1}^n$ is the Hessian matrix of S.

8. **Curie–Weiss model.** Consider the Ising model on the complete graph (also known as Curie–Weiss model). The complete graph has vertices $[N] = \{1, \dots, N\}$ and edges between all pairs of distinct vertices. The coupling constants of the Ising model are given by $J_{xy} = \beta/N$ for all $x \neq y$, i.e., for $\sigma \in \{\pm 1\}^N$,

$$\mathbb{P}^N_{\beta,h}(\sigma) \propto e^{-H_N(\sigma)}, \qquad H_N(\sigma) = -\sum_{xy \in E} J_{xy} \sigma_x \sigma_y - h \sum_x \sigma_x. \tag{0.6}$$

Observe that H_N is a function of the *mean spin (or mean field)* $m = \frac{1}{N} \sum_x \sigma_x$:

$$H_N(\sigma) = -\frac{\beta}{2} N m^2 - N h m. \tag{0.7}$$

Determine the set M_N of possible values that m can assume when $\sigma \in \{\pm 1\}^N$ and their multiplicities. Using these determine an explicit function $f_{\beta,h} : [-1, 1] \to \mathbb{R}$ such that

$$\sum_{\sigma \in \{\pm 1\}^N} e^{-H_N(\sigma)} = \sum_{m \in M_N} e^{-N f_{\beta,h}(m) + o(N)}. \tag{0.8}$$

[Hint: Stiring's formula gives $\log n! = n(\log n - 1) + o(n)$.] Show that the local minima m of $f_{\beta,h}$ satisfy

$$m = \tanh(\beta m + h). \tag{0.9}$$

Find the largest $\beta_c > 0$ such that $f_{\beta,h}$ has a unique minimum for all $\beta < \beta_c$ and all $h \in \mathbb{R}$. What happens for $\beta > \beta_c$?

9. **First Griffiths inequality for $O(n)$ model.** The $O(n)$ model is an analogue of the Ising model in which spins take values in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, i.e., its expectation is given by

$$\langle F(\sigma) \rangle^\Lambda_{\beta,h} = \int_{(\mathbb{S}^{n-1})^\Lambda} F(\sigma) e^{\beta \sum_{xy \in E} \sigma_x \cdot \sigma_y + h \sum_{x \in \Lambda} \sigma_x \cdot e} \prod_{x \in \Lambda} d\sigma_x \tag{0.10}$$

where is the uniform measure on $\mathbb{S}^{n-1}$ and e is the unit vector in the last (or any fixed) coordinate direction. Show that if $\beta \geqslant 0$ and $h \geqslant 0$ then

$$\langle \prod_{k=1}^K \sigma_{x_k}^{a_k} \rangle \geqslant 0 \tag{0.11}$$

for any $a_1, \dots, a_K \in \{1, \dots, n\}$ and any $x_1, \dots, x_K \in \Lambda$, where σ_x^a is the a-th component, $a \in \{1, \dots, n\}$, of the spin at $x \in \Lambda$.

10. **Domination by Ising model.** For μ an even Borel measure on $\mathbb{R}$, consider the generalised Ising model with single spin measure μ:

$$\langle F \rangle_{\mu,J} \propto \int F(\sigma)\, e^{\sum_{xy} J_{xy} \sigma_x \sigma_y} \prod_{x \in \Lambda} \mu(d\sigma_x).$$

Prove that if μ has support in $[-1, 1]$ then

$$\langle \sigma_A \rangle_{\mu,J} \leqslant \langle \sigma_A \rangle_{\text{Ising},J}$$

where $\sigma_A = \prod_{x \in A} \sigma_x$. *[Hint: Use Griffith's inequalities.]*

11. **Exponential decay.** Let Λ be a finite set, and let $J_{xy} = J_{yx} \geqslant 0$ for $x, y \in \Lambda$ be symmetric positive weights. Assume that $f : \Lambda \to \mathbb{R}_+$ satisfies, for $x \neq o$ where o is a fixed point in Λ,

$$f(x) \leqslant \sum_{y \in \Lambda} J_{xy} f(y),$$

and that there is $\mu > 0$ and a metric d on Λ such that

$$\sup_{x \in \Lambda} \sum_{y \in \Lambda} J_{xy} e^{\mu d(x,y)} \leqslant 1.$$

Show that then $f(x) \leqslant f(o) e^{-\mu d(o,x)}$. *[Hint: A probabilistic approach to this problem is to consider the discrete-time simple random walk with transition probabilities $p_{xy} = J_{xy} e^{\mu d(x,y) + \alpha_x}$ where the $\alpha_x \geqslant 0$ are defined to ensure $\sum_y p_{xy} = 1$, and use that $f(X_n) e^{-\mu d(X_0, X_n)}$ is a submartingale.]*

12. **Mean-field bound on critical temperature.** Show that the two-point function of the Ising model satisfies, for any $a \neq b$,

$$\langle \sigma_a \sigma_b \rangle^{\Lambda}_{\beta,0} \leqslant \sum_{x \sim a} \beta \langle \sigma_x \sigma_b \rangle^{\Lambda}_{\beta,0}. \tag{0.12}$$

Using the previous exercise deduce that $\langle \sigma_a \sigma_b \rangle^{\Lambda}_{\beta,0}$ decays exponentially when $\beta < 1/(2d)$ and $\Lambda \subset \mathbb{Z}^d$, and therefore that $\beta_c(d) \geqslant 1/(2d)$. Compare this with the bound from the high temperature expansion. *[Hint: Consider the Ising model with temperature β replaced by an edge-dependent temperature J_{xy} given by $J_{xy} = \beta$ for all edges xy not containing a and $J_{xy} = \lambda\beta$, $\lambda \in [0, 1]$, when xy contains a. What happens when $\lambda = 0$?]*