Linear Analysis (Michaelmas 2018)

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Primary references:
I. Leader, Linear Analysis, Lecture Notes
M. Dafermos, Linear Analysis, Lecture Notes
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B. Bollobás, Linear Analysis, Cambridge University Press

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1. Normed spaces and linear operators

Unless stated, vector spaces will be real or complex, and $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

1.1. Normed vector spaces

Defn. A normed vector space $(X, \|\cdot\|)$ is a vector space X with a norm $\|\cdot\| : X \to \mathbb{R}$, $x \mapsto \|x\|$ satisfying

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$ (pos.-def.);

(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in X$ (pos. homogeneity);

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle ineq.).

In particular, a metric is defined by $d(v,w) = \|v-w\|$.

Fact. The vector space operations and the norm are continuous maps

$$\mathbb{K} \times X \to X \quad (\lambda, x) \mapsto \lambda x$$
$$X \times X \to X \quad (x,y) \mapsto x+y$$
$$X \to \mathbb{R}, \quad x \mapsto \|x\|$$

and the metric is translation invariant:

$$d(x,y) = d(x+z, y+z) \quad \forall z \in X.$$

Proof. (only scalar multiplication). Since $\mathbb{K}$ and X are metric spaces, it suffices to check that $\lambda_j \to \lambda$ and $x_j \to x$ implies $\lambda_j x_j \to \lambda x$.

Indeed,

$$\|\lambda_j x_j - \lambda x\| = \|(\lambda_j - \lambda) x_j + \lambda (x_j - x)\|$$

$$\le \underbrace{|\lambda_j - \lambda|}_{\to 0} \; \underbrace{\|x_j\|}_{\substack{\text{bd. since} \\ x_j \to x}} + |\lambda| \, \underbrace{\|x_j - x\|}_{\to 0} \to 0$$

$\Rightarrow \|\lambda_j x_j - \lambda x\| \to 0$, i.e. $\lambda_j x_j \to \lambda x$.

Examples:

(a) $\ell_n^2 = (\mathbb{R}^n, \|\cdot\|_2)$ where $\|x\|_2 = \left(\sum_i |x_i|^2\right)^{1/2}$ (Euclidean norm)

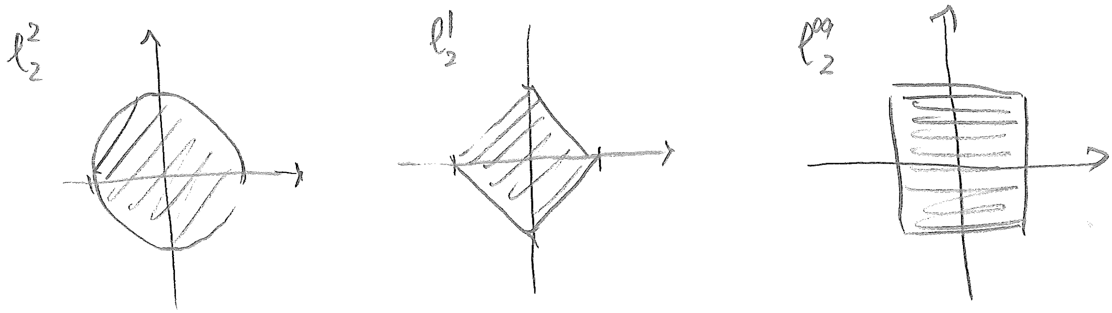
triangle ineq. follows from Cauchy-Schwarz inequality

(b) $\ell_n^1 = (\mathbb{R}^n, \|\cdot\|_1)$ where $\|x\| = \sum_i |x_i|$

triangle ineq. follows from $|x_i + y_i| \le |x_i| + |y_i|$ for all i.

(c) $\ell_n^\infty = (\mathbb{R}^n, \|\cdot\|_\infty)$ where $\|x\|_\infty = \max_i |x_i|$.

It often helps to look at the unit ball $B = B(X) = \{x \in X : \|x\| \le 1\}$.



Fact:

- B determines the norm through $\|x\| = \inf\{t > 0 : x \in tB\}$.
- B is convex: $x, y \in B$, $\lambda \in (0,1) \Rightarrow \lambda x + (1-\lambda) y \in B$.

Remark. Any set $B \subseteq \mathbb{R}^n$ which is closed, bounded, convex, and symmetric ($x \in B \Rightarrow -x \in B$), and a neighbourhood of 0 defines a norm by $\|x\| = \inf\{t > 0 : x \in tB\}$, and B is the unit ball of that norm.

1.2. The spaces ℓ^p

Let

$S = \{(x_i)_{i=1}^{\infty} \subset \mathbb{K}\}$ be the set of scalar sequences,

with

$$(x_i) + (y_i) = (x_i + y_i), \quad \lambda(x_i) = (\lambda x_i).$$

Defn. For $1 \leq p < \infty$,

$\ell^p = \{x \in S : \sum |x_n|^p < \infty\}$ with norm $\|x\|_p = \left(\sum |x_n|^p\right)^{1/p}$

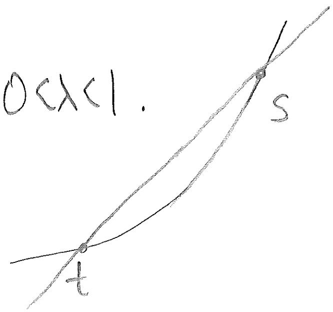
$\ell^\infty = \{x \in S : \sup |x_n| < \infty\}$ with norm $\|x\|_\infty = \sup_n |x_n|$

$c_0 = \{x \in S : x_n \to 0\}$ with norm $\|x\|_\infty = \sup_n |x_n|$.

That $\|x\|_p$ satisfies the triangle inequality follows from Minkowski's inequality, which we will discuss next.

Recall that: • $f: \mathbb{R}^+ \to \mathbb{R}$ is <u>convex</u> if

$$f(\lambda t + (1-\lambda)s) \le \lambda f(t) + (1-\lambda) f(s) \quad \forall s,t \in \mathbb{R}^+,\ 0<\lambda<1.$$



• f is <u>concave</u> if $-f$ is convex.

• $\log$ is a concave function $\mathbb{R}^+ \to \mathbb{R}$.

<u>Cor.</u> Let $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\frac{1}{p}|x|^p + \frac{1}{q}|y|^q \ge |x|\cdot|y|, \quad \forall x,y \in K.$$

<u>Proof.</u> Set $t=|x|^p$, $s=|y|^q$, $\lambda=\frac{1}{p}$. Then

$$\frac{1}{p}|x|^p + \frac{1}{q}|y|^q \ge |x|\cdot|y| \qquad (*)$$

$$\Leftrightarrow \lambda t + (1-\lambda)s \ge t^\lambda s^{1-\lambda}$$

$$\Leftrightarrow \log(\lambda t + (1-\lambda)s) \ge \lambda \log t + (1-\lambda)\log s$$

which holds since $\log$ is concave

<u>Thm.</u> (Hölder's inequality). Let $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $x \in \ell^p$ and $y \in \ell^q$. Then $xy = (x_n y_n) \in \ell^1$ and

$$\|xy\|_1 \le \|x\|_p \|y\|_q.$$

<u>Proof.</u> It suffices to assume that $\|x\|_p = 1 = \|y\|_q$. Then

$$\sum_{n=1}^N |x_n y_n| \le \frac{1}{p}\sum_{n=1}^N |x_n|^p + \frac{1}{q}\sum_{n=1}^N |y_n|^q \quad \text{by } (*)$$

Take $N\to\infty$: $\|xy\|_1 \le \frac{1}{p}+\frac{1}{q} = 1 = \|x\|_p \|y\|_q$.

Thm (Minkowski's inequality). Let $1<p<\infty$ and let $x,y\in\ell^p$. Then $x+y\in\ell^p$ and

$$\|x+y\|_p \le \|x\|_p+\|y\|_p .$$

Proof.

$$\underbrace{\sum_{i=1}^{N}|x_i+y_i|^r}_{\to\ \|x\|_r^r} = \sum_{i=1}^{N}|x_i+y_i|^{r-1}\underbrace{|x_i+y_i|}_{\le |x_i|+|y_i|}$$

$$\le \sum_{i=1}^{N}|x_i+y_i|^{r-1}|x_i| + \sum_{i=1}^{N}|x_i+y_i|^{r-1}|y_i|$$

Hölder with $p=\frac{r}{r-1}$, $q=r$

$$\le \underbrace{\left(\sum_{i=1}^{N}|x_i+y_i|^r\right)^{\frac{r-1}{r}}}_{\to\ \|x+y\|_r^{r-1}}\underbrace{\left(\sum_{i=1}^{N}|x_i|^r\right)^{1/r}}_{\to\ \|x\|_r} + \left(\sum_{i=1}^{N}|x_i+y_i|^r\right)^{\frac{r-1}{r}}\underbrace{\left(\sum_{i=1}^{N}|y_i|^r\right)^{1/r}}_{\to\ \|x\|_r}$$

$$\Rightarrow \left(\sum_{i=1}^{N}|x_i+y_i|^r\right)^{1/r} \le \left(\sum_{i=1}^{N}|x_i|^r\right)^{1/r} + \left(\sum_{i=1}^{N}|y_i|^r\right)^{1/r}$$

$N\to\infty$

$$\Rightarrow \|x+y\|_r \le \|x\|_r+\|y\|_r .$$

1.3. Banach spaces

Defn. A normed vector space is a Banach space if it is complete as a metric space, i.e. every Cauchy sequence converges.

Exercise. For $1 \le p \le \infty$, the space ℓ^p is complete, so a Banach space.

Examples.

(i) Any finite-dimensional normed space is a Banach space.

(ii) Let S be a set and $B(S)$ be the vector space of bounded $\mathbb{K}$-valued functions on S. Then $B(X)$ is a Banach space with norm

$$\|f\|_\infty = \sup_{s \in S} |f(s)|.$$

(iii) Let K be a compact Hausdorff space (e.g. $K = [0,1]$), and let $C(K)$ be the space of continuous functions on K. Then

$C(K) \subset B(K)$: every continuous function on K is bounded.

$C(K)$ is closed in $B(K)$: the uniform limit of a sequence of continuous functions is continuous

Thus $C(K)$ is a Banach space with norm $\|\cdot\|_\infty$.

(iv) Let $U \subset \mathbb{R}^n$ be open, bounded, and let $C^k(\overline{U})$ be the space of functions $f : U \to \mathbb{K}$ such that $D^\alpha f$ is continuous and bounded on U, for $|\alpha| \leq k$, where

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \quad |\alpha| = \sum \alpha_i .$$

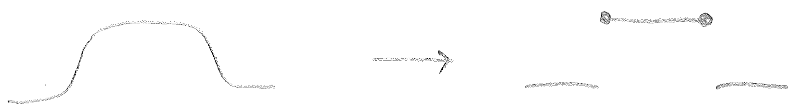
Then $C^k(\overline{U})$ is a Banach space with norm

$$\|f\|_{C^k(\overline{U})} = \max_{|\alpha| \leq k} \|D^\alpha f\|_\infty .$$

(v) Let $X = \{f : [0,1] \to \mathbb{R} \text{ continuous}\}$. Then, for $p \in [1, \infty)$,

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

is a norm on X. However, X is not complete in this norm!



$\notin X$

Its completion is L^p $\to$ measure theory.

(vi) Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $A(\overline{D})$ be the space of continuous functions $f : \overline{D} \to \mathbb{C}$ that are analytic on D. Then $A(\overline{D})$ is complete with $\|\cdot\|_\infty$ norm because a uniform limit of analytic functions is analytic.

Prop. Let X be a normed space and $Y \subset X$ a subspace.

(i) Y complete $\Rightarrow$ Y closed

(ii) X complete and Y closed $\Rightarrow$ Y complete

Proof. (i) Let $x \in \overline{Y}$. Then there is $(y_n) \subset Y$ s.t. $y_n \to x$.
So (y_n) is Cauchy. Thus $y_n \to y$ for some $y \in Y$ by completeness.
Thus $x = y$.

(ii) Suppose (y_n) is Cauchy in Y. Then (y_n) is Cauchy in X.
By completeness there is $x \in X$ s.t. $y_n \to x$.
Since Y is closed $x \in Y$.

Defn. A topological space is separable if it has a countable dense subset.

Exercise:
- For $1 \le p < \infty$, ℓ^p is separable.
- c_0 is separable.
- ℓ^∞ is not separable.

1.4. Bounded operators and the dual space

Thm. Let X, Y be normed spaces, $T: X \to Y$ linear. TFAE:

(i) T is continuous

(ii) T is continuous at 0

(iii) T is bounded, i.e., there is $C>0$ s.t. $\|Tx\| \le C\|x\|$ $\forall x \in X$.

Proof. (i) $\Rightarrow$ (ii) obvious

(ii) $\Rightarrow$ (iii) Since T is continuous at 0 and $\{y \in Y : \|y\| \le 1\}$ is a nbhd of $T(0)$, there is $\delta > 0$ s.t. $\|x\| \le \delta$ implies $\|Tx\| \le 1$.

For any $x \in X$, $x \neq 0$, by linearity thus

$$\|Tx\| = \frac{\|x\|}{\delta} \|T(\underbrace{\delta \tfrac{x}{\|x\|}}_{\|\cdot\| \le \delta})\| \le \frac{1}{\delta}\|x\|.$$

(iii) $\Rightarrow$ (i). Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{C}$. Then $\|x-y\| < \delta \Rightarrow \|T(x-y)\| \le \varepsilon$.

Thus T is (uniformly) continuous.

Defn. • For $T: X \to Y$ bounded, the operator norm is

$$\|T\| = \|T\|_{op} = \sup_{\|x\| \le 1} \|Tx\|.$$

$$(\text{thus } \|Tx\| \le \|T\| \, \|x\|)$$

• $B(X,Y) = \{T : X \to Y : T \text{ is linear and bounded}\}$

Fact: $B(X,Y)$ is a normed vector space with the operator norm.

Proof. Let $T, S \in B(X,Y)$. Then

$$\|(T+S)x\| = \|Tx+Sx\| \leq \|Tx\| + \|Sx\| \leq \big(\|T\| + \|S\|\big)\|x\|,$$

so $T+S \in B(X,Y)$ and $\|T+S\| \leq \|T\| + \|S\|$. The other properties are clear.

Examples. Let $p \in (1,\infty)$.

(i) Define $T: \ell^p \to \ell^p$ by $T(x_1, x_2, \dots) = (x_1, \dots, x_r, 0, 0, \dots)$ for some fixed r. Then $T \in B(\ell^p, \ell^p)$ with $\|T\| = 1$.

(ii) Define $T: \ell^p \to \ell^p$ by $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ [right shift]. Then $T \in B(\ell^p, \ell^p)$ with $\|T\| = 1$.
In fact, $\|Tx\| = \|x\|$ for all $x \in \ell^p$, i.e. T is an isometry, but clearly T is not surjective.

(iii) Define $S: \ell^p \to \ell^p$ by $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$ [left shift]. Then $S \in B(\ell^p, \ell^p)$ with $\|S\| = 1$.
Note that S is surjective, but not injective. $ST = id \neq TS$

(iv) Let $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Fix $y \in \ell^q$ and define $\phi_y: \ell^p \to \mathbb{R}$ by $\phi_y(x) = x \cdot y \overset{\text{def}}{=} \sum_n x_n y_n$.
By Hölder's inequality, this is well-defined and $\|\phi_y\| \leq \|y\|_q$.

(v) Let F be the space of finite sequences with $\|\cdot\|_1$-norm (real).
Define $T: F \to \mathbb{R}$ by $T(x_1, \dots, x_n, 0, 0, \dots) = \sum_{i=1}^{n} i x_i$.
Then T is not bounded (not continuous): $\|Te_n\| = n \to \infty$

$$e_n = (0, \dots, 0, \overset{(n)}{1}, 0, \dots, 0)$$

(vi) Define $T: \ell^1 \to \ell^2$ by $Tx = x$. Then $T \in B(\ell^1, \ell^2)$ with $\|T\| = 1$ (since $\sum |x_n| \leq 1 \Rightarrow \sum |x_n|^2 \leq 1$).
But $T\ell^1 \neq \ell^2$. Since $T\ell^1$ is also dense in ℓ^2, $T\ell^1$ is not closed in ℓ^2, thus not complete.

<u>Defn</u>. Let X, Y be normed spaces.

- An <u>isomorphism</u> from X to Y is a map $T: X \to Y$ that is a linear homeomorphism. Thus $T \in B(X,Y)$, $T^{-1} \in B(Y,X)$, i.e., there are C_1, C_2 s.t.

$$C_1 \|x\| \leq \|Tx\| \leq C_2 \|x\| \quad \text{for all } x \in X.$$

- A bijective linear map $T: X \to Y$ is an <u>isometric isomorphism</u> if $\|Tx\| = \|x\|$ for all $x \in X$.

<u>Defn</u>. Let X be a normed space. Its <u>dual space</u> is $X^* = B(X, \mathbb{K})$. A linear map $X \to \mathbb{K}$ is called a <u>functional</u>.

Thm. Let X, Y be normed spaces with Y complete. Then $B(X,Y)$ is complete. In particular, X^* is complete.

Proof. Let $(T_n) \subset B(X,Y)$ be a Cauchy sequence. Then for any $x \in X$, $(T_n x) \subset Y$ is a Cauchy sequence:

$$\|T_n x - T_m x\| \le \underbrace{\|T_n - T_m\|}_{< \frac{\varepsilon}{\|x\|} \text{ for } n,m \text{ large}} \|x\| < \varepsilon$$

Since Y is complete, there is $y \in Y$ s.t. $T_n x \to y$ $(n \to \infty)$.

Set $Tx = y$. Need to check that $T \in B(X,Y)$ and $\|T_n - T\| \to 0$.

- T is linear: $T(\lambda x + \mu y) = \lim_{n\to\infty} T_n(\lambda x + \mu y) = \lim_{n\to\infty} \lambda T_n x + \mu T_n y$
$$= \lambda Tx + \mu Ty$$

- T is bounded: for $\|x\| \le 1$, $\|Tx\| \le \underbrace{\|T_n x\|}_{\le \|T_n\|} + \underbrace{\|T_n x - Tx\|}_{< \varepsilon \text{ for } n > N}$
$$\le \|T_n\| + \varepsilon$$
$$\Rightarrow \|Tx\| \le \sup \|T_n\|$$

- $T_n \to T_n$ in norm: for $\|x\| \le 1$, $\|T_n x - Tx\| \le \underbrace{\|T_n x - T_m x\|}_{\le \|T_n - T_m\|} + \underbrace{\|T_m x - Tx\|}_{< \varepsilon \text{ for } m > M}$
$$\Rightarrow \|T_n - T\| \le \limsup_{m\to\infty} \|T_n - T_m\| \to 0 \quad (n \to \infty).$$

Example. Let $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Then the map

$$\phi : \ell^q \to (\ell^p)^*, \quad y \mapsto \phi_y \quad \text{where } \phi_y(x) = (x,y) := \sum_n x_n y_n, \quad \phi_y = (\cdot\,; y)$$

is an isometric isomorphism, i.e., '$(\ell^p)^* = \ell^q$'.

Proof. Clearly, ϕ is linear. We have already seen that $\|\phi_y\| \le \|y\|_q$

Claim: $\|\phi_y\| \ge \|y\|_q$.

Take $x_n = |y_n|^{q/p - 1}\bar{y}_n$. Then $\|x\|_p^p = \sum_n |y_n|^q = \|y\|_q^q < \infty$ so $x \in \ell^p$.

$$\Rightarrow \phi_y(x) = \sum_n |y_n|^{q/p+1} \underset{\uparrow}{=} \sum_n |y_n|^q = \|y\|_q^q = \|y\|_q \underbrace{\|y\|_q^{q-1}} = \|y\|_q \|x\|_p$$

$$\frac{q}{p}+1 = q\left(1-\frac{1}{q}\right)+1 = q \qquad \|x\|_p^{\frac{p}{q}(q-1)} = \|x\|_p^{p(1-\frac{1}{q})} = \|x\|_p^{p\cdot\frac{1}{p}} = \|x\|_p$$

$$\Rightarrow \|\phi_y\| \ge \|y\|_q.$$

Thus ϕ is an isometry. It remains to check that ϕ is surjective.

Let $T \in (\ell^p)^*$. Set $y_n = Te_n$ where $e_n = (0,\dots,0,\underset{\substack{\uparrow\\ n}}{1},0,\dots)$.

Claim: $y \in \ell^q$ and $\|y\|_q \le \|T\|$.

Define $x_n = \begin{cases} |y_n|^{q/p-1}\bar{y}_n & \text{if } n \le N \\ 0 & \text{if } n > N \end{cases}$ (N fixed).

$$\Rightarrow \|x\|_p = \left(\sum_{n=1}^N |y_n|^q\right)^{1/p}, \quad x \in \ell^p$$

$$Tx = \sum_{n=1}^N x_n Te_n = \sum_{n=1}^N x_n y_n = \sum_{n=1}^N |y_n|^q$$

$$\Rightarrow \sum_{n=1}^N |y_n|^q = Tx \le \|T\|\|x\|_p = \|T\|\left(\sum_{n=1}^N |y_n|^q\right)^{1/p} \overset{N\to\infty}{\Rightarrow} \|y\|_q \le \|T\|.$$

Claim: $T = \Phi_y$

For all n, we have $Te_n = \Phi_y(e_n)$.

Since T and Φ_y are both linear and continuous, thus $T = \Phi_y$ on $\overline{\text{span}\{e_n : n \geq 1\}} = \ell^p$.

Remark. Similarly, $(\ell^1)^* \cong \ell^\infty$ and $c_0^* = \ell^1$.

But it does not show that $(\ell^\infty)^* \cong \ell^1$ (which is false) since $\{e_n\}$ is not dense in ℓ^∞.

Cor. For $1 \leq p \leq \infty$, ℓ^p is complete (being the dual space of a normed space).

1.5. Finite-dimensional normed spaces

Fact. Any fin.-dim. vector space can be identified with $\mathbb{K}^n$ by choosing a basis. Here n is the dimension of the space.

Defn. Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space X are equivalent if there exists a constant $C>0$ s.t.

$$C^{-1}|||x||| \le \|x\| \le C|||x||| \quad \text{for all } x\in X,$$

i.e. $\mathrm{id}: (X,\|\cdot\|) \to (X, |||\cdot|||)$ is an isomorphism.

Thm. Let X be a fin.-dim. vector space. Then all norms on X are equivalent.

Proof. It suffices to show that any norm $\|\cdot\|$ on $\mathbb{K}^n$ is equivalent to $\|\cdot\|_2$.

Claim: $\|x\| \le C\|x\|_2$ for all x $\quad (*)$

$$\|x\| = \Big\|\sum_{i=1}^n x_i e_i\Big\| \le \sum_{i=1}^n |x_i|\,\|e_i\| \le n \max_i |x_i| \max_i \|e_i\| \le \underbrace{n\max\|e_i\|}_{C}\,\|x\|_2$$

Claim: $\|x\|_2 \le C\|x\|$ for all x

Let $S=\{x : \|x\|_2 = 1\}$ and $f: S\to\mathbb{R}$, $x\mapsto \|x\|$.

Then f is continuous: $|f(x)-f(y)| = \big|\|x\|-\|y\|\big| \le \|x-y\| \overset{(*)}{\le} C\|x-y\|_2$

And S is compact: closed and bounded.

$\Rightarrow$ f assumes minimum on S : $f(x)\ge \delta > 0\ \forall x\in S \Rightarrow \|x\| \ge \delta\|x\|_2\ \forall x$.

since $f(x)>0\ \forall x\in S$

Cor. Let X, Y be normed spaces with $\dim X < \infty$. Then every linear map $T: X \to Y$ is continuous.

Proof. Define a new norm on X by

$$|||x||| = \|x\| + \|Tx\|.$$

Since all norms on X are equivalent, there is $C > 0$ s.t

$$|||x||| \leq C\|x\| \qquad \forall x$$

$$\Leftrightarrow \|Tx\| \leq (C-1)\|x\| \qquad \forall x$$

Cor. Let X, Y be normed spaces and $T: X \to Y$ a linear bijection. Then T is an isomorphism. In particular, if $\dim X = \dim Y < \infty$ then X and Y are isomorphic.

Cor. (i) Every fin.-dim. normed space is complete.

(ii) Every fin.-dim. subspace of a normed space is closed.

Proof. (i) True in $\|\cdot\|_2$. (ii) complete $\Rightarrow$ closed.

Cor. Let X be a fin.-dim. normed space. Then $\overline{B(X)}$ is compact.

Proof. $\overline{B(X)}$ is closed and bounded in $\|\cdot\|_2$ (because closed and bounded in $\|\cdot\|$). Thus $\overline{B(X)}$ is compact in $\|\cdot\|_2$, thus compact in $\|\cdot\|$.

Thm. Let X be a normed vector space s.t. $\overline{B(X)}$ is compact. Then X is fin.-dim.

Proof. Since $\overline{B_1(0)} \overset{\text{def}}{=} \overline{B(X)}$ is compact, there are $x_1, \dots, x_n \in X$ s.t.

$$\overline{B_1(0)} \subset \bigcup_{i=1}^{n} B_{1/2}(x_i).$$

Let $Y = \operatorname{span}\{x_1, \dots, x_n\}$. Then $\dim Y \leq n$. Also

$$B_1(0) \subset Y + B_{1/2}(0)$$

$$\Rightarrow B_1(0) \subset Y + \tfrac{1}{2}(Y + B_{1/2}(0)) = Y + B_{\frac{1}{4}}(0)$$

By induction,

$$B_1(0) \subset Y + B_{2^{-m}}(0) \text{ for all } m \in \mathbb{N}$$

$$\Rightarrow B_1(0) \subset \overline{Y} = Y.$$

Since X is a vector space, thus $X \subset Y$. Thus $\dim X \leq n$.

1.6. Completion, products, quotients

Completion. Let X be a metric space. The completion of X is a complete metric space $\hat{X}$ containing a dense subset isometric to X, constructed as follows.

For two Cauchy sequences $(x_n) \subset X$, $(y_n) \subset X$, define

$(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0$.

This is an equivalence relation. Denote the equivalence class of a Cauchy sequence $x = (x_n)$ by $\tilde{x}$. Define

$\hat{X} = \{\tilde{x} : x \text{ is a Cauchy sequence in } X\}$.

Define

$\hat{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$.

The limit exists and is independent of the representatives of $\tilde{x}$ and $\tilde{y}$, because $d(x_n, y_n)$ is Cauchy in $\mathbb{R}$.

Then:

- $\hat{d}$ is a metric: if $d(\tilde{x}, \tilde{y}) = 0$ then $d(x_n, y_n) \to 0$, so $\tilde{x} = \tilde{y}$. Symmetry and triangle inequality follow from those for d.
- $X \hookrightarrow \hat{X}$: for $x \in X$ define $j(x)$ as the equivalence class of $(x, x, \dots)$. Then $\hat{d}(j(x), j(y)) = d(x, y)$, so $x \mapsto j(x)$ is an isometry. Also the image is dense since if x is Cauchy in X then $(j(x_n))$ is Cauchy in $\hat{X}$ and $j(x_n) \to \tilde{x}$.

- $\tilde{X}$ is complete: Let $(\tilde{x}^k)_k \subset \tilde{X}$ be Cauchy and let $(x_n^k)_n \subset X$ be a representative for $\tilde{x}^k$. Choose n_k s.t. $d(x_n^k, x_m^k) \le 2^{-k}$ for $n,m \ge n_k$. Define $x_k = x_{n_k}^k$.

Claim: $x = (x_k)_k \subset X$ is Cauchy, so $\tilde{x} \in \tilde{X}$, and $\tilde{x}^k \to \tilde{x}$ in $\tilde{X}$.

$$d(x_k, x_\ell) = d(x_{n_k}^k, x_{n_\ell}^\ell) \le \underbrace{d(x_{n_m}^k, x_{n_m}^\ell)}_{\to 0 \text{ as } (\tilde{x}^k) \text{ Cauchy}} + \underbrace{d(x_{n_m}^k, x_{n_k}^k)}_{\le 2^{-k}} + \underbrace{d(x_{n_m}^\ell, x_{n_\ell}^\ell)}_{\le 2^{-\ell}} \to 0$$

$$\tilde{d}(\tilde{x}^k, \tilde{x}) \le \tilde{d}(\tilde{x}^k, j(x_k)) + \tilde{d}(\tilde{x}, j(x_k)) = \lim_{n\to\infty} \Big[\underbrace{d(x_m^k, x_{n_k}^k)}_{\le 2^{-k}} + d(x_m, x_k) \Big] \xrightarrow{(k\to\infty)} 0$$

<u>Defn</u>. $\tilde{X}$ is called the <u>completion</u> of X and we regard $X \subset \tilde{X}$.

<u>Thm</u>. Let X be a normed space. Then there is a Banach space $\hat{X}$ containing X as a dense subspace.

<u>Proof</u>. Let $\hat{X}$ be the metric space completion of X. For $\tilde{x}, \tilde{y} \in \tilde{X}$, choose $(x_n) \subset X$, $(y_n) \subset X$ s.t. $x_n \to \tilde{x}$, $y_n \to \tilde{y}$.

For any $\lambda, \mu \in \mathbb{K}$, $\lambda x_n + \mu y_n$ is Cauchy. Set

$$\lambda \hat{x} + \mu \hat{y} = \lim_n (\lambda x_n + \mu y_n).$$

This makes $\hat{X}$ a vector space.

Moreover, $\|\hat{x}\| = \lim_{n\to\infty} \|x_n\|$ is a norm.

It satisfies $\|\tilde{x}\| = \tilde{d}(0, \tilde{x})$, so $\hat{X}$ is indeed a normed space whose induced norm is complete.

Remark. The completion $\tilde{X}$ is unique in the sense that if $\tilde{X}'$ is another completion then there is an isometric isomorphism between $\tilde{X}$ and $\tilde{X}'$ that is the identity on X.

Prop. Let X, Y be normed spaces, and let $T \in B(X,Y)$. Then there exists a unique $\tilde{T} \in B(\tilde{X}, \tilde{Y})$ s.t.

$$\tilde{T}|_X = T, \quad \|\tilde{T}\| \leq \|T\|.$$

Proof. For $\hat{x} \in \tilde{X}$, choose $(x_n) \subset X$ s.t. $x_n \to x$. Then (x_n) is Cauchy, and thus (Tx_n) is Cauchy as well. By completeness of $\tilde{X}$, there is $\tilde{y} \in \tilde{X}$ s.t. $Tx_n \to \tilde{y}$. Let $\tilde{T}(\hat{x}) = \tilde{y}$. Note that $\tilde{T}$ is well-defined, linear and $\tilde{T}|_X = T$. Also, $\tilde{T}$ is bounded since

$$\|\tilde{T}\hat{x}\| = \lim_{n\to\infty} \|Tx_n\| \leq \|T\| \lim_n \|x_n\| = \|T\| \|x\|.$$

Uniqueness: if $\tilde{T}'$ is another such map then $\tilde{T} = \tilde{T}'$ on X. Since $\tilde{T}$ and $\tilde{T}'$ are continuous thus $\tilde{T} = \tilde{T}'$ on $\tilde{X}$.