Linear Analysis (Michaelmas 2017)

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Primary references:
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B. Bollobás, Linear Analysis, Cambridge University Press

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1.1. Normed vector spaces

Unless stated, vector spaces will be real or complex. We write $\mathbb{K}$ for $\mathbb{R}$ or $\mathbb{C}$.

Defn. A normed vector space is a vector space V with a norm $\|\cdot\| : V \to \mathbb{R}$, $v \mapsto \|v\|$ satisfying

(i) $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ iff $v = 0$ (pos.-def.);

(ii) $\|\lambda v\| = |\lambda| \, \|v\|$ for every $\lambda \in \mathbb{K}$ and $v \in V$ (pos. homogeneity);

(iii) $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$ (triangle ineq.).

In particular, a metric is defined on V by $d(v,w) = \|v - w\|$.

Fact. The vector space operations are continuous maps

$$\mathbb{K} \times V \to V, \quad (\lambda, v) \mapsto \lambda v \quad \text{(scalar mult.)}$$
$$V \times V \to V, \quad (v, w) \mapsto v + w \quad \text{(addition)}$$

Proof. We only check that scalar multiplication is continuous. Since $\mathbb{K}$ and V are metric spaces, it suffices to check that $\lambda_j \to \lambda$ and $v_j \to v$ implies $\lambda_j v_j \to \lambda v$. But

$$\begin{aligned} \|\lambda v - \lambda_j v_j\| &= \|(\lambda - \lambda_j) v + \lambda_j (v - v_j)\| \\ &\leq \|(\lambda - \lambda_j) v\| + \|\lambda_j (v - v_j)\| \leq \underbrace{|\lambda - \lambda_j|}_{\to 0} \|v\| + \underbrace{|\lambda_j|}_{\text{bounded}} \underbrace{\|v_j - v\|}_{\to 0}. \end{aligned}$$

Thus $\|\lambda v - \lambda_j v_j\| \to 0$, i.e. $\lambda_j v_j \to \lambda v$.

Cor. Translations $(v \mapsto v + v_0)$ and dilatations $(v \mapsto \lambda v)$ are homeomorphisms.

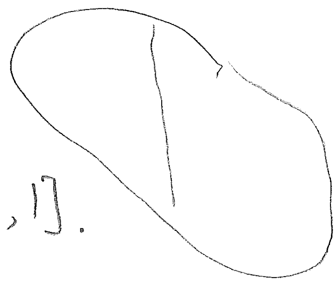
Defn. A topological vector space is a vector space together with a topology which makes the vector space operations continuous and points are closed.

(**Prop.** A topological vector space is a Hausdorff top. space.)

Defn. Let V be a vector space and $C \subset V$ a subset. We say that C is convex if

$$tC + (1-t)C \subset C \quad \text{for all } t \in [0,1],$$

i.e. $tx + (1-t)y \in C$ for all $x, y \in C$, $t \in [0,1]$.



Fact. Let V be a normed vector space. Then $B_1(0)$ is convex.

Fact. If C is convex, then $x + \lambda C$ is convex for all $\lambda \in \mathbb{K}$ and $v \in V$.

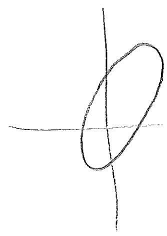
Defn. A topological vector space is locally convex if it's topology has a basis of convex sets, i.e., every open set can be written as a union of convex open sets.

Defn. Let V be a topological vector space and $B \subset V$. We say that B is bounded if for every open neighbourhood U of 0, there exists $t > 0$ s.t. $sU \supset B$ for all $s \geq t$.

Prop. Let V be a topological vector space and $C \subset V$ be a bounded and convex neighbourhood of 0. Then the topology on V is induced by a norm.

Lemma. If C is as in the proposition, there exists a balanced bounded convex neighbourhood $\tilde{C}$ of 0, i.e.,

$$\lambda \tilde{C} \subseteq \tilde{C} \text{ for all } |\lambda| \leq 1.$$



Proof of proposition. Let

$$\mu_{\tilde{C}}(v) = \inf\{t > 0 : v \in t\tilde{C}\} \quad (\text{Minkowski functional of } \tilde{C}).$$

We show that $\|v\| = \mu_{\tilde{C}}(v)$ is a norm on V and that the topology induced by is that of V.

(i) $\mu_{\tilde{C}}(v) = 0$ iff $v = 0$ since $\tilde{C}$ is bounded.

(ii) Since $\tilde{C}$ is balanced,

$$\mu_{\tilde{C}}(\lambda v) = \inf\{t > 0 : \lambda v \in t\tilde{C}\} \overset{\tilde{C} \text{ balanced}}{=} \inf\{t > 0 : v \in \tfrac{t}{|\lambda|}\tilde{C}\}$$
$$= \inf\{|\lambda| \cdot \tfrac{t}{|\lambda|} > 0 : v \in \tfrac{t}{|\lambda|} \in \tilde{C}\}$$
$$= |\lambda| \, \mu_{\tilde{C}}(v).$$

(iii) Given $v, w \in V$, write $v = \lambda v_0$ and $w = \mu w_0$ with $\lambda, \mu > 0$, $v_0, w_0 \in \tilde{C}$. Since $\tilde{C}$ is convex, $\dfrac{\lambda v_0 + \mu w_0}{\lambda + \mu} \in \tilde{C}$ and $\mu_{\tilde{C}}\left(\dfrac{\lambda v_0 + \mu w_0}{\lambda + \mu}\right) \leq 1$.

Therefore,

$$\mu_{\tilde{C}}(v+w) = (\lambda+\mu)\,\mu_{\tilde{C}}\left(\frac{\lambda v_0 + \mu w_0}{\lambda+\mu}\right) \leq \lambda+\mu.$$

$$\Rightarrow \mu_{\tilde{C}}(v+w) \leq \mu_{\tilde{C}}(v) + \mu_{\tilde{C}}(w).$$

Finally, to see that the topology is the same,

$$B_1(0) = \{x : \mu_{\tilde{C}}(x) < 1\} \subset \tilde{C} \subset \overline{B_1(0)}.$$

<u>Proof of lemma</u>. We consider the case $\mathbb{K}=\mathbb{C}$ only. Let $\mathbb{D}=\{z\in\mathbb{C} : |z|<1\}$ be the open unit disk. Let $S : \mathbb{C}\times V \to V$, $(\lambda, v) \mapsto \lambda v$. By continuity, $S^{-1}(C)$ contains an open nbhd of 0 in $\mathbb{C}\times V$. Thus it contains $t\mathbb{D}\times U$ for some $t>0$ and some open nbhd U of 0 in V. Thus $t\mathbb{D}U \subset C$. Take $\tilde{C}$ to be the convex hull of $t\mathbb{D}U$.

- $\tilde{C}$ is balanced since $t\mathbb{D}U$ is.
- Since C is convex, $\tilde{C}\subset C$ and since C is bounded, so is $\tilde{C}$.

<u>Cor.</u> A topological vector space is <u>normable</u> iff it is <u>locally bounded</u> and <u>locally convex</u>.

Defn. A <u>Banach space</u> is a normed vector space that is complete as a metric space, i.e., every Cauchy sequence converges.

<u>Examples</u>

(i) Any finite-dimensional vector space (such as $\mathbb{K}^n$) is a Banach space.

(ii) Let X be a set, and $B(X)$ be the <u>bounded</u> $\mathbb{K}$-valued functions on X. Then $B(X)$ is a Banach space with norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)| .$$

(iii) Let X be a compact Hausdorff space (such as $X = [0,1]$) and let $C(X)$ be the space of <u>continuous</u> functions on X. Then

$C(X) \subset B(X)$ (every continuous function on X is bounded)

$C(X)$ is a Banach space (the uniform limit of a sequence of cont. functions is continuous)

(iv) Let $U \subset \mathbb{R}^n$ be open, bounded, and let $C^k(\overline{U})$ be the space of functions $f : U \to \mathbb{K}$ such that $D^\alpha f$ is continuous and bounded on U, for $|\alpha| \leq k$. Here

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$ for a multi index $\alpha = (\alpha_1, \ldots, \alpha_n)$

$|\alpha| = \sum \alpha_i$

Then $C^k(\overline{U})$ is a Banach space with norm

$$\|f\|_{C^k(\overline{U})} = \max_{|\alpha|\le k} \|D^\alpha f\|_\infty$$

(v) For a sequence $x=(x_i)=(x_1,x_2,\dots)\subset \mathbb{K}$ define

$$\|x\|_p = \Big(\sum_i |x_i|^p\Big)^{1/p} \quad \text{for } p\in[1,\infty)$$

$$\|x\|_\infty = \sup_i |x_i|$$

Then $\ell^p = \{x=(x_i) : \|x\|_p<\infty\}$ is a Banach space, for $p\in[1,\infty]$.
Note: for $p<1$, $(\sum_i |x_i|^p)^{1/p}$ does not define a norm.

(vi) Let $U\subset\mathbb{R}^n$ be open (not necessarily bounded) and denote by $C(U)$ the space of continuous functions on U (not necessarily bounded). Then $C(U)$ is a topological vector space, with topology generated as follows.
Let $K_i\subset U$ be compact sets such that $K_i\subset K_{i+1}$ and $U=\bigcup_i K_i$.
Let

$$V(i,n) = \{f : \|f\|_{C(K_i)} < \tfrac{1}{n}\}.$$

The topology generated by $V(i,n)$ and its translates makes $C(U)$ a locally convex (but not locally bounded) topological vector space. The topology is given by the metric

$$d(f,g) = \sum_{i=1}^\infty 2^{-i} \frac{\|f-g\|_{C(K_i)}}{1+\|f-g\|_{C(K_i)}},$$

(vii) Let $X = \{f : [0,1] \to \mathbb{K} \text{ continuous}\}$. Then

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} \quad \text{for } p \in [1, \infty)$$

is a norm on X. However, X is not complete in this norm!

$\vdots$

More: Analysis of Functions.

1.2. Bounded linear maps and the dual space

Fact. In any topological vector spaces V, W, a linear map $T: V \to W$ is continuous iff it is continuous at 0.

Proof. Let T be continuous at 0 and $v \in V$. We show that then T is also continuous at v. Let $w = T(v)$ and $U \subset W$ an open nbhd of w. Then $U - w$ is an open nbhd of $0 \in W$. Since T is continuous at 0, $T^{-1}(U-w)$ contains an open nbhd $U' \subset V$ of 0. By linearity,

$$T(v+U') = T(v) + T(U') \subset T(v) + U - w = U.$$

Since $v + U'$ is an open nbhd of v, this means that T is continuous at v.

Defn. Let V, W be topological vector spaces and $T: V \to W$ linear. Then T is bounded if $T(B)$ is bounded for any bounded $B \subset V$.

Fact. If V, W are normed vector spaces, a linear map $T: V \to W$ is bounded iff there is $\lambda > 0$ s.t.

$$T(B_1(0)) \subseteq B_\lambda(0)$$

i.e. $\|Tv\| \leq \lambda$ for all $v \in V$ with $\|v\| \leq 1$.

Defn. Let V, W be normed v.s. The <u>operator norm</u> of a linear map $T: V \to W$ is

$$\|T\| = \sup_{\|v\|=1} \|Tv\| = \sup_{\|v\|\le 1} \|Tv\| = \sup_{\|v\|<1} \|Tv\|.$$

Denote by $\mathcal{L}(V,W)$ the space of linear maps $V \to W$, and by $\mathcal{B}(V,W)$ the space of bounded linear maps $V \to W$. These are clearly vector spaces.

Fact. The operator norm $\|\cdot\|$ is a norm on $\mathcal{B}(V,W)$.

(E.g. $\|\mu T\| = \sup_{\|v\|\le 1} \|\mu T v\| \le |\mu| \sup_{\|v\|\le 1} \|Tv\| = |\mu| \, \|Tv\|$.

The other properties follow similarly.)

Prop. Let V, W be normed v.s. Then a linear map $T: V \to W$ is bounded iff it is continuous. (The same is true is locally bounded top. v.s.).

Proof.

- T bounded $\Rightarrow$ T continuous:

Assume $\|v_k - v\| \to 0$. Then $\|Tv_k - Tv\| \le \|T\| \, \|v_k - v\| \to 0$

- T continuous $\Rightarrow$ T bounded:

Since T is continuous, $T^{-1}(B_1(0))$ contains an open ball $B_\varepsilon(0) \subset V$. Thus

$$T(B_1(0)) = \varepsilon^{-1} T(B_\varepsilon(0)) \subset \varepsilon^{-1} B_1(0) = B_{\varepsilon^{-1}}(0).$$

Thus T is bounded.

Defn. Let V be a top. v.s. The (topological) dual space of V is the space of all continuous linear maps $V \to \mathbb{K}$ and is denote by V^*. (In a normed space, $V^* = \mathcal{B}(V, \mathbb{K})$.) We call $\mathcal{L}(V, \mathbb{K})$ the algebraic dual of V.

Prop. Let V be a normed space and W a Banach space. Then $\mathcal{B}(V, W)$ is a Banach space.

Proof. Let $(T_i) \subset \mathcal{B}(V, W)$ be a Cauchy sequence. For any $v \in V$, then $(T_i v)$ is a Cauchy sequence in W:

$$\|T_i v - T_j v\| \leq \underbrace{\|T_i - T_j\|}_{< \varepsilon \text{ for } i,j > N} \|v\|$$

Since W is complete, there is $w \in W$ s.t. $T_i v \to Tv$. We need to verify that the map $v \mapsto Tv$ is in $\mathcal{B}(V, W)$.

- T is linear: $T(\lambda v + \mu w) = \lim_{i \to \infty} T_i(\lambda v + \mu w) = \lim_{i \to \infty} (\lambda T_i v + \mu T_i w)$

$$= \lambda Tv + \mu Tw.$$

- T is bounded: for $\|v\| \leq 1$, $\|Tv\| \leq \|T_i v\| + \underbrace{\|T_i v - Tv\|}$

$$\leq \underbrace{\|T_i\|}_{\text{bounded since } (T_i) \text{ is Cauchy}} + \varepsilon \quad < \varepsilon \text{ for } i > N.$$

- $T_i \to T$ in norm: for $\|v\| \leq 1$, $\|T_i v - Tv\| \leq \underbrace{\|T_i v - T_j v\|}_{\leq \|T_i - T_j\|} + \underbrace{\|T_j v - Tv\|}_{< \varepsilon}$

$\Rightarrow \|T_i - T\| \to \infty$.

Cor. Let V be a normed vector space. Then V^* is a Banach space. (V does not need to be a Banach space).

Defn. Let V, W be normed vector spaces, and $T \in \mathcal{B}(V,W)$. Then the adjoint map $T^*: W^* \to V^*$ is defined by

$$[T^*f]v = f(Tv) \quad \text{for } f \in W^*,\ v \in V.$$

Fact. T^*f is indeed in $V^* = \mathcal{B}(V,\mathbb{K})$ and $\|T^*\| \leq \|T\|$

(Later: $=$ by Hahn-Banach)

(Indeed,

$$\|[T^*f]v\| = \|f(Tv)\| \leq \|f\| \, \|Tv\| \leq \|f\| \, \|T\| \, \|v\|$$

$\Rightarrow \|T^*f\| \leq \|f\| \, \|T\|$

$\Rightarrow \|T^*\| \leq \|T\|$.)

Defn. Let V be a normed vector space. The double dual of V is the dual space of V^*, i.e., $V^{**} = (V^*)^*$.

Fact. The map $\phi: V \to V^{**}$, $v \mapsto \tilde{v}$ where $\tilde{v}(f) = f(v)$ is bounded and linear.

Remark. The Hahn-Banach Theorem implies that ϕ is isometric (injective), so V can be considered a subspace of V^{**}. In general, ϕ is not surjective.

Defn. A Banach space is reflexive if ϕ is a bijection.

Examples

(i) Let V, W be finite-dimensional vector spaces. Then any linear map $T: V \to W$ is bounded with respect to any norms on V, W. It can be represented by a matrix and T^* is represented by the transpose of the matrix. Fin.-dim. v.s. are reflexive.

(ii) The map $T: \ell^p \to \ell^p$, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ is bounded with $\|T\| = 1$, injective, but not surjective.

(iii) The map $D: C^1[0,1] \to C^0[0,1]$ is

bounded as a map $C^0[0,1], \|\cdot\|_\infty \to C^1[0,1], \|\cdot\|_\infty + \|\cdot'\|_\infty$

unbounded as a map $C^0[0,1], \|\cdot\|_\infty \to C^1[0,1], \|\cdot\|_\infty$

(iv) The map $\mathrm{id}: C^1[0,1], \|\cdot\|_\infty \to C^1[0,1], \|\cdot\|_\infty + \|\cdot'\|_\infty$ is unbounded.

1.3. Finite-dimensional vector spaces

Fact. Any finite-dimensional vector space can be identified with $\mathbb{K}^n$ by choosing a basis. Here n is the dimension.

Defn. Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space V are equivalent iff there exists a constant $C>0$ s.t.

$$C^{-1}|||v||| \leq \|v\| \leq C|||v||| \quad \text{for all } v\in V.$$

Prop. All norms on a finite-dimensional vector space are equivalent.

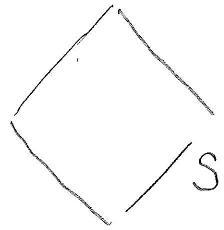
Proof. It suffices to show that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_1$ where

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|.$$

(i)

$\|v\| \leq C\|v\|_1$: Let $e_i = (0,\dots,0,1,0,\dots)$. Then

$$\|v\| = \Big\|\sum_{i=1}^{n} v_i e_i\Big\| \leq \sum_{i=1}^{n} |v_i|\|e_i\| \leq \underbrace{\left(\max_i \|e_i\|\right)}_{C} \|v\|_1 .$$



$\|v\|_1 \leq C\|v\|$: Let $f : \underbrace{\{v\in\mathbb{R}^n : \|v\|_1 = 1\}}_{S} \to \mathbb{R}, \ v \mapsto \|v\|$.

- f is continuous w.r.t. $\|\cdot\|_1$-topology:

$$|f(v)-f(w)| = \big|\|v\|-\|w\|\big| \underset{\substack{\uparrow \\ \text{triangle ineq.}}}{\leq} \|v-w\| \underset{\substack{\uparrow \\ \text{above}}}{\leq} C\|v-w\|_1 .$$

- S is compact w.r.t. $\|\cdot\|_1$-topology.

Let $(v^i) \subset S$ be a sequence. We need to find a convergent subsequence.

Let $(v_1^i)_{i\in\Lambda_1}$, $\Lambda_1 \subset \mathbb{N}$ be a subsequence s.t. $v_1^i \xrightarrow{i\in\Lambda_1} \tilde{v}_1$, for some $\tilde{v}_1 \in \mathbb{K}$. Such a subsequence exists since (v_1^i) is bounded.

Let $(v_2^i)_{i\in\Lambda_2}$, $\Lambda_2 \subset \Lambda_1$ be a subsequence s.t. $v_i^2 \xrightarrow{\Lambda_2} \tilde{v}_2$.

$\vdots$

Then, for $i \in \Lambda_n$, $i \to \infty$, $\|v_i - \tilde{v}\|_1 \leq \sum_{k=1}^{n} |v_k^i - \tilde{v}_k| \to 0$.

In summary, f is a continuous function on a compact set.

$\Rightarrow$ f attains its minimum.

Since $\|\cdot\|$ is a norm, this minimum must be strictly pos.

$\Rightarrow$ $\|v\| = f(v) \geq f(v_0) = c$ for all $\|v\|_1 = 1$.

For arbitrary $v \in \mathbb{K}^n$, therefore

$$\|v\| = \|v\|_1 \left\| \frac{v}{\|v\|_1} \right\| \geq \|v\|_1 c.$$

Cor. In any f.d. normed vector space, the closed unit ball is compact.

Cor. Every finite-dimensional normed vector space is a Banach space.

Proof. Let (v_i) be Cauchy. Then (v_i) is bounded, $(v_i) \subset \overline{B_R(0)}$ for some R. But $\overline{B_R(0)}$ is compact, so (v_i) converges.

Cor. Let V be a normed vector space and $W \subset V$ be a finite-dimensional subspace. Then W is closed.

Cor. Let V be a normed vector space, W a f.d. normed space, and $T: W \to V$ linear. Then T is bounded.

Proof. Since $\operatorname{im} T$ is finite-dimensional, we can assume without loss of generality that V is finite-dimensional. Moreover, we can assume that $V = \mathbb{K}^m, \|\cdot\|_\infty$, $W = \mathbb{K}^n, \|\cdot\|_1$.

Let (T_{ij}) be the matrix associated to T.

$$T(w_1, \dots, w_n) = \left(\sum_{i=1}^n T_{1i} w_i, \dots, \sum_{i=1}^n T_{mi} w_i\right)$$

$$\Rightarrow \|T(w_1, \dots, w_n)\|_\infty = \max_j \left|\sum_{i=1}^n T_{ji} w_i\right| \le \max_{i,j} |T_{ji}| \sum_{i=1}^n |w_i|$$

$$\le C(T) \|w\|_1 .$$

Thm. Let V be a normed vector space s.t. $\overline{B_1(0)}$ is compact. Then V is finite-dimensional.

Proof. Since $\overline{B_1(0)}$ is compact there are $w_1, \dots, w_n \in V$ s.t.

$$\overline{B_1(0)} \subset \bigcup_{i=1}^n B_{1/2}(w_i).$$

Let $W = \operatorname{span}\{w_1, \dots, w_n\}$. Note that $\dim W \le n$. Also

$$B_1(0) \subset W + B_{\frac{1}{2}}(0).$$

$$\Rightarrow B_1(0) \subset W + \tfrac{1}{2}(W + B_{1/2}(0)) = W + B_{\frac{1}{4}}(0).$$

By induction,

$$B_1(0) \subset W + B_{2^{-i}}(0) \quad \text{for all } i \in \mathbb{N}.$$

$$\Rightarrow B_1(0) \subset \overline{W} = W.$$

Since V is a vector space, thus $V \subset W$. Thus $\dim V \leq n$.

2. The Hahn-Banach Theorem

Defn. Given vector spaces $W \subset V$, linear maps $g: W \to \mathbb{K}$, $f: V \to \mathbb{K}$, we say that f extends g if $f|_W = g$.

When can one extend a map specified on a subspace in a continuous way?

2.1. Finite codimension

Let V be a real vector space.

Defn. A map $p: V \to \mathbb{R}$ is sublinear if

(i) $p(\alpha v) = \alpha p(v)$ for all $v \in V$, $\alpha \geq 0$;

(ii) $p(v+w) \leq p(v) + p(w)$ for all $v, w \in V$.

Example. Any norm is sublinear.

Lemma. Let $W \subset V$ be a subspace of codimension 1, i.e., there exists $v_1 \in V \setminus W$ such that

$$V = \{v + tv_1 : v \in W, t \in \mathbb{R}\} = W \oplus \mathbb{R}v_1.$$

Let $p: V \to \mathbb{R}$ be sublinear and $g: W \to \mathbb{R}$ linear with

$$g(v) \leq p(v) \quad \text{for all } v \in W.$$

Then there is a linear map $f: V \to \mathbb{R}$ that extends g and

$$f(v) \leq p(v) \quad \text{for all } v \in V.$$

Proof. We will find $\alpha \in \mathbb{R}$ s.t. $f_\alpha : V \to \mathbb{R}$ defined by

$$f_\alpha(v + t v_1) = g(v) + t\alpha \quad \text{for all } v \in W,\ t \in \mathbb{R}$$

is the asserted extension of g. Let

$$\alpha = \sup_{v \in W} \big(g(v) - p(v - v_1)\big).$$

Claim: $\alpha < \infty$

By linearity of g and sublinearity of p,

$$g(v) + g(w) = g(v+w) \le p(v+w) \le p(v - v_1) + p(w + v_1) \qquad (*)$$

$$\Rightarrow g(v) - p(v - v_1) \le -g(w) + p(w + v_1)$$

$$\Rightarrow \alpha < \infty$$

Claim: $f_\alpha(v - v_1) \le p(v - v_1)$ for all $v \in W$

$$f_\alpha(v - v_1) = g(v) - \alpha \le p(v - v_1) \text{ by defn. of } \alpha$$

Claim: $f_\alpha(w + v_1) \le p(w + v_1)$ for all $w \in W$

$$f_\alpha(w + v_1) = g(w) + \alpha \overset{(*)}{\le} \alpha - \big(g(v) - p(v - v_1)\big) + p(w + v_1)$$

Taking the infimum over $v \in W$,

$$f_\alpha(w + v_1) \le p(w + v_1).$$

Claim: $f_\alpha(v + t v_1) \le p(v + t v_1)$ for all $v \in W$, $t \in \mathbb{R}$.

By linearity of f and positive homogeneity of p, for all $t > 0$,

$$f_\alpha(v \pm t v_1) = t f\!\left(\frac{v}{t} \pm v_1\right) \underset{\substack{\uparrow \\ \text{previous cases.}}}{\le} t\, p\!\left(\frac{v}{t} \pm v_1\right) = p(v \pm t v_1)$$

Cor. The same statement holds if $W \subset V$ has finite codimension.

2.2. Zorn's Lemma

Defn. (i) A partially ordered set (poset) is a set P with a binary relation $\leq$ s.t. for all $x, y \in P$ either $x \leq y$ or $x \not\leq y$, and

$$x \leq x \qquad \text{(reflexive)}$$
$$x \leq y,\ y \leq z \Rightarrow x \leq z \qquad \text{(transitive)}$$
$$x \leq y,\ y \leq x \Rightarrow x = y \qquad \text{(antisymmetric)}$$

(ii) Let P be a poset. A subset $T \subset P$ is called totally ordered (or a chain) if

$$x \not\leq y \Rightarrow y \leq x,$$

i.e.

$$\text{either } x \leq y \text{ or } y \leq x.$$

(iii) Let P be a poset and $U \subseteq P$ a subset. Then

- $b \in P$ is an upper bound for U if $x \leq b$ for all $x \in U$;
- $l \in P$ is a least upper bound for U if l is an upper bound and any other upper bound b for U satisfies $l \leq b$.

(iv) Let P be a poset. An element $m \in P$ is maximal if

$$m \leq x \Rightarrow m = x.$$

Zorn's Lemma. Let P be a poset with the property that every non-empty totally ordered subset has a least upper bound. Then P has at least one maximal element.

Remark. (i) Zorn's Lemma is trivial in finite posets.

(ii) In infinite posets, it is equivalent to the Axiom of Choice (assuming the usual other axioms of set theory).

Recall that in a vector space V, elements $v_1, \dots, v_k$ are linearly independent if

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = 0.$$

A set $S \subset V$ is linearly independent if any finite subset is. A basis of V is a set $B \subset V$ that is linearly independent and such that every element of V is a finite linear combination of elements in B.

Prop. Let $V \neq \{0\}$ be a vector space and $S \subset V$ linearly independent. Then V has a basis B s.t. $S \subset B$.

Proof. Let

$$P = \{T \subseteq V \text{ subset} : T \supseteq S,\ T \text{ is linearly independent}\}.$$

Then P is a poset with partial order $\subset$, i.e. $T_1 \leq T_2$ iff $T_1 \subseteq T_2$. For any $L \subseteq P$ that is totally ordered, set

$$T_0 = \bigcup_{T \in L} T.$$

Claim: $T_b \in P$ and T_b is an upper bound for L.

Clearly, $S \subset T_b$. Let $\alpha_i \in \mathbb{K}$ and $v_i \in T_b$ s.t. $\sum_{i=1}^{m} \alpha_i v_i = 0$. Since L is totally ordered and $m < \infty$ there must be $T \in L$ s.t. $v_1, \dots, v_m \in T$. Since T is linearly independent therefore $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$. Thus T_b is linearly independent. Thus $T_b \in P$ and clearly T_b is a leact upper bound for L.

Zorn's Lemma $\Rightarrow$ there exists $B \in P$ maximal

Claim: B is a basis

By definition, B is independent. We need to show that $\operatorname{span} B = V$. Suppose otherwise. Then there is $v \in V \setminus \operatorname{span} B$ and $B \cup \{v\}$ is linearly independent. This contradicts the maximality of B.

2.3. The Hahn-Banach Theorem

Thm (Hahn-Banach). Let V be a real vector space, $W \subset V$ a subspace, $p: V \to \mathbb{R}$ sublinear, $g: W \to \mathbb{R}$ linear s.t.

$$g(v) \leq p(v) \quad \text{for all } v \in W.$$

Then there is $f: V \to \mathbb{R}$ linear s.t. $f|_W = g$ and

$$f(v) \leq p(v) \quad \text{for all } v \in V.$$

Proof. Let

$$P = \{(N,h) : N \subset V \text{ linear}, W \subseteq N, h: N \to \mathbb{R} \text{ linear}, \\ h|_W = g, h(v) \leq p(v) \text{ for } v \in N\}.$$

For $(N,h), (N',h') \in P$, set

$$(N,h) \leq (N',h') \text{ iff } N \subset N' \text{ and } h'|_N = h.$$

Then P is a poset, $(W,g) \in P$, so $P \neq \emptyset$. We want to apply Zorn's Lemma. Let $(N_i, h_i)_{i \in I} \subset P$ be a totally ordered subset. Set

$$N = \bigcup_{i \in I} N_i, \quad h(v) = h_i(v) \text{ if } v \in N_i.$$

This well-defined since $(N_i, h_i)_{i \in I}$ is totally ordered. Here N is linear and h is linear. Also, $h(v) \leq p(v)$ for all $v \in N$. Thus $(N,h) \in P$. Also, (N,h) is a least upper bound for $(N_i, h_i)_{i \in I}$. Zorn's Lemma gives a maximal element (M,f). Then $M = V$; otherwise the finite codimension case gives (M_1, f_1) with $(M,f) \lneq (M_1, f_1)$, contradiction.

Cor. (Hahn-Banach Theorem). Let V be a normed vector space (real or complex), $W \subset V$ a subspace. For any $g \in W^*$ there exists $f \in V^*$ s.t. $f|_W = g$ and $\|f\| \leq \|g\|$.

Proof. Assume that V is real; the complex case is treated on the example sheet. Then $p(v) = \|v\|$ is sublinear and the claim follows directly from the Hahn-Banach Theorem.

Cor. Let V be a normed vector space and $v \in V$. Then there exists $f_v \in V^*$ s.t. $\|f_v\| = 1$ and $f_v(v) = \|v\|$.

Here f_v is called a support functional for v.

Proof. Let $W = \langle v \rangle$. Define $g \in W^*$ by $g(tv) = t\|v\|$, $t \in \mathbb{K}$. Then $\|g\| = 1$ and $g(v) = \|v\|$. By Hahn-Banach there is $f = f_v$ as desired.

Remark. In concrete examples, one can often construct f_v by hand $\rightarrow$ Example sheet.

Cor. Let V be a normed vector space and $v \in V$. Then

$$v = 0 \Leftrightarrow f(v) = 0 \text{ for all } f \in V^*.$$

In particular, $V^* \neq 0$.

Cor. Let V be a normed vector space, $v, w \in V$, $v \neq w$. Then there exists $f \in V^*$ s.t. $f(v) \neq f(w)$.

Proof. Take $f = f_{v-w}$.

Prop. The map $\phi : V \to V^{**}$ is an isometry: $\|\phi(v)\| = \|v\|$ for all $v \in V$.

Proof. We have seen $\|\phi(v)\| \leq \|v\|$. Now

$$|\phi(v) f_v| = |f_v(v)| = \|v\| \Rightarrow \|\phi(v)\| \geq \|v\|.$$

Prop. Let V, W be normed vector spaces. For any $T \in \mathcal{B}(V, W)$, the dual map $T^* \in \mathcal{B}(W^*, V^*)$ satisfies $\|T^*\| = \|T\|$.

Proof. Again we have seen $\|T^*\| \leq \|T\|$. Now, with $f = f_{Tv}$,

$$[T^* f] v = f(Tv) = \|Tv\|$$

$$\Rightarrow \|T^*\| = \sup_{\substack{f \in W^* \\ \|f\| \leq 1}} \|T^* f\| \geq \|T^* f_{Tv}\| = \sup_{\substack{v \in V \\ \|v\| \leq 1}} [T^* f] v \geq \sup_{\|v\| \leq 1} \|Tv\| = \|T\|.$$

3. Completeness and the Baire category

3.1. Baire Category

Recall that if X is a metric space, then $Y \subset X$ is dense if $\overline{Y} = X$, i.e. $Y \cap B_r(x) \neq \emptyset$ for all $x \in X$, $r > 0$.

Thm. (Baire Category Theorem). Let X be a complete metric space. For any sequence of open dense subsets $U_j \subset X$,

$$\bigcap_j U_j \text{ is dense in } X.$$

Proof. Let $U = \bigcap_j U_j$. Given any $x \in X$, $r > 0$, we need to show that $B_r(x) \cap U \neq \emptyset$.

Since U_1 is dense and open, there is $x_1 \in X$, $r_1 \in (0,1)$ s.t.

$$\overline{B_{r_1}(x_1)} \subset B_{2r_1}(x_1) \subset U_1 \cap B_r(x).$$

Likewise, choose $x_2 \in X$, $r_2 \in (0, \frac{1}{2})$ s.t.

$$\overline{B_{r_2}(x_2)} \subset U_2 \cap B_{r_1}(x_1),$$

and $x_n \in X$, $r_n \in (0, \frac{1}{n})$ s.t.

$$\overline{B_{r_n}(x_n)} \subset U_n \cap B_{r_{n-1}}(x_{n-1}).$$

Then $r_n \to 0$ and $B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \cdots$. Thus $d(x_n, x_m) < r_n$ if $m \geq n$, i.e. (x_n) is Cauchy. Since X is complete, there is $y \in X$ s.t. $x_n \to y$. Note that $y \in \overline{B_{r_k}(x_k)} \subset U_k$ for all k. Thus

$$y \in \bigcap_j U_j \text{ and } y \in \overline{B_{r_1}(x_1)} \subset B_r(x).$$

Cor. Let X be a complete metric space. Let $A_j \subset X$ be a sequence of closed subsets s.t. $\bigcup_j A_j$ has nonempty interior, i.e. it contains some ball. Then at least one of the A_j has nonempty interior.

Proof. Let $U_j = X \setminus A_j$. Since $\bigcup A_j$ has nonempty interior,

$$X \setminus \bigcup_j A_j = \bigcap_j U_j$$

is not dense. Since the U_j are open, by the BCT, at least one of them must not be dense, say U_k. Thus $A_k = X \setminus U_k$ has nonempty interior.

Defn. Let X be a metric space.

(i) A subset $Y \subset X$ is nowhere dense if $\mathrm{int}(\overline{Y}) = \emptyset$, i.e. if Y is not dense in any ball.

(ii) A subset $Z \subset X$ is meagre or of the first category if there are countably many sets $Y_j \subset X$ that are nowhere dense and $Z = \bigcup_j Y_j$.

(iii) A subset $U \subset X$ is nonmeagre or of the second category if it is not meagre.

(iv) A subset $R \subset X$ is residual if its complement is meagre.

Fact. $Y \subset X$ is nowhere dense $\Leftrightarrow$ $\overline{Y}$ is nowhere dense
$\Leftrightarrow$ $X \setminus \overline{Y}$ is open dense.

Example: • $\mathbb{Q} = \bigcup_{x\in\mathbb{Q}} \{x\} \subset \mathbb{R}$ is meagre in $\mathbb{R}$.

• Any countable union of meagre sets is meagre.

Remark. There is similarity of the concepts of meagre, nonmeagre, residual with those of null sets, set of positive measure, and set of full measure in measure theory. For metric spaces that are also measure spaces, such as $\mathbb{R}$ with the Lebesgue measure, one could ask if there is a closer correspondence. The answer is negative, in general. There exists a meagre set A and a Lebesgue null set B such that $\mathbb{R} = A \cup B$.

Cor. Let X be a complete metric space. Then X is of the second category.

Proof. Let $Y_j \subset X$ be nowhere dense sets. It suffices to show that $X \neq \bigcup \overline{Y_j}$. But $U_j = X \setminus \overline{Y_j}$ is open dense. So $\bigcap U_j = X \setminus \bigcup \overline{Y_j}$ is dense, in particular nonempty.

Cor. Let X be a complete metric space. Then residual sets are nonmeagre and dense.

Proof. Let $Z \subset X$ be meagre and suppose that $R = X \setminus Z$ was meagre. Then $X = Z \cup R$ would be meagre as a union of two meagre sets. But since X is complete, it is not. So R is nonmeagre.

To show that R is dense, we can suppose that $Z = \bigcup_j Y_j$ with Y_j nowhere dense. Then $U_j = X \setminus \overline{Y_j}$ is open and dense. So $R \supset \bigcap U_j$ is dense by the BCT.

<u>Cor</u>. Let X be a complete metric space and $U \subset X$ open. Then $U = \emptyset$ or U is of the second category.

<u>Proof</u>. Assume that U is open and meagre. Then $X \setminus U$ is closed and residual, so dense. Thus $X \setminus U = X$, i.e. $U = \emptyset$.

3.2. Principle of uniform boundedness

Thm. (Principle of uniform boundedness). Let X be a complete metric space. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of continuous functions $f_\lambda : X \to \mathbb{R}$. If $(f_\lambda)_{\lambda\in\Lambda}$ is pointwise bounded, i.e.,

$$\sup_{\lambda\in\Lambda} |f_\lambda(x)| < \infty \quad \text{for every } x \in X,$$

then there is a ball $B_r(x_0) \subset X$ s.t. (f_λ) is uniformly bounded on $B_r(x_0)$, i.e.,

$$\sup_{\lambda\in\Lambda} \sup_{x\in B_r(x_0)} |f_\lambda(x)| < \infty.$$

Proof. Let

$$A_k = \{x\in X : |f_\lambda(x)| \le k \text{ for all } \lambda\in\Lambda\} = \bigcap_{\lambda\in\Lambda} \{x\in X : |f_\lambda(x)| \le k\}.$$

Since the f_λ are continuous, A_k is closed as the intersection of closed sets. Since (f_λ) is pointwise bounded,

$$\bigcup_{k\in\mathbb{N}} A_k = X.$$

By the BCT, at least one of the A_k must contain a ball. Thus (f_λ) is uniformly bounded on that ball.

Thm. (Banach–Steinhaus). Let V be a Banach space and W a normed vector space. Let $(T_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}(V, W)$ be pointwise bounded, i.e.,

$$\sup_{\lambda\in\Lambda} \|T_\lambda v\| < \infty \quad \text{for all } v\in V.$$

Then (T_λ) is uniformly bounded, i.e.,

$$\sup_{\lambda\in\Lambda} \|T_\lambda\| < \infty.$$

<u>Proof</u>. Set $f_\lambda : V \to \mathbb{R}$, $v \mapsto \|T_\lambda v\|$. Then f_λ is continuous and (f_λ) is pointwise bounded. By the principle of uniform boundedness, there is a ball $B_r(v_0) \subset V$ s.t. (f_λ) is uniformly bounded on $B_r(v_0)$, i.e.,

$$\sup_{\lambda\in\Lambda} \sup_{\|v-v_0\|\le r} \|T_\lambda v\| < \infty.$$

But since the T_λ are linear, for any $v \in V$ with $\|v\| = 1$,

$$\|T_\lambda v\| = \frac{1}{r} \|T_\lambda(v_0 + rv) - T_\lambda v_0\|$$

$$\le \underbrace{\frac{1}{r} \sup_{\lambda\in\Lambda} \sup_{\|v-v_0\|\le r} \|T_\lambda v\| + \frac{1}{r}\sup_{\lambda} \|T_\lambda v_0\|}_{=:\, M < \infty \text{ (independent of } \lambda \text{ and } v \text{ with } \|v\|=1\text{)}}.$$

Thus $\sup_{\lambda\in\Lambda} \|T_\lambda\| \le M$.

3.3. Open Mapping Theorem

Defn. A map between topological spaces is open iff it maps open sets to open sets.

Example. (i) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is continuous but not open.
(ii) $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x + \operatorname{sign}(y)$ is open but not continuous.

Thm. (Open Mapping Theorem). Let V, W be Banach spaces and $T \in \mathcal{B}(V,W)$.
(i) If T is surjective, then T is open.
(ii) If T is bijective, then $T^{-1} \in \mathcal{B}(W,V)$.

Lemma. Let V, W be normed spaces. Then $T: V \to W$ is open if $T(B_1(0)) \supset B_r(0)$ for some $r > 0$.

Proof. Let $U \subset V$ be open and $v \in U$. Using that U is open, choose $\delta > 0$ s.t. $v + B_\delta(0) \subset U$. Then

$$T(U) \supset T(v + B_\delta(0)) = Tv + T(B_\delta(0)) = Tv + \delta T(B_1(0))$$

Thus if $T(B_1(0)) \supset B_r(0)$ then

$$T(U) \supset Tv + r\delta B_1(0).$$

Thus $T(U)$ contains an open ball around any element Tv. Thus T is open.

Lemma. Let V be a Banach space, W a normed space, and $T \in \mathcal{B}(V, W)$. Then $\overline{T(B_1(0))} \supset B_1(0)$ implies $T(B_1(0)) \supset B_1(0)$.

Proof. Let $w_0 \in B_1(0) \subset W$. We need to find $v \in B_1(0)$ s.t. $Tv = w$. We construct v as the limit of a Cauchy sequence. Let $v_1 \in B_{\frac{1}{2}}(0) \subset V$ be s.t.

$$\|Tv_1 - w_0\| < \frac{1}{2}.$$

This is possible since there is $\tilde{w}_0 \in B_{1/2}(0)$ with $\|\tilde{w}_0 - w_0\| < \frac{1}{2}$ and we can find $v_1 \in B_{1/2}(0)$ s.t. $\|Tv_1 - \tilde{w}_0\|$ is arbitrily small by density of $T(B_{1/2}(0))$ in $B_{1/2}(0)$.

Set $w_1 = w_0 - Tv_1 \in B_{1/2}(0)$. By induction, if $w_1, \dots, w_k$ and $v_1, \dots, v_k$ are given s.t.

$$\|v_i\| < 2^{-i}, \quad w_i = w_{i-1} - Tv_i \in B_{2^{-i}}(0) \subset W, \quad 1 \le i \le k,$$

choose $v_{k+1} \in B_{2^{-k-1}}(0) \subset V$ s.t.

$$w_{k+1} := w_k - Tv_{k+1} \in B_{2^{-k-1}}(0) \subset W.$$

$\Rightarrow \sum_{k=1}^{\infty} \|v_k\| < 1$ and $v = \sum_{k=1}^{\infty} v_k \in B_1(0)$ exists since V is complete.

$$\Rightarrow w_0 - Tv = \lim_{n \to \infty} \left(w_0 - \sum_{k=1}^{n} Tv_k \right) = \lim_{n \to \infty} \left(w_1 - \sum_{k=2}^{n} Tv_k \right)$$

$$= \dots = \lim_{n \to \infty} w_n = 0.$$

$\Rightarrow w_0 \in T(B_1(0))$ for any $w_0 \in B_1(0)$. $\Rightarrow T(B_1(0)) \supset B_1(0)$.

Proof of Open Mapping Theorem.

(i) By the previous two lemmas, it suffices to prove that $\overline{T(B_1(0))} \supset B_r(0)$ for some $r>0$. We use the BCT to do this. Since T is surjective,

$$W = \bigcup_{k=1}^{\infty} \overline{T(B_k(0))}.$$

Since W is complete, the BCT implies that there is $k_0 \in \mathbb{N}$ s.t. $\overline{T(B_{k_0}(0))}$ has nonempty interior, i.e., there is $r_0>0$, $w_0 = Tv_0$ s.t.

$$B_{r_0}(w_0) \subset \overline{T(B_{k_0}(0))}.$$

By linearity,

$$\begin{aligned} B_{r_0}(0) = B_{r_0}(w_0) - Tv_0 &\subset \overline{T(B_{k_0}(0))} - Tv_0 \\ &= \overline{T(B_{k_0}(-v_0))} \\ &\subset \overline{T(B_{k_0+\ell_0}(0))} \quad \text{if } \ell_0 \geq \|v_0\| \\ &= (k_0+\ell_0)\, \overline{T(B_1(0))} \end{aligned}$$

$\Rightarrow$ $B_r(0) \subset \overline{T(B_1(0))}$ for $r = r_0/(k_0+\ell_0)$.

(ii) If T is bijective, that T is open means that T^{-1} is continuous.

Remark. The completeness of V and W is necessary.
→ Example sheet

Example. Let $c_c = \{(x_n) : x_n = 0$ except for finitely many $n\}$, with norm $\|x\|_\infty = \max_n |x_n|$. Define $T : c_c \to c_c$ by $(Tx)_n = \frac{x_n}{n}$. Then T is continuous and bijective. But $(T^{-1}x)_n = n x_n$ is unbounded.

Remark. The basic problem in linear PDE is the following one. Given $f \in W$, e.g. $W = L^2(\Omega)$ for a bounded nice domain $\Omega \subset \mathbb{R}^n$, and a linear partial differential operator $L : V \to W$, say $V = H_0^2(\Omega)$ and $L = \Delta$, is there a unique solution $u \in V$ to

$$Lu = f \, ?$$

The typical procedure is to show that for f nice, say $f \in C^\infty(\Omega)$, spanning a dense subspace of W, there exists a unique solution with

$$\|u\| \leq C\|f\|.$$

Such an a priori estimate allows to solve $Lu = f$ for general $f \in W$ by approximation. This implies that L is surjective. The Open Mapping Theorem guarantees that this strategy works if L is surjective.

3.4. Closed Graph Theorem

<u>Thm</u> (Closed Graph Theorem). Let V, W be <u>Banach</u> spaces, $T: V \to W$ linear. Then T is bounded iff the graph $\Gamma = \{(v, Tv) : v \in V\} \subset V \times W$ is closed.

<u>Proof</u>. Let T be bounded and $(v_k, w_k) \subset \Gamma$ be a sequence s.t.

$$v_k \to v, \quad w_k = Tv_k \to w.$$

Since T is continuous, then $w = Tv$, i.e. $(v, w) \in \Gamma$. So Γ is closed.

Conversely, assume that Γ is closed. Since $V \times W$ is a Banach space with norm $\|(v,w)\| = \|v\| + \|w\|$, and since Γ is closed, it is also a Banach space with induced norm. The projections

$$\pi_V : \Gamma \to V, \quad (v, Tv) \mapsto v$$
$$\pi_W : \Gamma \to W, \quad (v, Tv) \mapsto Tv$$

are continuous, and π_V is also a bijection. By the Open Mapping Theorem, therefore $\pi_V^{-1} \in \mathcal{B}(V, \Gamma)$. Thus

$$T = \pi_W \circ \pi_V^{-1} \in \mathcal{B}(V, W).$$

<u>Remark</u>. As a consequence, to prove that $T: V \to W$ is bounded, if V, W are Banach spaces, it suffices to check

$$(v_k \to v, \ Tv_k \to w) \Rightarrow Tv = w \quad \text{instead of} \quad v_k \to v \Rightarrow \begin{cases} Tv_k \to w \\ w = Tv. \end{cases}$$

4. The space of continuous functions on a compact space

4.1. Normal topological spaces

Recall that a topological space X is <u>Hausdorff</u> iff for any $x, y \in X$, $x \neq y$, there exist open nbhds U of x and V of y such that $U \cap V = \emptyset$.

<u>Prop.</u> Let X be Hausdorff and $K_1, K_2 \subset X$ compact sets with $K_1 \cap K_2 = \emptyset$. Then there exist open $U_1 \supset K_1$ and $U_2 \supset K_2$ s.t. $U_1 \cap U_2 = \emptyset$.

<u>Proof.</u> For any $x \in K_1$, $y \in K_2$, let U_{xy} and V_{xy} be open nbhds s.t. $x \in U_{xy}$, $y \in V_{xy}$ and $U_{xy} \cap V_{xy} = \emptyset$ (use that X is Hausdorff).

Then $\bigcup_{x \in K_1} U_{xy} \supset K_1$. Since K_1 is compact, there are finitely many points $x_1, \dots, x_n \in K_1$ such that $\bigcup_{i=1}^{n} U_{x_i y} \supset K_1$. Set

$$U_y = \bigcup_{i=1}^{n} U_{x_i y} \quad \text{and} \quad V_y = \bigcap_{i=1}^{n} V_{x_i y}.$$

Then $U_y \cap V_y = \emptyset$ and $U_y \supset K_1$ and $y \in V_y$. Thus $\bigcup_{y \in K_2} V_y \supset K_2$. Again by compactness, there are $y_1, \dots, y_m \in K_2$ s.t. $\bigcup_{i=1}^{m} V_{y_i} \supset K_2$. Set

$$V = \bigcup_{i=1}^{m} V_{y_i} \quad \text{and} \quad U = \bigcap_{i=1}^{m} U_{y_i}.$$

These sets are open, $V \cap U = \emptyset$, and $U \supset K_1$, $V \supset K_2$.

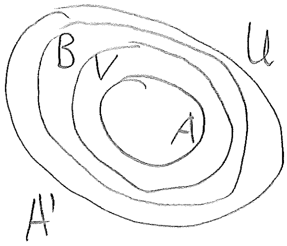
<u>Defn.</u> A topological space X is <u>normal</u> if for any closed sets $A_1, A_2 \subset X$ s.t. $A_1 \cap A_2 = \emptyset$ there exist open sets $U_1, U_2 \subset X$ s.t. $A_1 \subset U_1$, $A_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Cor. Any compact Hausdorff space is normal.

Proof. Closed subsets of a compact space are also compact. Thus the claim follows from the previous proposition.

Fact. Let X be normal. Then for every closed $A \subset X$ and open $U \supset A$, there exists an open set V and a closed set B such that

$$A \subset V \subset B \subset U.$$



Proof. Set $A' = X \setminus U$. Then A and A' are closed and disjoint. So there open sets V, V' s.t.

$$V \supset A, \quad V' \supset A', \quad V \cap V' = \emptyset.$$

Take $B = X \setminus V'$. Then $A \subset V \underset{V \cap V' = \emptyset}{\subset} B \underset{V' \supset X \setminus U}{\subset} U$.

Prop. (Urysohn's Lemma). Let X be normal. For every closed $A \subset X$ and open $U \supset A$, there exists a continuous $f : X \to [0,1]$ s.t.

$$f(x) = 1 \quad \forall x \in A$$
$$f(x) = 0 \quad \forall x \notin U.$$

Proof. Let $A_1 = A$ and $U_0 = U$. Since $A_1 \subset U_0$ there exists an open set $U_{1/2}$ and a closed set $A_{1/2}$ s.t.

$$\underbrace{A_1 \subset U_{1/2}} \subset \underbrace{A_{1/2} \subset U_0}.$$

Applying this procedure again, there are open $U_{1/4}, U_{3/4}$ and closed $A_{1/4}, A_{3/4}$ s.t.

$$\underbrace{A_1 \subset U_{3/4}} \subset \underbrace{A_{3/4} \subset U_{1/2}} \subset \underbrace{A_{1/2} \subset U_{1/4}} \subset \underbrace{A_{1/4} \subset U_0}.$$

Iterating this procedure, there are open sets U_q and closed A_q, for $q \in \{m 2^{-n} : n \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 < m < 2^n\}$ s.t. $U_q \supset U_{q'}$, $A_q \supset A_{q'}$ if $q' \leq q$,

$$U_q \subset A_q \quad \text{for all } q$$

$$A_{q'} \subset U_q \quad \text{for } q' > q.$$

Define $f(x) = \sup\{q : x \in U_q\} = \inf\{q : x \notin A_q\}$ where $\inf \emptyset = 1$, $\sup \emptyset = 0$. Clearly, $0 \leq f \leq 1$. If $x \notin U = U_0$ then $f(x) = 0$. If $x \in A = A_1$, then also $x \in U_q$ for all q and $f(x) = 1$. To show continuity, note that, for any $t \in \mathbb{R}$,

$$\{f(x) > t\} = \bigcup_{q > t} U_q \quad \text{is open,}$$

and that

$$\{f(x) < t\} = \bigcup_{q > t} X \setminus A_q \quad \text{is open.}$$

Thus the preimage of any open interval is open. Thus f is continuous.

<u>Cor.</u> Let X be normal and $A_0, A_1 \subset X$ closed and disjoint. Then there exists $f : X \to [0,1]$ continuous s.t.

$$f|_{A_0} = 0, \quad f|_{A_1} = 1.$$

<u>Proof.</u> Take $A = A_1$ and $U = X \setminus A_0$ in Urysohn's Lemma.

Cor. Let K be a compact Hausdorff space. Then $C(K)$ separates points; i.e., for all $x, y \in K$, $x \neq y$, there is $f \in C(K)$ s.t. $f(x) \neq f(y)$.

Thm (Tietze-Urysohn Extension Theorem). Let X be normal, $A \subset X$ closed, $g: A \to \mathbb{K}$ continuous. Then there exists a continuous extension $f: X \to \mathbb{K}$ s.t. $f|_A = g$ and $\|f\|_\infty \leq \|g\|_\infty$.

Proof. We first assume that g takes values in $[0,1]$. Let $g = g_0$. Let $A_0 = g^{-1}([0, \frac{1}{3}])$, $B_0 = g^{-1}([\frac{2}{3}, 1])$. Then A_0, B_0 are closed. By the corollary above, there is $h_0: X \to [0, \frac{1}{3}]$ continuous s.t.

$$h_0|_{A_0} = 0, \quad h_0|_{B_0} = \frac{1}{3}.$$

Let $g_1 = g_0 - h_0|_A$. Then $g_1(x) \in [0, \frac{2}{3}]$ for all $x \in A$.

By induction, assume that $g_i : A \to [0, (\frac{2}{3})^i]$ is given and set $A_i = g_i^{-1}([0, \frac{1}{3}(\frac{2}{3})^i])$, $B_i = g_i^{-1}([\frac{2}{3}(\frac{2}{3})^i, (\frac{2}{3})^i])$, and $h_i : X \to [0, (\frac{1}{3})(\frac{2}{3})^i]$ a continuous function with $h_i|_{A_i} = 0$ and $h_i|_{B_i} = \frac{1}{3}(\frac{2}{3})^i$. Set $g_{i+1} = g_i - h_i|_A$.

$$\Rightarrow g = g_0 = \sum_{i=0}^{\infty} h_i|_A.$$

Set $\tilde{f} = \sum_{i=0}^{\infty} h_i$. The convergence is uniform, so $\tilde{f}$ is continuous.

If g takes values in $\mathbb{R}$, we can apply the above to $\frac{1}{2} + \frac{1}{2\pi} \arctan \circ g$ which takes values in $[\frac{1}{4}, \frac{3}{4}]$ to obtain an extension $\hat{f}$.

If g takes values in $\mathbb{C}$, we can apply this to the real and imaginary part to obtain an extension $\tilde{f}$.

Finally, define

$$f(x) = \begin{cases} \tilde{f}(x) & \text{if } |\tilde{f}(x)| \leq \|g\|_\infty \\ e^{i \arg \tilde{f}(x)} \|g\| & \text{if } |\tilde{f}(x)| > \|g\|_\infty . \end{cases}$$

Then f is still an extension of g and $\|f\|_\infty \leq \|g\|_\infty$.

4.2. The Arzelà–Ascoli Theorem

When is a subset of $C(K)$ compact?

Prop./Defn. A metric space X is compact if any of the following equivalent conditions hold.

(i) X has the Heine-Borel property: any open cover of X has a finite subcover.

(ii) X is sequentially compact, i.e., any sequence in X has a convergent subsequence.

(iii) X is complete and totally bounded, i.e., for any $\varepsilon>0$ there exists a finite ε-net for X. This is a finite set $M\subset X$ s.t. for any $x\in X$ there exists $m\in M$ with $d(m,x)<\varepsilon$.

Proof that (iii) $\Rightarrow$ (ii). Let $(x_n)\subset X$ be a sequence and M_n a finite $1/n$-net for X. Let $m_1\in M_1$ be s.t. $B_1(m_1)$ contains infinitely many of the x_n. Let n_1 be the first n s.t. $x_n\in B_1(m_1)$.

Given $m_1\in M_1,\dots,m_k\in M_k$, $n_1,\dots,n_k$ s.t. $B_{1/j}(n_j)$ contains infinitely many points from $(x_n)\cap B_{1/i}(m_i)$ for all $i\le j$ and $x_{n_\ell}\in\bigcap_{j=1}^{\ell} B_{1/j}(m_j)$ for $\ell\le k$, let

$$\begin{cases} m_{k+1} \text{ be s.t. } B_{1/(k+1)}(m_{k+1}) \text{ contains inf. pts. from } (x_n)\cap\bigcap_{j=1}^{k} B_{1/j}(m_j) \\ n_{k+1} \text{ be the first } n\ge n_k \text{ s.t. } x_{n_{k+1}}\in\bigcap_{j=1}^{k+1} B_{1/j}(m_j). \end{cases}$$

It follows that

$$d(x_{n_k}, x_{n_\ell}) \le d(x_{n_k}, m_k) + d(m_k, x_{n_\ell}) \le \frac{1}{k} + \frac{1}{k} \quad \text{for } \ell \ge k.$$

$\Rightarrow (x_{n_k})_k$ is Cauchy

Since X is complete, (x_n) has a convergent subsequence.

Cor. Let X be complete. Then $Y \subset X$ is relatively compact (has compact closure) iff Y is totally bounded.

For K compact, we always assume that $C(K)$ has $\|\cdot\|_\infty$-norm, making it a Banach space.

Thm. (Arzelà–Ascoli). Let K be compact Hausdorff and $\mathcal{F} \subset C(K)$. Then the following are equivalent.

(i) $\mathcal{F}$ is relatively compact.

(ii) $\mathcal{F}$ is bounded and equicontinuous, i.e.,

$\sup_{f \in \mathcal{F}} \|f\|_\infty < \infty$ and $\forall \varepsilon > 0, x \in K, \exists$ nbhd U of x s.t. $\forall f \in \mathcal{F}$

$$|f(x) - f(y)| < \varepsilon \text{ for all } y \in U.$$

Proof. (i) $\Rightarrow$ (ii). Let $\mathcal{F}$ be relatively compact, i.e., totally bounded. Thus for any $\varepsilon > 0$ there exist $f_1, \dots, f_n \in \mathcal{F}$ s.t. $\forall f \in \mathcal{F}$

$$\min_i \|f - f_i\| < \varepsilon.$$

Thus $\|f\| \le \varepsilon + \max_i \|f_i\|$ for all $f \in \mathcal{F}$, so $\mathcal{F}$ is bounded.

Let $\varepsilon>0$, $x\in K$. Since the f_i are continuous, there exist nbhds U_i of x s.t. $|f_i(x)-f_i(y)|<\varepsilon$ for $y\in U_i$.

Let $U=\bigcap_{i=1}^{n} U_i$. Then U is a nbhd of x and for $y\in U$,

$$|f(x)-f(y)| \leq \underbrace{|f(x)-f_i(x)|}_{<\varepsilon} + \underbrace{|f_i(x)-f_i(y)|}_{<\varepsilon} + \underbrace{|f_i(y)-f(y)|}_{<\varepsilon} < 3\varepsilon.$$

for all $f\in\mathcal{F}$, where i is chosen s.t. $\|f-f_i\|<\varepsilon$.

(ii) $\Rightarrow$ (i). Let $\mathcal{F}$ be bounded and equicontinuous. For any $\varepsilon>0$, we construct a finite 3ε-net for $\mathcal{F}$. Let $\varepsilon>0$.

For $x\in K$, let U_x be an open nbhd of x s.t. $|f(x)-f(y)|<\varepsilon$ if $f\in\mathcal{F}$, $y\in U_x$. Since K is compact, there are $x_1,\dots,x_n$ s.t.

$$K=\bigcup_{i=1}^{n} U_{x_i}.$$

Since $\mathcal{F}$ is uniformly bounded, the vector $(f(x_1),\dots,f(x_n))\in\mathbb{K}^n$ is bounded in any norm on $\mathbb{K}^n$, say the $\|\cdot\|_\infty$-norm. Thus

$$F=\{(f(x_1),\dots,f(x_n))\in\mathbb{K}^n : f\in\mathcal{F}\} \text{ is relatively compact in } \mathbb{K}^n.$$

Thus there are $f_1,\dots,f_m\in\mathcal{F}$ s.t. $\{(f_i(x_1),\dots,f_i(x_n)) : 1\leq i\leq m\}$ is a finite ε-net for F.

Claim: $f_1,\dots,f_m$ are a finite 3ε-net for $\mathcal{F}$. Indeed,

$$|f(x)-f_i(x)| \leq \underbrace{|f(x)-f(x_j)|}_{<\varepsilon \text{ if } x\in U_{x_j}} + \underbrace{|f(x_j)-f_i(x_j)|}_{\substack{<\varepsilon \text{ for some}\\ i\in\{1,\dots,m\}}} + \underbrace{|f_i(x_j)-f_i(x)|}_{<\varepsilon \text{ if } x\in U_{x_j}} < 3\varepsilon.$$

Example. Let $K=[0,1]$ and $\mathcal{F}=\{f\in C^1[0,1] : \|f\|_\infty+\|f'\|_\infty\le 1\}$.
Then $\mathcal{F}$ is relatively compact: Indeed, $\mathcal{F}$ is bounded and

$$|f(x)-f(y)| \le \|f'\|_\infty |x-y| \le |x-y| < \varepsilon. \quad \text{whenever } |x-y|<\varepsilon,$$

for any $f\in\mathcal{F}$.

Since $\mathcal{F}$ is the unit ball in $C^1[0,1]$, the Arzelà-Ascoli Thm implies that the identity map $i: C^1[0,1]\to C^0[0,1]$ is a compact embedding: it is continuous and maps bounded sets to relatively compact sets.

4.3. Application: Peano Existence Theorem

Thm. (Picard-Lindelöf). Let $f: \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous. Then for any $x_0 \in \mathbb{R}$ there exists a maximal interval (T_1, T_2) with $T_1 = -\infty$ and $T_2 = +\infty$ allowed, s.t.

$$\begin{cases} x'(t) = f(x(t)) \\ x(0) = x_0 \end{cases} \qquad (*)$$

has a unique C^1 solution $x: (T_1, T_2) \to \mathbb{R}$ (that is not the restriction of such a soln to a larger interval).
Moreover, if $T_2 \neq +\infty$, for any bounded $K \subset \mathbb{R}$ there exists $t < T_2$ s.t. $x([t, T_2)) \cap K = \emptyset$ and similarly if $T_1 \neq -\infty$.

Thm (Peano). Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Then for any $x_0 \in \mathbb{R}$, there exists $\varepsilon > 0$ and a solution $x: (-\varepsilon, \varepsilon) \to \mathbb{R}$ to $(*)$.

Remark. The solution is not necessarily unique, e.g., for $f(x) = \sqrt{|x|}$.

Lemma. Assume that $b > 0$, $M > 0$ are s.t.

$$|f(x)| \leq M \quad \text{for } |x - x_0| \leq b. \qquad (@)$$

Then, if $T \leq b/M$ and x is any C^1-solution to $(*)$ for all $|t| \leq T$, it follows that $|x(t) - x_0| \leq b$, $|x'(t)| \leq M$.

Proof. Assume that x is a C^1-soln to $(*)$ for $|t| \leq T' < T$ s.t. $|x(t) - x_0| \leq b$.
$\Rightarrow |x'(t)| = |f(x(t))| \leq M$, $|x(t) - x_0| = \left|\int_0^t f(x(s))\,ds\right| \leq Mt < b$ for $|t| \leq T'$. (†)
This allows to extend the soln beyond T' by 'continuous induction':

Let $I = \{T' \in [0,T] : |x(t) - x_0| \le b$ for $|t| \le T'\}$. Note $I \ne \emptyset$ and that I is closed.

Claim: $\sup I = T$.

Otherwise $|x(t) - x_0| < b$ by (†) for $|t| \le \sup I$. But by continuity a nbhd of $\sup I$ is then also in I, a contradiction. So $I = [0,T]$.

Proof of thm. Let $B = \{f + \tilde{g} : \tilde{g} \in C^0, \|\tilde{g}\|_\infty \le 1\}$ and choose $M, b > 0$ s.t. (@) holds for all $g \in B$. For any $f \in B \cap C^1$ there is a local soln by the Picard–Lindelöf Theorem. The lemma implies that these are defined on all on $[-T,T]$ with T as in the lemma.
Define

$$
\begin{aligned}
S : B \cap C^1 &\longrightarrow C^1[-T,T] \\
f &\longmapsto x \quad \text{where } x \text{ is the soln. to } (*)
\end{aligned}
$$

By the lemma, $S(B \cap C^1)$ is bounded in $C^1[-T,T]$ with norm $\|x\|_\infty + \|\dot{x}\|_\infty$.
By Arzelà–Ascoli, the embedding $C^1[-T,T] \to C^0[-T,T]$ is compact, so $S(B \cap C^1)$ is relatively compact in $C^0[-T,T]$.

Let $f_i \in B \cap C^1$ be s.t. $f_i \to f$ in C^0, i.e., $\|f - f_i\|_\infty \to 0$. (The existence of such f_i follows from the next section.)

By relative compactness of $S(B \cap C^1)$ there is a subsequence s.t. $x_i = Sf_i$ converges to some $x \in C^0[-T,T]$ in $\|\cdot\|_\infty$-norm.

Claim: $x \in C^1[-T,T]$ and $(*)$ holds.

Since $f_i \xrightarrow{C^0} f$ and $x_i \xrightarrow{C^0} x$ (along the subseq.), also $f_i \circ x_i \xrightarrow{C^0} f \circ x$.
$\Rightarrow x_i' = f_i \circ x_i \longrightarrow f \circ x$ uniformly in $|t| \le T$. $\Rightarrow x \in C^1$ and $x' = f \circ x$.

4.4. Stone-Weierstraß Theorem

Thm (Weierstraß Approximation Theorem). The set of polynomials with real coefficients is dense in $C([a,b],\mathbb{R})$, in the uniform topology.

We will prove Stone's far reaching generalisation. But before consider the following example which we will use in the proof of the general case.

Lemma. There is a sequence of polynomials $P_n : [-1,1] \to [0,1]$ s.t.

$$P_n(s) \longrightarrow |s| \text{ uniformly in } s \in [-1,1] \text{ as } n \to \infty.$$

Proof. We use the Babylonian method to construct square roots. Note that if $q : [0,1] \to [0,1]$ is a function with

$$(*) \quad q(t) = \frac{1}{2}(t + q(t)^2)$$

$$\Rightarrow \quad (1-q(t))^2 = 1-t \Rightarrow 1-q(t) = \sqrt{1-t} \Rightarrow |t| = 1-q(1-t^2).$$

To approximate q, define polynomials $Q_n : [0,1] \to [0,1]$ by

$$Q_0(t) = 0, \quad Q_n(t) = \frac{1}{2}(t + Q_{n-1}(t)^2).$$

If $Q_n(t)$ converges to some $q(t)$ then $q(t) \in [0,1]$ and $(*)$ must hold. To show that it converges, note that, for any $t \in [0,1]$,

$$Q_{n+1}(t) \geq Q_n(t).$$

Indeed,

$$Q_{n+1}(t) - Q_n(t) = \frac{1}{2}\left(Q_n(t)^2 - Q_{n-1}(t)^2\right) = \frac{1}{2}\underbrace{\left(Q_n(t) - Q_{n-1}(t)\right)}_{\substack{\geq 0 \text{ by} \\ \text{induction}}}\underbrace{\left(Q_n(t) + Q_{n-1}(t)\right)}_{\geq 0}.$$

Also, $Q_n(t)$ is increasing in $t \in [0,1]$, and the last equality implies that also $Q_{n+1}(t) - Q_n(t)$ is increasing in $t \in [0,1]$.

$$\Rightarrow \; Q_{n+1}(t) - Q_n(t) \leq Q_{n+1}(1) - Q_n(1)$$

$$\Rightarrow \; Q_m(t) - Q_n(t) \leq Q_m(1) - Q_n(1) \quad \text{for } m \geq n, \; t \in [0,1].$$

Let $m \to \infty$:

$$0 \leq 1 - \sqrt{1-t^2} - Q_n(t) \leq 1 - Q_n(1).$$

$$\Rightarrow \; \| 1 - Q_n - \sqrt{1-t} \|_\infty \leq 1 - Q_n(1) \to 0.$$

Set $P_n(t) = 1 - Q(1-t^2)$. Then $\| P_n - |\cdot| \|_\infty \to 0$.

<u>Exercise</u>. Using the lemma, prove the Weierstraß Approximation Theorem for piecewise linear functions. Then deduce the general case.

<u>Defn</u>. (i) A real (complex) <u>algebra</u> is a real (complex) vector space A with a bilinear map $A \times A \to A$, $(a,b) \mapsto ab$ called product s.t.

$$(ab)c = a(bc) \quad \text{for all } a,b,c \in A \quad \text{(associativity)}.$$

(ii) If A is a normed vector space and

$$\|ab\| \leq \|a\| \, \|b\| \quad \text{for all } a,b \in A$$

then A is called a <u>normed algebra</u>. If A is a Banach space then A is called a <u>Banach algebra</u>.

(iii) If $ab = ba$ for all $a,b \in A$ then A is <u>commutative</u>.

(iv) If there exists an element $1 \in A \setminus \{0\}$ s.t. $1a = a1 = a$ for all $a \in A$ then A is <u>unital</u>.

<u>Example.</u>

- $C(K,\mathbb{R})$ is a commutative unital Banach algebra with product $(fg)(x) = f(x)\,g(x)$ and unit $1(x) = 1$ for all $x \in K$.
- $\mathcal{B}(V,V)$ where V is a normed vector space is a normed unital algebra with product given by composition and unit given by the identity operator. If V is a Banach space, then $\mathcal{B}(V,V)$ is a Banach algebra. It is noncommutative.

<u>Thm</u>. (Stone-Weierstraß). Let $A \subset C(K,\mathbb{R})$ be a subalgebra that

- separates points: for all $x,y \in K$, $x \neq y$, there is $f \in A$ s.t. $f(x) \neq f(y)$.
- vanishes nowhere: for all $x \in K$, there is $f \in A$ s.t. $f(x) \neq 0$.

Then A is dense in $C(K,\mathbb{R})$, i.e., $\overline{A} = C(K,\mathbb{R})$.

Example. Let $U \subset \mathbb{R}^n$ be bounded and open. Let A be the set of polynomials in $x_1, \ldots, x_n$. Then A is an algebra, separates points, and contains the constant polynomials, so vanishes nowhere. Thus the Stone-Weierstraß Theorem implies $\bar{A} = C(\bar{U})$.
In particular, $C^\infty(\bar{U})$ is dense in $C(\bar{U})$.

Defn. A lattice is a poset L with the property that for any $u, v \in L$ there is a least upper bound (join) $u \vee v$ and a greatest lower bound (meet) $u \wedge v$.

Example. $C(K, \mathbb{R})$ is a lattice with partial order

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for all } x \in K,$$

and

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

Lemma. Let $\bar{A} \subset C(K, \mathbb{R})$ be a closed subalgebra. Then $\bar{A}$ is a lattice in $C(K, \mathbb{R})$.

Proof.

$$(f \vee g)(x) = \tfrac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$
$$(f \wedge g)(x) = \tfrac{1}{2}\left(f(x) + g(x) - |f(x) - g(x)|\right)$$

Thus it suffices to show that if $f \in \bar{A}$ then $|f| \in \bar{A}$.

Let $f \in \bar{A} \setminus \{0\}$ and $\varepsilon > 0$. Replacing f by $f/\|f\|_\infty$ we may assume that f takes values in $[-1,1]$. By the lemma at the beginning of this section, there is a polynomial $P: [-1,1] \to [0,1]$ s.t.

$$\sup_{|t| \le 1} |P(t) - |t|| < \varepsilon.$$

$$\Rightarrow \|P \circ f - |f|\| < \varepsilon$$

Since $P \circ f \in \bar{A}$ and since $\bar{A}$ is closed therefore $|f| \in \bar{A}$.

<u>Lemma</u>. Let $L \subset C(K, \mathbb{R})$ be a sublattice and $g \in C(K, \mathbb{R})$ s.t.

$$\forall \varepsilon > 0 : \forall x, y \in K : \exists f \in L \text{ s.t. } |f(x) - g(x)| < \varepsilon \text{ and } |f(y) - g(y)| < \varepsilon. \quad (*)$$

Then $g \in \bar{L}$. In particular, if this condition holds for all $g \in C(K, \mathbb{R})$ then $\bar{L} = C(K, \mathbb{R})$.

<u>Proof</u>. Let $g \in C(K, \mathbb{R})$ be as in the assumption and $\varepsilon > 0$. We construct $f \in L$ s.t. $\|f - g\|_\infty < \varepsilon$.

For $x, y \in K$, let $f_{xy} = f$ be as in $(*)$. By continuity, the sets

$$U_{xy} = \{z \in K : f_{xy}(z) < g(z) + \varepsilon\}, \quad V_{xy} = \{z \in K : f_{xy}(z) > g(z) - \varepsilon\}$$

are open and $\{x, y\} \subset U_{xy} \cap V_{xy}$.

For any x, $\{U_{xy}\}_y$ is an open cover of K, so by compactness there are $y_1, \dots, y_n$ s.t. $\bigcup_{i=1}^n U_{xy_i} = K$.

Set $V_x = \bigcap_{i=1}^{n} V_{xy_i}$ and

$$f_x = f_{xy_1} \wedge \cdots \wedge f_{xy_n} \in L.$$

$$\Rightarrow \quad f_x(y) < g(y) + \varepsilon \quad \forall y \in K \qquad (<)$$

$$f_x(y) > g(y) - \varepsilon \quad \forall y \in V_x \qquad (>)$$

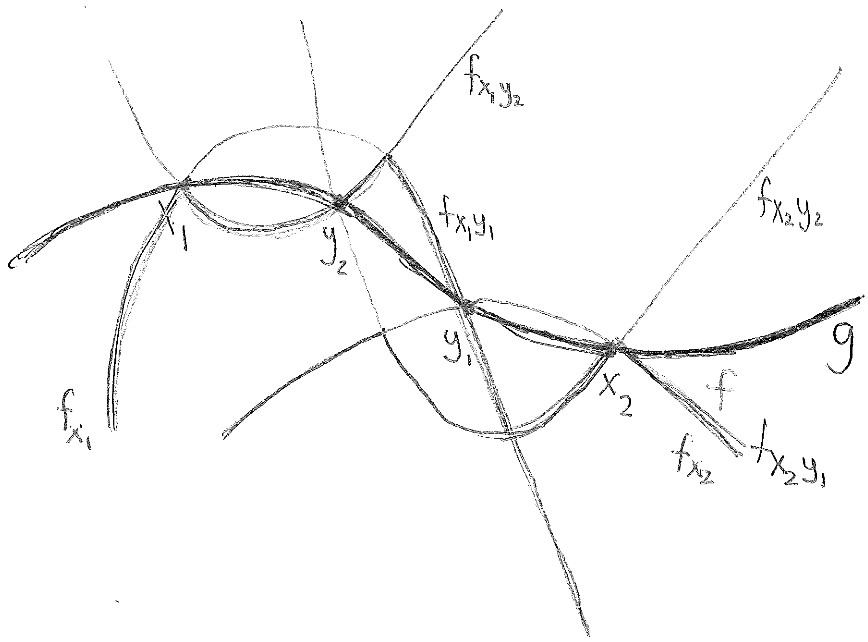
Now $\{V_x\}_x$ is an open cover for K. Choose a finite subcover $\{V_{x_j}\}_{j=1}^{m}$ by compactness of K. Set

$$f = f_{x_1} \vee \cdots \vee f_{x_m} \in L$$

$$\Rightarrow \quad f(y) < g(y) + \varepsilon \quad \forall y \in K \text{ by } (<)$$

$f(y) > g(y) - \varepsilon \quad \forall y \in K$ by $(>)$ and since $\{V_{x_j}\}$ is a cover

Thus $|f(y) - g(y)| < \varepsilon$ for all $y \in K$.



Proof of Stone-Weierstraß Theorem. By continuity of addition and multiplication, the closure $\bar{A}$ is a sublattice of $C(K,\mathbb{R})$.

We apply the last lemma. Let $g \in C(K,\mathbb{R})$, $x, y \in K$. We will find $f \in A$ s.t. $f(x) = g(x)$, $f(y) = g(y)$. In particular, (*) holds.

By assumption, A vanishes nowhere and separates points:

$\forall x \in K \; \exists f_x \in A$ s.t. $f_x(x) \neq 0$

$\forall x, y \in K, \; x \neq y, \; \exists f_{xy} \in A$ s.t. $f_{xy}(x) \neq f_{xy}(y)$.

If $x = y$, we can take $f = \dfrac{g(x)}{f_x(x)} f_x \in A$. Thus assume $x \neq y$.

Claim: There are $\alpha, \beta, \gamma \in \mathbb{R}$ s.t. $h \overset{(\dagger)}{=} \alpha f_x + \beta f_y + \gamma f_{xy}$ satisfies

$$h(x) \neq 0, \quad h(y) \neq 0, \quad h(x) \neq h(y).$$

Indeed, if $f_{xy}(x) \neq 0$ and $f_{xy}(y) \neq 0$ we can take $h = f_{xy}$. Otherwise, without loss of generality $f_{xy}(y) = 0$ and rescaling we can assume

$$f_{xy}(x) = 1 \qquad f_y(x) = C$$
$$f_{xy}(y) = 0 \qquad f_y(y) = 1$$

Take $\alpha = 0$, $\beta = 1$, $\gamma = 2 - C$ in $(\dagger)$. This gives the claim since

$$h(x) = C + \gamma = 2, \qquad h(y) = 1.$$

The claim implies that $(h(x), h(y))$ and $(h(x)^2, h(y)^2)$ are linearly independent in $\mathbb{R}^2$. Thus there are $t, s \in \mathbb{R}$ s.t.

$$(g(x), g(y)) = t(h(x), h(y)) + s(h(x)^2, h(y)^2) = (f(x), f(y))$$

with $f = th + sh^2 \in A$. Thus (*) holds for all $g \in C(K,\mathbb{R})$.

Cor. Let $U \subset \mathbb{R}^n$ be a bounded open domain. Then $C(\overline{U})$ is separable, i.e., there is a countable dense subset.

Proof. By the Weierstraß approximation theorem, the set of polynomials with real coefficients is dense in $C(\overline{U})$. But every polynomial with real coefficients can be approximated by one with rational coefficients. These are countable.

Remark. For $p \in [1, \infty)$, ℓ^p is separable, but ℓ^∞ is not.

Proof that ℓ^∞ is not separable. Let
$$X = \{(x_n) \in \ell^\infty : x_n \in \{0,1\} \text{ for all } n\}.$$
Then X is uncountable (binary expansion gives bijection with $\{x \in \mathbb{R} : 0 \le x < 1\}$).

Note that $\|x-y\|_\infty = 1$ for any $x \neq y$, $x, y \in X$. Assume that ℓ^∞ was separable, i.e. there is a dense countable subset $(y^k)_{k \in \mathbb{N}}$. Then for any $x \in X$, there is $k \in \mathbb{N}$ s.t. $\|x - y^k\| < \frac{1}{2}$ and $\|y - y^k\| > \frac{1}{2}$ for all $y \in X$, $y \neq x$. This gives a bijection $X \to \mathbb{N}$, a contradiction.

Exercise (Example Sheet 3). Let K be a compact Hausdorff space. Then $C(K)$ is separable iff K is metrisable.

4.5. Complex Stone-Weierstraß Theorem

Thm. Let $A \subset C(K, \mathbb{C})$ be a subalgebra s.t.

- A separates points
- A vanishes nowhere
- A is closed under complex conjugation, i.e., $\bar{f} \in A$ if $f \in A$.

Defn. A C^*-algebra is a complex unital Banach algebra A with antilinear involution $a \mapsto a^*$ satisfying

$$(ab)^* = b^* a^*, \quad 1^* = 1, \quad a^{**} = a, \quad (\lambda a)^* = \bar{\lambda} a^*, \quad \|a^*\| = \|a\|.$$

Example. • $C(K, \mathbb{C})$ is a commutative C^*-algebra, with $f^* = \bar{f}$

• $\mathcal{B}(H, H)$ where H is a Hilbert space is a C^*-algebra ($\to$ later).

Cor. If $A \subset C(K, \mathbb{C})$ is a C^*-subalgebra that separates points then $\bar{A} = C(K, \mathbb{C})$.

Proof of Thm. Note:

$$f \in A \Rightarrow \operatorname{Re} f = \tfrac{1}{2}(f + \bar{f}) \in A, \quad \operatorname{Im} f = \tfrac{1}{2i}(f - \bar{f}) \in A.$$

Let $A_{\mathbb{R}}$ be the subalgebra of $C(K, \mathbb{R})$ generated by $\operatorname{Re} f$, $\operatorname{Im} f$, $f \in A$. Then $A_{\mathbb{R}}$ vanishes nowhere and separates points, since A does, so the real version of the Stone-Weierstraß Theorem implies $\bar{A}_{\mathbb{R}} = C(K, \mathbb{R})$. Let $f = u + iv \in C(K, \mathbb{C})$. Then there are u_j and $v_j \in A_{\mathbb{R}}$ s.t. $u_j + iv_j \to f$. Since $u_j + iv_j \in A$, hence $\bar{A} = C(K, \mathbb{C})$.

Example (Hardy Space). The assumption that A is closed under complex conjugation is necessary. Let $K = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disc. Then

$$A = \{f \in C(\overline{\mathbb{D}}, \mathbb{C}) : f \text{ is analytic in } \mathbb{D}\}$$

is an algebra. It separates points and vanishes nowhere. But $\overline{A} \neq C(\overline{\mathbb{D}}, \mathbb{C})$ since $\bar{z} \notin A$.

4.6. Application: Convergence of Fourier Series

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle, i.e., the interval $[-\pi, \pi]$ with its endpoints identified. Let A be the vector space spanned by $\{e^{inx}\}_{n\in\mathbb{Z}}$. Its elements are called trigonometric polynomials. It is an algebra, separates points and contains the constants, so vanishes nowhere. It is also closed under complex conjugation since $\overline{e^{inx}} = e^{-inx}$. Thus $\overline{A} = C(\mathbb{T}, \mathbb{C})$ by the complex Stone-Weierstraß Theorem.

Example (Example Sheet). There exists $f \in C(\mathbb{T})$ s.t. $S_n f(0) \not\to f(0)$, where $S_n f$ is the partial Fourier sum,

$$S_n f = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}, \quad \hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx.$$

Thus the trigonometric polynomials that provide a <u>uniform</u> approximation to a given $f \in C(\mathbb{T})$ can not always be taken to be the partial Fourier sum.

However, we can deduce that the partial Fourier sum of f converges to f in L^2.

Prop. For every $f \in C(\mathbb{T})$,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f - S_n f|^2\, dx = 0.$$

Proof. By the complex Stone-Weierstraß Theorem, for any $\varepsilon>0$, there is a trigonometric polynomial $P\in A$ s.t.

$$\|P-f\|_\infty < \varepsilon.$$

Note that $S_nP = P$ if $n \ge \deg P$, where $\deg P$ is the largest n s.t. P contains $e^{\pm inx}$.

$\Rightarrow |f-S_nf| \le |f-P| + |S_nf - S_nP| \quad$ for $n \ge \deg P$

$\Rightarrow |f-S_nf|^2 \le 2|f-P|^2 + 2|S_nf-S_nP|^2 \quad$ since $(a+b)^2 \le 2a^2+2b^2$

$\Rightarrow \int_{-\pi}^{\pi} |f-S_nf|^2 \le 4\int_{-\pi}^{\pi} |f-P|^2\,dx \le 8\pi\varepsilon^2$

The first inequality follows from the next claim, which we will prove later.

Claim (Bessel's inequality). For $g\in C(\mathbb{T})$,

$$\int_{-\pi}^{\pi} |S_ng|^2\,dx \le \int_{-\pi}^{\pi} |g|^2\,dx.$$

4.7. Aside: Radon measures and the weak-$*$ topology

<u>Defn</u>. An element $\mu \in C(K)^*$ is called <u>positive</u> if $\mu(f) \geq 0$ for every $f \in C(K)$ with $f(x) \geq 0$ for all $x \in K$.

<u>Example</u>. Let $K = [0,1]$. Then

- $\mu(f) = \int_0^1 f(x)\, dx$ defines a positive functional;
- for any $x \in K$, $\delta_x(f) = f(x)$ defines a positive element in $C(K)^*$.

<u>Thm</u>. (Riesz-Markov, without proof). Let K be compact Hausdorff. Then for any positive $\mu \in C(K)^*$ there is a unique regular Borel measure such that

$$\mu(f) = \int_K f(x)\, d\mu(x) \quad \text{for all } f \in C(K).$$

<u>Example</u>. Let $x, y \in K$, $x \neq y$. Then $\|\delta_x - \delta_y\| = 2$ (use Urysohn's Theorem). In particular, that map $x \in K \mapsto \delta_x \in C(K)^*$ is not continuous in the norm-topology on $C(K)^*$.

<u>Defn</u>. Let V be a normed vector space.

- The topology induced by the sets $\Omega_{\ell,U} = \ell^{-1}(U)$, $\ell \in V^*$, $U \subset \mathbb{R}$ open is called the <u>weak topology</u> on V.
- The topology induced by $\Omega_{v,U} = f_v^{-1}(U)$, $v \in V$, $U \subset \mathbb{R}$ open is called the <u>weak-$*$ topology</u> on V^*. Here $f_v \in V^{**}$, $f_v(\ell) = \ell(v)$.

The weak topology makes V a locally convex top. vector space. It is the weakest topology that makes every bounded linear $\ell : V \to \mathbb{K}$ continuous. It does not come from a norm.

The weak-$*$ topology on V^* is the weakest topology that makes the maps $f_v \in V^{**}$ continuous for all $v \in V$.

Example. Let K be compact Hausdorff. Then the map $x \in K \mapsto \delta_x \in C(K)^*$ is continuous w.r.t. the weak-$*$ topology.

Thm. A normed vector space V is separable iff the weak-$*$ topology on the closed unit ball in V^* is metrisable.
(very similar to Example Sheet 3, Problem 9.)

Thm. (Banach-Alaoglu). Let V be a (separable) normed space. Then the closed unit ball in V^* is compact w.r.t. the weak-$*$ top.
(in separable case not very different from proof of Arzelà-Ascoli.)

Thm. Let V be a (separable) normed vector space and let K be the closed unit ball in V^* with the weak-$*$ topology. Then

$$\Phi : V \longrightarrow C(K), \quad v \mapsto f_v \quad \text{where} \quad f_v(\ell) = \ell(v)$$

is an isometric isomorphism.

Proof. That $f_v \in C(K)$ holds exactly by defn. of the weak-$*$ topology. Linearity is clear. That Φ is isometric follows from Hahn-Banach:

$$\|f_v\| = \sup_{\ell \in K} |f_v(\ell)| = \sup_{\|\ell\| \leq 1} |\ell(v)| = \|v\|.$$

Example. Let K be a compact metric space and

$$P(K) = \{ \mu \in C(K)^* : \mu \text{ is positive}, \ \mu(\underset{\uparrow}{1}) = 1 \}$$

be the set of Borel probability measures. constant function $1(x) = 1$

Let $\phi : K \to K$ be a homeomorphism. Then there exists a ϕ-invariant probability measure, i.e., $\mu \in P(K)$ s.t.

$$\mu(f) = \mu(f \circ \phi) \quad \text{for any } f \in C(K).$$

Proof. Fix $x_0 \in K$ and define $\mu_n \in P(K)$ by

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(\underbrace{\phi \circ \cdots \circ \phi}_{k \text{ times}}(x_0)) \quad \text{for } f \in C(K).$$

Since K is metric, $C(K)$ is separable. By the Banach-Alaoglu Thm, there exists a weak-$*$ convergent subsequence $(\mu_n)_{n \in \Lambda}$. Its limit satisfies

$$\mu(1) = \lim_{\substack{n \to \infty \\ n \in \Lambda}} \mu_n(1) = 1, \quad \mu(f) \geq 0 \text{ if } f \geq 0.$$

Thus $\mu \in P(K)$. Moreover, for any $f \in C(K)$,

$$\mu(f \circ \phi) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\underbrace{\phi \circ \cdots \circ \phi}_{k}(x_0)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\phi \circ \cdots \circ \phi(x_0)) = \mu(f).$$

5. Euclidean vector spaces and Hilbert spaces

5.1. Definitions and Examples

Defn. Let V be a vector space (real or complex). Then an inner product is a map $(\cdot,\cdot): V \times V \to \mathbb{K}$ s.t.

(i) $(v,w) = \overline{(w,v)}$ for all $v,w \in V$ ((skew-) symmetric)

(ii) $(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 (v_1, w) + \lambda_2 (v_2, w)$ for all $v_i, w \in V$, $\lambda_i \in \mathbb{K}$ (linear in first argument)

(iii) $(v,v) \geq 0$ with $(v,v) = 0$ iff $v = 0$ (positive definite)

A vector space together with an inner product is called an inner product space.

Remark. In the real case, $(\cdot,\cdot)$ is bilinear. In the complex case, $(\cdot,\cdot)$ is linear in the first argument and antilinear in the second argument. In physics, the usual convention is to reverse the role of the first and second argument.

Prop. (Cauchy-Schwarz inequality). Let V be an inner product space. Then

$$|(v,w)| \leq (v,v)^{1/2} (w,w)^{1/2} \text{ for all } v,w \in V$$

with equality iff $v = \lambda w$ for some $\lambda \in \mathbb{K}$.

Proof. We may assume that $(v,v)=1=(w,w)$ and that $(v,w)\geq 0$. Then, for $t>0$,

$$0\leq (v-tw,\ v-tw)=(v,v)-2t(v,w)+t^2(w,w)$$
$$=(1+t^2)-2t(v,w)$$

$$\Rightarrow (v,w)\leq \inf_{t>0}\frac{1+t^2}{2t}=1.$$

Cor. Let V be an inner product space. Then $\|v\|=(v,v)^{1/2}$ is a norm.

Proof. Positive definiteness and positive homogeneity of $\|\cdot\|$ hold by definition of $(\cdot,\cdot)$ and

$$\|\lambda v\|=(\lambda v,\lambda v)^{1/2}=(\lambda\bar{\lambda})^{1/2}(v,v)^{1/2}=|\lambda|\,\|v\|.$$

For the triangle inequality,

$$\|v+w\|^2=(v+w,\ v+w)=\underbrace{(v,v)+(w,w)}_{\|v\|^2+\|w\|^2}+\underbrace{(v,w)+(w,v)}_{2\,\mathrm{Re}(v,w)\,\leq\, 2|(v,w)|}$$

$$\leq 2\,\|v\|\,\|w\|$$

Cauchy-Schwarz

$$\leq (\|v\|+\|w\|)^2$$

$$\Rightarrow \|v+w\|\leq \|v\|+\|w\|.$$

Defn. A normed vector space is called Euclidean if its norm arises from some inner product. A Banach space that is Euclidean is called a Hilbert space.

Fact. Let V be Euclidean. Then there is a unique inner product such that $\|v\| = (v,v)^{1/2}$.

Proof. $\|v\|^2 = (v,v)$ implies

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2 + 2\mathrm{Re}(v,w).$$

If V is a real vector space, then $(v,w) = \frac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2)$.

If V is a complex vector space, then also

$$\|v+iw\|^2 = \|v\|^2 + \|w\|^2 + 2\underbrace{\mathrm{Re}(v,iw)}_{-\mathrm{Re}\, i(v,w) = +\mathrm{Im}(v,w)}$$

$$\Rightarrow (v,w) = \mathrm{Re}(v,w) + i\,\mathrm{Im}(v,w) \overset{(*)}{=} \frac{1}{2}\left(\|v+w\|^2 + i\|v+iw\|^2 - (1+i)(\|v\|^2 + \|w\|^2)\right).$$

Fact. Let V be Euclidean. Then $(\cdot,\cdot) : V \times V \to \mathbb{K}$ is continuous.

Proof. Immediate from $(*)$ and the continuity of $\|\cdot\|$.

Examples.

- $\ell^2 = \{(x_n) \subset \mathbb{K} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ is a Hilbert space with inner product $(x,y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$.

- $C[a,b]$ with norm
$$\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$$
is a Euclidean vector space but not complete. Its completion can be identified with the Lebesgue space $L^2[a,b]$.

Remark. Any metric space X can be completed. For two Cauchy sequences $(x_n) \subset X$, $(y_n) \subset X$, define
$$(x_n) \sim (y_n) \quad \text{if } d(x_n, y_n) \to 0.$$
This is an equivalence relation. Denote the equivalence class of a Cauchy sequence $x = (x_n)$ by $\bar{x}$. Define
$$\overline{X} = \{\bar{x} : x \text{ is a Cauchy sequence in } X\}.$$
Then $\bar{d}(\bar{x}, \bar{y}) = \lim_{n\to\infty} d(x_n, y_n)$ exists since $(d(x_n, y_n))_n$ is a Cauchy sequence in $\mathbb{R}$, it is well-defined, i.e., independent of the sequences representing $\bar{x}$ and $\bar{y}$, and one can also check that it is a metric on $\overline{X}$.

The metric space $\bar{X}$ is complete: if $(\bar{x}^k) \subset \bar{X}$ is a Cauchy sequence in $\bar{X}$ and $(x_n^k)_n \subset X$ is a representative for $\bar{x}^k$, choose n_k s.t. $d(x_i^k, x_j^k) \leq 2^{-k}$ for $i,j \geq n_k$. Define $x_k = x_{n_k}^k$. Then $x = (x_k) \subset X$ is a Cauchy sequence and $\bar{x}^k \to \bar{x}$ in $\bar{X}$.

The space X is isometrically embedded in $\bar{X}$ by the map mapping $x \in X$ to the equivalence class of the constant sequence $(x, x, \ldots)$. Thus we regard X as a subset of $\bar{X}$. One can check that X is then dense in $\bar{X}$.

<u>Exercise</u>. Let X be a normed vector space or a Euclidean vector space. Then the norm respectively inner product can be extended uniquely to the completion of X, making $\bar{X}$ a Banach space respectively a Hilbert space.

<u>Example</u>.

- $L^2[a,b]$ is the completion of $C[a,b]$ with inner product $(f,g) = \int_a^b f\bar{g}\,dx$. Thus its elements are equivalence classes of Cauchy sequences. However, these can be identified with equivalence classes of Lebesgue measurable functions, where two functions are equivalent if they differ on a set of Lebesgue measure 0.
- The Sobolev space $H^1[a,b]$ is the completion of $C^\infty[a,b]$ with inner product $(f,g) = \int f\bar{g}\,dx + \int f'\bar{g}'\,dx$.

5.2. Orthogonal complements and projections

Defn. Let V be a Euclidean vector space.

- $v, w \in V$ are orthogonal if $(v,w)=0$.
- the orthogonal complement of a set $S \subset V$ is
$$S^\perp = \{ v \in V : (v,w)=0 \text{ for all } w \in S \}.$$

Fact. If $v, w \in V$ are orthogonal, then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.

Fact. $S^\perp$ is a closed subspace of V and $\overline{\operatorname{span} S}^\perp = S^\perp$.

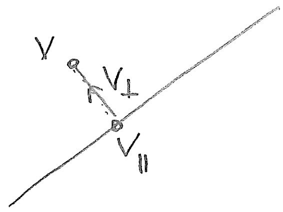
Proof. $S^\perp = \bigcap_{w \in S} f_w^{-1}(0)$ where $f_w(v) = (v,w)$ is continuous. Thus $S^\perp$ is the intersection of closed sets, so closed. Clearly,
$$S^\perp \supseteq (\operatorname{span} S)^\perp \supseteq \overline{\operatorname{span} S}^\perp.$$
On the other hand, let $v \in S^\perp$ and $w \in \overline{\operatorname{span} S}$, i.e. $w = \lim_{k\to\infty} w_k$ with $w_k \in \operatorname{span} S$. Then $(v,w) = \lim_{k\to\infty} (v, w_k) = 0$, so $v \in \overline{(\operatorname{span} S)}^\perp$.

For $W \subset V$ a subspace, $W^\perp \cap W = \{0\}$. Thus the sum $W + W^\perp$ is direct and we write $W + W^\perp = W \oplus W^\perp$.

Example. • If V is finite-dimensional, then $V = W \oplus W^\perp$.

- Let $V = C[0,1]$ with $(f,g) = \int f\bar{g}\,dx$ and $W = C^1[0,1] \subset C[0,1]$. Then $W^\perp = \{0\}$ since $\int f\bar{g}\,dx = 0 \ \forall g \in C^1$ implies $f=0$. Thus $V \neq W \oplus W^\perp$.

Thm. Let $W \subset V$ be a complete subspace. Then $W \oplus W^\perp = V$. Moreover, given $v \in V$, its unique decomposition $v = v_\parallel + v_\perp$, $v_\parallel \in W$, $v_\perp \in W^\perp$, is characterised by



$$\|v_\perp\| = \|v - v_\parallel\| = \inf_{w \in W} \|v - w\|. \qquad (*)$$

The assumptions hold in particular if V is a Hilbert space and $W \subset V$ a closed subspace.

Proof. Let $v \in V$ and $D = \inf_{w \in W} \|v - w\|$. Choose any sequence $(w_k) \subset W$ s.t.

$$\|v - w_k\| \to D.$$

Claim: (w_k) is Cauchy.

By the parallelogram identity, $\|x-y\|^2 + \|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$, with $x = v - w_j$ and $y = v - w_k$,

$$\|w_j - w_k\|^2 + \|2v - w_j - w_k\|^2 = 2\|v - w_j\|^2 + 2\|v - w_k\|^2$$

$$\Rightarrow \quad \|w_j - w_k\|^2 = 2 \underbrace{\|v - w_j\|^2}_{\to D^2} + 2 \underbrace{\|v - w_k\|^2}_{\to D^2} - 4 \underbrace{\|v - \tfrac{1}{2}(w_j + w_k)\|^2}_{\geq D^2}$$

$$\leq 4(D^2 + \varepsilon) - 4D^2 = 4\varepsilon \quad \text{for } j,k \text{ sufficiently large.}$$

By completeness of W, the claim implies $w_j \to v_\parallel$ for some $v_\parallel \in W$. By continuity of the norm, $\|v - v_\parallel\| = D$.

Claim: $v_\perp := v - v_\| \in W^\perp$

Suppose $v_\perp \notin W^\perp$. Then there must be $\tilde{w} \in W$ s.t. $(\tilde{w}, v_\perp) > 0$.

$$\Rightarrow \|v_\perp - t\tilde{w}\|^2 = \|v_\perp\|^2 - 2t(v_\perp, \tilde{w}) + t^2\|\tilde{w}\|^2$$

$$= D^2 - t\underbrace{\left(2(v_\perp, \tilde{w}) - t\|\tilde{w}\|^2\right)}_{\substack{>0 \text{ for } t>0 \\ \text{small enough}}} < D^2 \quad \text{for } t<0 \text{ small.}$$

But $\|v_\perp - t\tilde{w}\|^2 = \|v - (\underbrace{v_\| + t\tilde{w}}_{\in W})\|^2 \geq D^2$ — a contradiction.

Thus $v_\perp \in W^\perp$ and

$$v = v_\| + v_\perp, \quad v_\| \in W, \quad v_\perp \in W^\perp.$$

To show that the decomposition is characterised by (*), suppose that $v = \tilde{v}_\perp + \tilde{v}_\|$ for some $\tilde{v}_\| \in W$, $\tilde{v}_\perp \in W^\perp$. Then

$$\tilde{v}_\| = v_\| + w, \qquad w = v_\perp - \tilde{v}_\perp \in W^\perp$$

but also $w = \tilde{v}_\| - v_\| \in W$. Thus $w \in W \cap W^\perp = \{0\}$.

<u>Thm</u> (Riesz representation theorem). Let H be a <u>Hilbert space</u>. Then for any $\ell \in H^*$ there is a unique $v_\ell \in H$ s.t. $\ell(w) = (w, v_\ell)$ for all $w \in H$ and $\|\ell\| = \|v_\ell\|$.

<u>Proof</u>. Let $\ell \in H^*$, $\ell \neq 0$. Then $\ker \ell$ is closed and the previous theorem implies $H = \ker \ell \oplus (\ker \ell)^\perp$. Since $\ell \neq 0$, $(\ker \ell)^\perp \neq \{0\}$.

Claim: There is $v_0 \in H$ s.t. $(\ker \ell)^\perp = \operatorname{span}\{v_0\}$. and $\|v_0\|=1$.

Let $v_0 \in (\ker \ell)^\perp$, $\|v_0\|=1$. Then for any $w \in H$,

$$w = \underbrace{\left(w - \frac{\ell(w)}{\ell(v_0)} v_0\right)}_{\in \ker \ell} + \underbrace{\frac{\ell(w)}{\ell(v_0)} v_0}_{\in \operatorname{span}\{v_0\} \subset (\ker \ell)^\perp}$$

Claim: Define $v_\ell = \overline{\ell(v_0)}\, v_0$. Then $\ell(v) = (v, v_\ell)$ for any $v \in H$.

If $v \in \ker \ell$, then $\ell(v) = 0$ and $(v, v_\ell) = \ell(v_0)(v_0, v) = 0$.

If $v \in (\ker \ell)^\perp$, i.e. $v = \lambda v_0$, $\lambda \in \mathbb{K}$, then

$$(v, v_\ell) = \lambda\, \ell(v_0)\,(v_0, v_0) = \lambda \ell(v_0) = \ell(v).$$

Since ℓ and $(\cdot, v_\ell)$ are linear and agree on $\ker \ell$ and $(\ker \ell)^\perp$, they must also agree on $\ker \ell \oplus (\ker \ell)^\perp = H$.

Uniqueness: if $(v, v_\ell) = (v, \tilde{v}_\ell)$ for all $v \in H$, then

$$(v, v_\ell - \tilde{v}_\ell) = 0 \quad \forall v$$
$$\Rightarrow (v_\ell - \tilde{v}_\ell, v_\ell - \tilde{v}_\ell) = 0 \Rightarrow \tilde{v}_\ell = v_\ell.$$

Norm: $\|\ell\| = \sup_{\|v\| \le 1} |\ell(v)| = \sup_{\|v\| \le 1} |(v, v_\ell)| \overset{\text{take } v = \frac{v_\ell}{\|v_\ell\|}}{=} \|v_\ell\|$.

<u>Cor</u>. The antilinear map $H \to H^*$, $v \mapsto (\cdot, v)$ is bijective and isometric.

Thus H and H^* can be identified. In particular, H^* is a Hilbert space. For $\ell = (\cdot, v) \in H^*$, $\ell' = (\cdot, v') \in H^*$, the inner product on H^* is

$$(\ell, \ell') = (v', v).$$

Defn. Let V be a Euclidean vector space.

- A linear operator $P: V \to V$ is a <u>projection</u> if $P^2 = P$.
- P is an <u>orthogonal projection</u> if $P^2 = P$ and

$$(Pv, w) = (v, Pw) \quad \text{for all } v, w \in V \quad (P \text{ is self-adjoint}).$$

Fact. Let P be an orthogonal projection. Then $\|P\| = 1$ or $\|P\| = 0$.

Proof. Let $v \in V$, $Pv \neq 0$. Then

$$\|Pv\| = \frac{(Pv, Pv)}{\|Pv\|} = \frac{(v, P^2 v)}{\|Pv\|} = \frac{(v, Pv)}{\|Pv\|} \leq \|v\|.$$

$\Rightarrow \|P\| \leq 1$.

On the other hand, if $\|P\| \neq 0$, there is v s.t. $Pv \neq 0$. Let $w = Pv$.

$\Rightarrow \|Pw\| = \|Pv\| = \|w\| \Rightarrow \|P\| \geq 1$.

Cor. Let $W \subset V$ be complete (i.e. closed if V is a Hilbert space). Then there is an orthogonal projection P with

$$\operatorname{im} P = W \text{ and } \ker P = W^\perp.$$

Proof. Let $Pv = v_\parallel$ with $v_\parallel \in W$ and $v_\perp = v - v_\parallel \in W^\perp$ as in the theorem. Then P is linear since

$$v = v_\parallel + v_\perp, \quad w = w_\parallel + w_\perp, \quad v_\parallel, w_\parallel \in W, \quad v_\perp, w_\perp \in W^\perp$$

$$\Rightarrow \lambda v + \mu w = \underbrace{(\lambda v_\parallel + \mu w_\parallel)}_{\in W} + \underbrace{(\lambda v_\perp + \mu w_\perp)}_{\in W^\perp} \Rightarrow P(\lambda v + \mu w) = \lambda Pv + \mu Pw$$

by uniqueness of the orthogonal decomposition.

Clearly, $P^2 = P$ and also

$$(Pv, w) = (v_\parallel, w_\parallel + w_\perp) = (v_\parallel, w_\parallel) = (v_\parallel + v_\perp, w_\parallel) = (v, Pw)$$

so P is orthogonal.

<u>Cor.</u> Let H be a Hilbert space and $S \subset H$. Then

$$\overline{\operatorname{span} S} = (\overline{\operatorname{span} S^\perp})^\perp = (S^\perp)^\perp.$$

<u>Example.</u> Let $V = C(\mathbb{T}, \mathbb{C})$ with inner product $(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}\,dx$.
Then $S_n : V \to V$ defined by the partial Fourier sum

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}$$

is the orthogonal projection with image $W = \operatorname{span}\{e^{ikx}\}_{|k|\leq n}$.
(which is finite-dimensional and thus complete).

<u>Proof</u>. Let $e_k(x) = e^{+ikx}$. Then $\hat{f}_k = (f, e_k)$, so $S_n f = \sum_{k=-n}^{n} e_k (f, e_k)$.
If $f \in W$, i.e., $f = \sum_{|k|\leq n} a_k e_k$, then

$$S_n f = \sum_{|k|\leq n} e_k \Big(\sum_{|\ell|\leq n} a_\ell e_\ell, e_k\Big) = f \quad \text{since } (e_k, e_\ell) = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

If $f \in W^\perp$ then $(f, e_k) = 0$ for all $|k| \leq n$, so $S_n f = 0$.
$\Rightarrow S_n$ is a projection with $\operatorname{im} S_n = W$. Also,

$$(S_n f, g) = \sum \hat{f}_k (e_k, g) = \sum \hat{f}_k \overline{(g, e_k)} = \sum \hat{f}_k \bar{\hat{g}}_k = (f, S_n g),$$

so S_n is orthogonal.

5.3. Orthonormal systems

Defn. Let V be a Euclidean vector space. A set $\{e_\alpha\}_\alpha \subset V$ of unit vectors is an orthonormal system if $(e_\alpha, e_\beta) = 0$ for all $\alpha \neq \beta$. It is called maximal if it cannot be extended to a larger orthonormal system.

Defn. Let H be a Hilbert space. Then a maximal ONS is called a complete ONS or a Hilbert basis (or simply a basis).

Fact. Let H be a Hilbert space and S an orthonormal system. Then S is a Hilbert basis iff $\overline{\operatorname{span} S} = H$.

Proof. Let $W = \overline{\operatorname{span} S}$. Then W is complete, so $H = W \oplus W^\perp$. Suppose that $W^\perp \neq \{0\}$. Then there is $0 \neq v \in W^\perp = S^\perp$, i.e., $S \cup \{v\}$ is an ONS. Thus

$$\overline{\operatorname{span} S} \neq H \Leftrightarrow W^\perp \neq \{0\} \Leftrightarrow S \text{ is not maximal.}$$

Fact (Gram-Schmidt). Let V be Euclidean and $\{v_i\}_{i=1}^N \subset V$ linearly independent (with $N = \infty$ permitted). Then there is an ONS $\{e_i\}_{i=1}^N$ with $\operatorname{span}\{v_i\}_{i=1}^k = \operatorname{span}\{e_i\}_{i=1}^k$ for all $k \leq N$.

Proof. Let $e_1 = \frac{x_1}{\|x_1\|}$ and given $e_1, \dots, e_k$, set $e_{k+1} = \dfrac{x_{k+1} - \sum_{i=1}^k e_i (x_{k+1}, e_i)}{\| \text{---} '' \text{---} \|}$.

Cor. Let H be a separable Hilbert space. Then there is a countable Hilbert basis.

Thus from now on, we will always take Hilbert bases to be countable if H is separable.

Prop. (Bessel inequality). Let V be a Euclidean vector space and $\{e_i\}_{i=1}^N$ an ONS (with $N=\infty$ permitted). Then

$$\sum_{i=1}^{N} |(v,e_i)|^2 \le \|v\|^2 \quad \text{for all } v \in V.$$

In particular, if $N=\infty$, then $(x_i) \in \ell^2$ where $x_i = (x, e_i)$.

Proof. By taking a limit, we may assume that $N < \infty$. Define $Pv = \sum_{i=1}^{N} (v,e_i)\, e_i$. Then $P^2 = P$ and

$$(Pv, w) = \sum_{i=1}^{N} (v,e_i)\overline{(w,e_i)} = (v, Pw).$$

Thus P is an orthonormal projection.

$$\Rightarrow \sum_{i=1}^{N} |(x,e_i)|^2 = \|Pv\|^2 \le \|v\| \quad \text{for all } x.$$

Prop. Let H be a separable infinite-dimensional Hilbert space with Hilbert basis $\{e_i\}_{i=1}^{\infty}$. Then:

(i) For any $x \in H$, set $x_i = (x, e_i)$. Then $(x_i) \in \ell^2$ and

$$x = \sum_{i=1}^{\infty} x_i e_i$$

(ii) Conversely, if $(x_i) \in \ell^2$ then there is $x \in H$ s.t. $(x, e_i) = x_i$ for all i.

(iii) For any $x, y \in H$,

$$(x,y) = \sum_{i=1}^{\infty} x_i \overline{y_i} \quad \text{(Parseval identity)}.$$

In particular, the map

$$\phi : H \to \ell^2, \quad x \mapsto ((x, e_i))_{i=1}^{\infty}$$

is an isometric isomorphism (Riesz-Fisher Theorem).

<u>Proof</u>. (i) Let $s_n = \sum_{i=1}^{n} x_i e_i$. Then (s_n) is Cauchy: for $m \geq n$,

$$\|s_m - s_n\|^2 = \|\sum_{i=n}^{m} x_i e_i\| = \sum_{i=n}^{m} |x_i|^2 \leq \sum_{i=n}^{\infty} |x_i|^2 \to 0 \text{ as } n \to \infty$$

since $(x_i) \in \ell^2$ by Bessel's inequality. By completeness of H, there is $s \in H$ s.t. $s_n \to s$.

Claim: $s = x$

For any i, $(s - x, e_i) = \lim_{n\to\infty} (s_n - x, e_i) = x_i - x_i = 0$.

$\Rightarrow s - x \in \text{span}\{e_i\}^{\perp} = \overline{\text{span}\{e_i\}}^{\perp} = H^{\perp} = \{0\} \Rightarrow s = x$.

Thus (i) follows.

(ii) If $(x_i) \in \ell^2$, the sum $x = \sum_{i=1}^{\infty} x_i e_i$ converges by the same argument.

$\Rightarrow (x, e_i) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} x_j e_j, e_i\right) = x_i$.

(iii) Similarly,

$$(x,y) = \lim_{n\to\infty} \lim_{m\to\infty} \left(\sum_{i=1}^{n} x_i e_i , \sum_{j=1}^{m} y_j e_j \right) = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

where the infinite sum converges absolutely since $(x_i), (y_i) \in \ell^2$.

6. Spectral Theory

From now on, Banach and Hilbert spaces are always complex.

6.1. Spectrum and resolvent

Defn. Let X be a (complex) Banach space and $T \in \mathcal{B}(X)$, where from now on $\mathcal{B}(X) = \mathcal{B}(X,X)$.

- The resolvent set of T is
$$\rho(T) = \{z \in \mathbb{C} : T - z \text{ is bijective and } (T-z)^{-1} \in \mathcal{B}(X)\}.$$
- The spectrum of T is
$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$
- The resolvent of T is the map $R_T : \rho(T) \to \mathcal{B}(X)$,
$$R_T(z) = (T-z)^{-1}.$$

Remark. If T is bounded (as above) and bijective then the condition $(T-z)^{-1} \in \mathcal{B}(X)$ is automatic by the open mapping theorem. For unbounded operators, which we do not discuss, it needs to included in the definition.

Prop. Let $z_0 \in \rho(T)$. Then $\rho(T)$ contains the disk
$$D = \{z \in \mathbb{C} : |z - z_0| \, \|R_T(z_0)\| < 1\}.$$

In particular, $\varrho(T)$ is open, $\sigma(T)$ is closed. Moreover, the map $R_T : \varrho(T) \to \mathcal{B}(X)$ is analytic (can be represented by an absolutely convergent power series in any small enough disk.)

<u>Lemma.</u> Let $T \in \mathcal{B}(X)$ with $\|T\| < 1$. Then the series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{B}(X)$ and

$$(1-T)^{-1} = \sum_{n=0}^{\infty} T^n, \quad \|(1-T)^{-1}\| \leq \frac{1}{1-\|T\|}.$$

<u>Proof.</u> Let $S_n = \sum_{k=0}^{n} T^k$. Then (S_n) is a Cauchy sequence in $\mathcal{B}(X)$:

$$\| S_m - S_n \| \leq \sum_{k=n}^{\infty} \|T\|^k \to 0 \quad \text{since } \|T\| < 1.$$

Thus the limit $S = \lim_{n\to\infty} S_n$ exists and $\|S\| \leq \lim_{n\to\infty} \sum_{k=0}^{n} \|T\|^k = \frac{1}{1-\|T\|}$.

Moreover,

$$S(1-T) = \sum_{k=0}^{\infty} T^k - \sum_{k=1}^{\infty} T^k = \mathrm{id}.$$

<u>Proof of proposition.</u> For $z \in D$,

$$T - z = (T - z_0) - (z - z_0) = (T - z_0)(\mathrm{id} - \underbrace{R_T(z_0)(z - z_0)}_{\|\cdot\| < 1 \text{ for } z \in D})$$

$$\Rightarrow (\mathrm{id} - R_T(z_0)(z - z_0))^{-1} = \sum_{n=0}^{\infty} (z - z_0)^n R_T(z_0)^n \in \mathcal{B}(X)$$

$$\Rightarrow (T - z)^{-1} = \sum_{n=0}^{\infty} (z - z_0)^n R_T(z_0)^{n+1} \in \mathcal{B}(X) \Rightarrow z \in \varrho(T).$$

Thus $D \subset \varrho(T)$ and R_T is analytic on $\varrho(T)$.

Cor. $\sigma(T) \neq \emptyset$ and $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| \leq \|T\|\}$.

Proof. For any $|z| > \|T\|$,

$$R_T(z) = \frac{1}{z} \frac{1}{1 - T/z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{T^n}{z^n} \in \mathcal{B}(X).$$

Thus $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| \leq \|T\|\}$ and $\|R_T(z)\| \to 0$ as $|z| \to \infty$.

If $\sigma(T) = \emptyset$ then $R_T : \mathbb{C} \to \mathcal{B}(X)$ would be entire. By Liouville's Theorem it would have to be constant, thus 0.

But $R_T(z)$ is not identically 0, since e.g.

$$z R_T(z) \xrightarrow{|z| \to \infty} \mathrm{id}.$$

6.2. Classification of spectrum

Prop. Let X be a Banach space, Y a normed space, $T \in \mathcal{B}(X,Y)$. Then $T^{-1} \in \mathcal{B}(Y,X)$ iff $\operatorname{im} T$ is dense in Y and T is bounded below: $\exists \varepsilon > 0$ s.t. $\forall x \in X$: $\|Tx\| > \varepsilon \|x\|$.

Proof. The direction $T^{-1} \in \mathcal{B}(Y,X) \Rightarrow (\operatorname{im} T$ dense and T bounded below$)$ is clear. Thus assume that $T \in \mathcal{B}(X,Y)$ is such that $\operatorname{im} T$ is dense and T bounded below. Then T is injective, so bijective onto its image. Let $S : \operatorname{im} T \to X$ be its inverse. Since T is bounded below, S is bounded.

Since $\operatorname{im} T$ is dense in Y, for every $y \in Y$, there are $(y_k) \subset \operatorname{im} T$ s.t. $y_k \to y$. Define

$$T^{-1}y = \lim_{k\to\infty} Sy_k.$$

The limit exists since S is bounded and X is complete, $T^{-1}y$ is linear in y and independent of the approximating sequence. Moreover, $T^{-1} \in \mathcal{B}(Y, X)$,

$$TT^{-1}y = \lim_{k\to\infty} TSy_k = \lim_{k\to\infty} y_k = y,$$

as claimed.

Defn.

$\sigma_p(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not injective}\}$

i.e. there exists $x \in X$ s.t. $Tx = \lambda x$

is the point spectrum or the set of eigenvalues.

$\sigma_c(T) = \{\lambda \in \sigma(T) : T-\lambda \text{ is injective and } \operatorname{im}(T-\lambda) \text{ is dense}\}$

is the continuous spectrum.

$\sigma_r(T) = \{\lambda \in \sigma(T) : T-\lambda \text{ is injective and } \operatorname{im}(T-\lambda) \text{ is not dense}\}$

is the residual spectrum.

Remark. By the previous proposition, if $\lambda \in \sigma_c(T)$ then T is not bounded below. Thus there exists a sequence $(x_k) \subset X$ with $\|x_k\|=1$ and

$$Tx_k - \lambda x_k \longrightarrow 0 : \quad \lambda \text{ is an } \underline{\text{approximate eigenvalue}}.$$

The set

$$\sigma_{ap}(T) = \{\lambda \in \sigma(T) : \lambda \text{ is an approximate eigenvalue}\}$$

is the <u>approximate point spectrum</u>.

Example. Let X be finite-dimensional. Then equivalently

- $T-\lambda$ is injective
- $T-\lambda$ is surjective

so $\sigma(T) = \sigma_p(T)$. Moreover,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \det(T-\lambda) = 0\} \text{ contains at most } n \text{ points}.$$

In particular, $\rho(T)$ is dense in $\mathbb{C}$.

6.3. Normal linear operators

Recall. Given $T: H \to H$ we defined the adjoint $T^*: H^* \to H^*$ by $(T^*\ell)(v) = \ell(Tv)$ for any $\ell \in H^*$, $v \in H$. In a Hilbert space, $H \cong H^*$ and we can regard T^* as a map $H \to H$ by the canonical isomorphism.

Defn. $T \in \mathcal{B}(H)$ is

- normal if $TT^* = T^*T$
- self-adjoint if $T = T^*$
- unitary if $TT^* = T^*T = \mathrm{id}$

In particular, self-adjoint and unitary operators are normal.

Exercise. Let $T \in \mathcal{B}(H)$ be normal. Then $\|Tx\| = \|T^*x\|$ for all $x \in H$ and $\ker T = \ker T^* = (\operatorname{im} T)^\perp = (\operatorname{im} T^*)^\perp$ ($\to$ Example Sheet 4). Also,

$$\overline{\operatorname{im} T} = (\operatorname{im} T)^{\perp\perp} = (\ker T)^\perp = (\ker T^*)^\perp = \overline{\operatorname{im} T^*}.$$

closure

Cor. For T normal, $\sigma_r(T) = \varnothing$.

Cor. For T normal, if $Tx = \lambda x$ then $T^*x = \bar{\lambda} x$. In particular $\overline{\sigma_p(T)} = \sigma_p(T^*)$ and similarly $\overline{\sigma_c(T)} = \sigma_c(T^*)$.

complex conjugate

Proof. If T is normal, then so is $T-\lambda$ and $(T-\lambda)^* = T^* - \bar{\lambda}$. Thus

$$\|(T-\lambda)x\| = \|(T-\lambda)^* x\| = \|(T^*-\bar{\lambda})x\|$$

so $Tx = \lambda x \Leftrightarrow T^* x = \bar{\lambda} x$ and more generally $(T-\lambda)x_j \to 0 \Leftrightarrow (T^*-\bar{\lambda})x_j \to 0$.

Cor. Let T be self-adjoint. Then $\sigma(T) \subset \mathbb{R}$.

Exercise. Let T be unitary. Then $\sigma(T) \subset S^1$.

Example. Let $T \in \mathcal{B}(H)$ be self-adjoint. Then, for any $t \in \mathbb{R}$,

$$e^{itT} = \sum_{n=0}^{\infty} \frac{1}{n!}(itT)^n$$

converges in $\mathcal{B}(H)$ and $U(t) = e^{itT}$ is characterised by the ODE

$$-i\frac{\partial}{\partial t}U(t) = TU(t), \quad U(0) = \mathrm{id}.$$

Any solution to the Schrödinger equation

$$-i\frac{\partial}{\partial t}\Psi(t) = T\Psi(t), \quad \Psi(0) = \Psi_0$$

is given by $\Psi(t) = U(t)\Psi_0$. For any $t \in \mathbb{R}$, e^{itT} is unitary.

In Quantum Mechanics, T is called the Hamiltonian and Ψ the wave function.

Lemma. Let $T \in \mathcal{B}(H)$ be self-adjoint. Then

$$\|T\| \overset{(1)}{=} \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |(Tx,y)| \overset{(2)}{=} \sup_{\|x\|\leq 1} |(Tx,x)|.$$

Proof. Assume $T \neq 0$. By definition,

$$\|T\| = \sup_{\|x\|\leq 1} (Tx, Tx)^{1/2}.$$

Let $(x_i) \subset H$, $\|x_i\| = 1$ be s.t. $(Tx_i, Tx_i) \to \|T\|^2$.

$$\Rightarrow \|T\| = \frac{1}{\|T\|} \lim_{i\to\infty} (Tx_i, Tx_i) = \frac{1}{\|T\|} \lim_{i\to\infty} (x_i, T^2 x_i)$$

$$= \lim_{i\to\infty} \frac{1}{\|Tx_i\|} (x_i, T^2 x_i) = \lim_{i\to\infty} (x_i, Ty_i)$$

$$y_i = \frac{Tx_i}{\|Tx_i\|}.$$

$$\Rightarrow \|T\| \leq \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} (x, Ty).$$

On the other hand $|(x,Ty)| \leq \|T\|$ for $\|x\| \leq 1$, $\|y\| \leq 1$. $\Rightarrow$ (1).

To show (2), $\sup_{\|x\|\leq 1} |(Tx,x)| \leq \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |(Tx,y)|$ is clear.

$$|(x, Ty)| = \frac{1}{4}\Big|\big((T(x+y), x+y) - (T(x-y), x-y)\big)\Big|$$

$$\leq \frac{1}{4} \sup_{\|x\|\leq 1} |(Tx,x)| \big(\underbrace{\|x+y\|^2 + \|x-y\|^2}_{\leq 2\|x\|^2 + 2\|y\|^2} \big)$$

$$\leq \sup_{\|x\|\leq 1} |(Tx,x)| \text{ for } \|x\|, \|y\| \leq 1. \Rightarrow (2).$$

Lemma. Let T be self-adjoint. Then at least one of $\|T\|$ and $-\|T\|$ must be an approximate eigenvalue.

Proof. Replacing T by $-T$ if necessary, assume

$$\|T\| = \sup_{\|x\|\leq 1} |(x,Tx)| = \sup_{\|x\|\leq 1} (x,Tx).$$

Then there is $(x_i) \subset H$, $\|x_i\|=1$ s.t. $(x_i, Tx_i) \to \|T\| = \lambda$.

$$\Rightarrow \|Tx_i - \lambda x_i\|^2 = \underbrace{\|Tx_i\|^2}_{\leq \lambda^2} - 2\lambda \underbrace{(x, Tx_i)}_{\to \lambda} + \lambda^2 \to 0.$$

6.4. Spectral theorem for compact self-adjoint operators.

Defn. Let X, Y be normed vector spaces. Then $T \in \mathcal{B}(X,Y)$ is compact if $T(B)$ is relatively compact for any bounded $B \subset X$.

Lemma. Let T be compact. Then any nonzero approximate eigenvalue is an eigenvalue.

Proof. Assume that $Tx_i - \lambda x_i \to 0$ with $\|x_i\| = 1$, $\lambda \neq 0$. By compactness, there is a subsequence Λ s.t. $Tx_i \to y$ $(i \in \Lambda, i \to \infty)$.

$\Rightarrow Ty = T\left(\lim Tx_i\right) = \lim T(\lambda x_i) = \lambda y.$

Moreover, if $\lambda \neq 0$ then $y \neq 0$.

Cor. Let $T \in \mathcal{B}(H)$ be self-adjoint and compact. Then $\|T\|$ or $-\|T\|$ is an eigenvalue.

Notation: $E_\lambda = \ker(T - \lambda)$.

Strategy: repeat corollary with H replaced by $E_{\lambda_1}^\perp$.

Lemma. Let $T \in \mathcal{B}(H)$ be self-adjoint. Then

(i) For any eigenvalues $\lambda \neq \mu$, the spaces E_λ and E_μ are orthogonal.

(ii) For any eigenvalues $\{\lambda_i\}_{i \in I}$, $T\left(\left(\bigoplus_{i \in I} E_{\lambda_i}\right)^\perp\right) \subset \left(\bigoplus_{i \in I} E_{\lambda_i}\right)^\perp$.

Proof. (i) Assume $Tx = \lambda x$ and $Ty = \mu y$, $\lambda \neq 0$. Then

$$(x,y) = \frac{1}{\lambda}(Tx,y) = \frac{1}{\lambda}(x, Ty) = \frac{\mu}{\lambda}(x,y)$$

$\Rightarrow \mu = \lambda$ or $(x,y) = 0$.

(ii) Let $y \in \left(\bigoplus_{i\in I} E_{\lambda_i}\right)^{\perp}$. Then for any $x \in E_{\lambda_i}$,

$$0 = (x,y) = \frac{1}{\lambda_i}(Tx,y) = \frac{1}{\lambda_i}(x,Ty) \Rightarrow (x,Ty) = 0 \ \forall x \in \bigoplus_{i\in I} E_{\lambda_i}.$$

$$\Rightarrow T\left(\left(\bigoplus_{i\in I} E_{\lambda_i}\right)^{\perp}\right) \subset \left(\bigoplus_{i\in I} E_{\lambda_i}\right)^{\perp}.$$

Lemma. Let $T \in \mathcal{B}(H)$ be self-adjoint and compact.
Then, for any $\varepsilon > 0$,

$$\bigoplus_{\substack{\lambda \in \sigma_p(T) \\ |\lambda| \geq \varepsilon}} E_\lambda$$

is finite-dimensional.

Proof. Otherwise there are infinitely many eigenvectors (x_i) s.t. $\|x_i\| = 1$ and $(x_i, x_j) = 0$ for $i \neq j$ and

$$\|Tx_i - Tx_j\|^2 = \|Tx_i\|^2 + \|Tx_j\|^2 \geq 2\varepsilon.$$

This is a contradiction to compactness of T.

Thm (Hilbert-Schmidt). Let $T \in \mathcal{B}(H)$ be <u>self-adjoint</u> and <u>compact</u>. Then there are at most countably many eigenvalues (λ_i) which can only accumulate at 0. The eigenspaces E_{λ_i} and E_{λ_j} are orthogonal for $i \neq j$, E_{λ_i} is finite-dimensional for $\lambda_i \neq 0$, and

$$T = \sum_{j=0}^{\infty} \lambda_j P_{\lambda_j} \text{ in } \mathcal{B}(H), \qquad H = \ker T \oplus E_{\lambda_1} \oplus \cdots$$

where P_{λ_j} is the orthogonal projection onto E_{λ_j}.

Proof. By the previous lemmas, there is an eigenvalue λ_1 s.t. $|\lambda_1| = \|T\|$. Given $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k|$ s.t. T has no eigenvalue $> |\lambda_k|$, let $H_k = \left(\bigoplus_{i=1}^{k} E_{\lambda_i}\right)^{\perp}$. Then $H_k \subset H$ is closed, so itself a Hilbert space. Also,

$$T|_{H_k} : H_k \to H_k, \qquad \|T|_{H_k}\| \leq |\lambda_k|.$$

Thus there is an eigenvalue λ_{k+1} different from the λ_i, $i \leq k$, with $|\lambda_{k+1}| = \|T_{H_k}\| \leq |\lambda_k|$; and there is no other eigenvalue μ with $|\lambda_{k+1}| < |\mu| \leq |\lambda_k|$. This defines a sequence (λ_i) with $|\lambda_{k+1}| \leq |\lambda_k|$. Since $\bigoplus_{|\lambda_i| > \varepsilon} E_{\lambda_i}$ is finite-dimensional, the sequence can only accumulate at 0. In particular, (λ_k) is finite or countable with $\lambda_k \to 0$.

Since $T|_{\left(\overline{\bigoplus_{\lambda\neq 0} E_\lambda}\right)^\perp}$ cannot have a nonzero eigenvalue, $\left\|T|_{\left(\overline{\bigoplus_{\lambda\neq 0} E_\lambda}\right)^\perp}\right\| = 0$

and

$$H = \ker T \oplus \left(\overline{\bigoplus_{\lambda\neq 0} E_\lambda}\right).$$

$P_n x = \sum_{i=1}^n P_{\lambda_i} x$

It follows that

$$\left\|Tx - \sum_{i=1}^n \lambda_i P_{\lambda_i} x\right\| = \|Tx - TP_n x\| \le |\lambda_n| \, \|x\|$$

$$\Rightarrow \left\|T - \sum_{i=1}^n \lambda_i P_{\lambda_i}\right\| \le |\lambda_n| \longrightarrow 0 \Rightarrow T = \sum_{i=1}^\infty \lambda_i P_{\lambda_i}$$

Cor. Let H be separable and $T \in \mathcal{B}(H)$ self-adjoint and compact. Then there is an orthonormal basis for H of T eigenfunctions.

(Apply Gram-Schmidt to the eigenspaces.)

Cor. $\sigma(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\}$.

Proof. Let $T_n = \sum_{i=1}^n \lambda_i P_{E_{\lambda_i}}$. Then for $\mu \notin \sigma_p(T) \cup \{0\}$,

$$T_n - \mu = \sum_{i=1}^n (\lambda_i - \mu) P_{\lambda_i} - \mu \sum_{i=n+1}^\infty P_{\lambda_i}$$

$$\Rightarrow (T_n - \mu)^{-1} = \sum_{i=1}^n (\lambda_i - \mu)^{-1} P_{\lambda_i} - \mu^{-1} \sum_{i=n+1}^\infty P_{\lambda_i} \text{ exists}$$

$$\|(T_n - \mu)^{-1}\| \le \max\{|\mu|^{-1}, |\lambda_i - \mu|^{-1}\}.$$

$$\Rightarrow T - \mu = T_n - \mu + T - T_n = (T_n - \mu)\Big(1 + \underbrace{\underbrace{(T_n - \mu)^{-1}}_{< C}\underbrace{(T - T_n)}_{\to 0}}_{< 1}\Big)$$

$$\Rightarrow (T - \mu)^{-1} \in \mathcal{B}(H) \Rightarrow \mu \notin \sigma(T).$$

6.5 Application: Boundary value problem

Let T be the 1D Schrödinger operator acting on $C^2[a,b]$ by

$$Tu(x) = -u''(x) + V(x)\,u(x)$$

with boundary condition

$$u(a) = u(b) = 0$$

and where V is a continuous function.

(More generally, we could consider a Sturm-Liouville operator

$$Tu(x) = -\frac{d}{dx}\left(a(x)\frac{d}{dx}u(x)\right) + V(x)$$

where $a(x) > 0$ is in C^1, V is continuous, and

$$A_1 u(a) + A_2 u'(a) = 0, \quad (A_1, A_2) \neq (0,0)$$
$$B_1 u(b) + B_2 u'(b) = 0, \quad (B_1, B_2) \neq (0,0).\ )$$

Thm. There exists a continuous function (Green function) $k : [a,b]^2 \to \mathbb{R}$ s.t. the unique solution $u \in C^2[a,b]$ to the boundary value problem

$$Tu(x) = f(x), \quad f \text{ continuous}$$
$$u(a) = u(b) = 0$$

is given by $u(x) = \int_a^b k(x,y)\, f(y)\, dy$ and $k(x,y) = k(y,x)$.

Proof By the theory of linear ODE, there are C^2 functions $u_a : [a,b] \to \mathbb{R}$ and $u_b : [a,b] \to \mathbb{R}$ s.t. any solution to

$$-u''(x) + V(x)u(x) = 0, \qquad u(a) = 0$$

is a multiple of u_a and any solution to

$$-u''(x) + V(x)u(x) = 0, \qquad u(b) = 0$$

is a multiple of u_b. Define

$$k(x,y) = \begin{cases} u_a(y)\, u_b(x) & \text{if } a \le y \le x \le b \\ u_a(x)\, u_b(y) & \text{if } a \le x \le y \le b \end{cases}$$

Claim: $u(x) = \int k(x,y) f(y)$ solves $(*)$.

$$u(x) = \int_a^x u_a(y) u_b(x) f(y)\, dy + \int_x^b u_a(x) u_b(y) f(y)\, dy$$

$$\Rightarrow u'(x) = \cancel{u_a(x) u_b(x) f(x)} + \int_a^x u_a(y) u_b'(x) f(y)\, dy$$
$$\cancel{- u_a(x) u_b(x) f(x)} + \int_x^b u_a'(x) u_b(y) f(y)\, dy$$

$$\Rightarrow u''(x) = u_a(x) u_b'(x) f(x) + \int_a^x u_a(y) u_b''(x) f(y)\, dy$$
$$- u_a'(x) u_b(x) f(x) + \int_x^b u_a''(x) u_b(y) f(y)\, dy$$

$$= \underbrace{(u_a u_b' - u_a' u_b)(x)}_{=: W(x)} f(x) + V(x)\, u(x) = C f(x) + V(x) u(x)$$

$=: W(x)$ and $W'(x) = (u_a u_b'' - u_a'' u_b)(x)$
$$= V(x)(u_a u_b - u_a u_b)(x) = 0 \Rightarrow W(x) = C$$

Multiplying u_a by a constant, we can assume $C=1$. Thus

$$-u''(x) + V(x)\,u(x) = f(x).$$

Also

$$u(a) = \int_a^b k(a,y)\,f(y) = u_a(a)\int_a^b u_b(y)\,f(y) = 0$$

$$u(b) = \int_a^b k(b,y)\,f(y) = u_b(b)\int_a^b u_a(y)\,f(y) = 0\,;$$

and $k(x,y) = k(y,x)$.

<u>Lemma</u>. Let $k : [a,b]^2 \to \mathbb{R}$ be continuous. Then the integral operator defined by

$$Kf(x) = \int_a^b k(x,y)\,f(y)\,dy$$

is bounded and compact from $C[a,b], \|\cdot\|_2$ to $C[a,b], \|\cdot\|_\infty$.

<u>Proof</u>. By Cauchy-Schwarz,

$$\|Kf\|_\infty \le \sup_x \int |k(x,y)|\,|f(y)|\,dy \le \underbrace{\sup_x \left(\int |k(x,y)|^2\,dy\right)^{\frac{1}{2}}}_{\le C} \|f\|_2$$

so K is bounded. Also, $\{Kf : f \in C[a,b],\ \|f\|_2 \le 1\}$ is equicontinuous:

$$|Kf(x) - Kf(y)| \le \underbrace{\left(\int_a^b |k(x,z) - k(y,z)|^2\,dz\right)^{\frac{1}{2}}}_{\to 0 \text{ as } x \to y} \|f\|_2 \to 0$$

uniformly in f.

Thus compactness follows from the Arzelà-Ascoli Theorem.

Cor. K is compact from $C[a,b]$, $\|\cdot\|_2$ to $C[a,b]$, $\|\cdot\|_2$.

Proof. This follows from

$$\|Kf\|_2 = \left(\int_a^b |Kf(x)|^2 \, dx \right)^{1/2} \leq \sqrt{|a-b|} \, \|Kf\|_\infty .$$

Let $H = L^2[a,b]$ be the completion of $C[a,b]$, $\|\cdot\|_2$. In particular, $C[a,b]$ is dense in $L^2[a,b]$.

Fact (Example Sheet 4). Let X, Y be Banach spaces and $D \subset X$ a dense subspace. Then a bounded (compact) operator $T: D \to Y$ extends uniquely to a bounded (compact) operator $T: X \to Y$ with the same norm.

Cor. K extends uniquely to a compact self-adjoint operator $K: H \to H$. Moreover, $Kf \in C[a,b]$ for any $f \in H$.

Proof. That K is compact follows from the above fact. That K is self-adjoint follows from the symmetry of k.

That $Kf \in C[a,b]$ for any $f \in H$ follows from the fact that K is also bounded from $C[a,b]$, $\|\cdot\|_2$ to $C[a,b]$, $\|\cdot\|_\infty$ and thus from H to $C[a,b]$, $\|\cdot\|_\infty$.

By the spectral theorem, hence there exists an ONS $(f_n) \subset H$, $(\lambda_n) \subset \mathbb{R}$ with $\mu_n \to 0$ s.t.

$$Kf = \sum_{n=1}^{\infty} \mu_n (f, f_n) f_n \quad \text{in } H.$$

By the last corollary, if $\mu_n \neq 0$,

$$f_n = \frac{1}{\mu_n} K f_n \in C[a,b]$$

and in fact then $f_n \in C^2[a,b]$ since $Kf \in C^2$ if $f \in C$.

Assuming $\ker(K) = 0$, there is thus a Hilbert basis of C^2 eigenfunctions of K. With $\lambda_n = \frac{1}{\mu_n}$, then

$$T f_n = \lambda_n TK f_n = \lambda_n f_n,$$

so these f_n are also eigenfunctions of T, and $\lambda_n \to \infty$.

In QM, the vectors $f \in H$ describe the state of a system and $|f(x)|^2$ the probability density of finding a particle at $x \in [a,b]$.

6.6. Continuous functional calculus

Let $T \in \mathcal{B}(H)$ be self-adjoint and compact with spectral resolution

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda .$$

For $f \in C(\sigma(T))$ define

$$f(T) = \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda \in \mathcal{B}(H).$$

Thm. The map $\Phi : C(\sigma(T)) \longrightarrow \mathcal{B}(H)$, $f \mapsto f(T)$ is a $*$-homomorphism (a homomorphism of C^* algebras):

$$(*) \quad \begin{array}{l} \Phi(fg) = \Phi(f)\Phi(g), \quad \Phi(\lambda f) = \lambda \Phi(f), \quad \Phi(f)^* = \Phi(\bar{f}) \\ \Phi(1) = \mathrm{id} \end{array}$$

and it has the following properties:

(a) $\|\Phi(f)\| \leq \|f\|$ (in particular Φ is continuous);

(b) if $f(t) = t$ for all t then $f(T) = T$;

(c) if $Tx = \lambda x$ then $f(T)x = f(\lambda)x$;

(d) $\sigma(f(T)) = \{ f(\lambda) : \lambda \in \sigma(T) \}$;

(e) if $f \geq 0$ then $f(T) \geq 0$, i.e., $(x, f(T)x) \geq 0$ for all $x \in H$.

Remark. For any $x \in H$, $f \in C(\sigma(T)) \mapsto \mu_x(f) = (x, f(T)x)$ defines a positive linear functional on $C(\sigma(T))$, i.e. a measure. This is the spectral measure of $x \in H$.

Thm. Let $T \in \mathcal{B}(H)$ be <u>self-adjoint</u> (but not necessarily compact). Then there is a unique $*$-homomorphism $\Phi : C(\sigma(T)) \to \mathcal{B}(H)$ with all properties as in the last theorem.

<u>Idea</u>: $(*)$ and (b) determine Φ uniquely for any polynomial f. Since the set of polynomials is dense in $C(\sigma(T))$, the main task is to prove (a). Assuming (a), the existence and uniqueness follow by approximation (BLT theorem).

<u>Lemma</u>. Let $P(t) = \sum_{n=0}^{N} a_n t^n$, $P(T) = \sum_{n=0}^{N} a_n T^n$. Then (d) holds:

$$\sigma(P(T)) = \{P(\lambda) : \lambda \in \sigma(T)\} = P(\sigma(T)).$$

<u>Proof</u>: Let $\lambda \in \sigma(T)$ and write $P(t) - P(\lambda) = (t-\lambda) Q(t)$ for a polynomial Q.

$$\Rightarrow P(T) - P(\lambda) = \underbrace{(T-\lambda)}_{\text{not invertible since } \lambda \in \sigma(T)} Q(T) = Q(T)(T-\lambda)$$

$\Rightarrow P(T) - P(\lambda)$ is not invertible $\Rightarrow P(\lambda) \in \sigma(P(T))$.

Let $\mu \in \sigma(P(T))$ and let $\lambda_1, \dots, \lambda_n$ be the roots of $P(\lambda) - \mu$.

$\Rightarrow P(t) - \mu = a_N (t-\lambda_1) \cdots (t - \lambda_n)$.

If $\lambda_1, \dots, \lambda_n \notin \sigma(T)$ then

$$(P(T) - \mu)^{-1} = a_N^{-1} (T-\lambda_1)^{-1} \cdots (T-\lambda_n)^{-1} \in \mathcal{B}(H), \text{ i.e. } \mu \notin \sigma(P(T)).$$

$\Rightarrow$ some $\lambda_i \in \sigma(T) \Rightarrow \mu = P(\lambda_i)$ for some $\lambda_i \in \sigma(T)$, i.e. $\mu \in P(\sigma(T))$.

Lemma. Let $T \in \mathcal{B}(H)$ be self-adjoint. Then

$$\|T\| = \sup_{\lambda \in \sigma(T)} |\lambda|. \qquad (\dagger)$$

In particular, together with the previous lemma,

$$\|P(T)\| = \sup_{\lambda \in \sigma(T)} |P(\lambda)|.$$

Proof. We have proved before that $\|T\| \in \sigma(T)$ or $-\|T\| \in \sigma(T)$. Thus $\|T\| \leq \sup_{\lambda \in \sigma(T)} |\lambda|$. We also saw that $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$. Thus $\|T\| = \sup_{\lambda \in \sigma(T)} |\lambda|$. Now:

$$\begin{aligned}
\|P(T)\|^2 = \sup_{\|x\| \leq 1} \|P(T)x\|^2 &= \sup_{\|x\| \leq 1} (x, P(T)^* P(T) x) \\
&= \|P(T)^* P(T)\| \\
&= \|(\bar{P} P)(T)\| \\
&\overset{(\dagger)}{=} \sup_{\lambda \in \sigma(\bar{P}P(T))} |\lambda| \\
&\overset{\text{(Lemma)}}{=} \sup_{\lambda \in \sigma(T)} |\bar{P}P(\lambda)| = \left(\sup_{\lambda \in \sigma(T)} |P(\lambda)| \right)^2.
\end{aligned}$$

Proof of theorem. Let $A \subset C(\sigma(T), \mathbb{C})$ be the algebra of complex polynomials. For $P(A)$, set $\Phi(P) = P(T)$. Then

$$\|\Phi(P)\|_{\mathcal{B}(H)} = \|P\|_{C(\sigma(T))}.$$

By the complex Stone-Weierstraß Theorem, A is dense in $C(\sigma(T), \mathbb{C})$. Thus Φ extends uniquely to a map $\Phi : C(\sigma(T), \mathbb{C}) \to \mathcal{B}(H)$.

Properties (*), (0), (b) hold by construction (and approximation by polynomials). Also, (c) is clearly true if f is a polynomial and thus by approximation for $f \in C(\sigma(T), \mathbb{C})$: if $Tx = \lambda x$ then for $f_n \in A$,

$$f(T)x = \lim_{n\to\infty} f_n(T)x = \lim_{n\to\infty} f_n(\lambda)x = f(\lambda)x.$$

For (d), we have for any $f \in C(\sigma(T), \mathbb{C})$:

$$f(T)^* f(T) = \bar{f}(T) f(T) = \bar{f}f(T) = f\bar{f}(T) = f(T) f(T)^*$$

so $f(T)$ is normal (self-adjoint if f is real-valued). To show that $\sigma(f(T)) \subseteq f(\sigma(T))$ define for $\mu \notin f(\sigma(T))$ the continuous function

$$g(\lambda) = \frac{1}{\mu - f(\lambda)}, \quad \lambda \in \sigma(T).$$

$$\Rightarrow \mathrm{id} = g(T)(\mu - f(T)) \Rightarrow \mu \notin \sigma(f(T)).$$

To show $\sigma(f(T)) \subseteq f(\sigma(T))$ take $f_n \in A$ s.t. $f_n \to f$. By the previous lemma then $f_n(\sigma(T)) = \sigma(f_n(T))$. Thus for $\lambda \in \sigma(T)$ there are (x_n) with $\|x_n\| = 1$ s.t. $f_n(T)x_n - f_n(\lambda)x_n \longrightarrow 0$.

$$\Rightarrow (f(T) - f(\lambda))x_n = \underbrace{(f_n(T) - f_n(\lambda))x_n}_{\to 0} + \underbrace{(f(T) - f_n(T))}_{\to 0}x_n + \underbrace{(f_n(\lambda) - f(\lambda))}_{\to 0}x_n \to 0$$

$$\Rightarrow f(\lambda) \in \sigma(f(T)) \Rightarrow f(\sigma(T)) \subseteq \sigma(f(T)).$$

Cor. Let $T \geq 0$, i.e. $\sigma(T) \subset [0,\infty)$. Then there is $\sqrt{T} \in \mathcal{B}(H)$ s.t. $\sqrt{T}\sqrt{T} = T$.

Cor. The resolvent satisfies $\| R_T(z) \| = \dfrac{1}{\operatorname{dist}(z, \sigma(T))}$.