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Throughout the following exercises, H is a complex Hilbert space.

1. For any closed subspace $L \subset H$, show that $(L^\perp)^\perp = L$. For any set $S \subset H$, show that S has dense linear span in H iff $S^\perp = \{0\}$.

2. Given $v \in \ell^\infty$, define the multiplication operator $V : \ell^2 \to \ell^2$ by $(Vx)_n = v_n x_n$ for $x \in \ell^2$. Show that $V \in \mathcal{B}(\ell^2)$ with $\|V\| = \|v\|_\infty$. Find the eigenvalues, the approximate eigenvalues, and the spectrum of V. Show that V is compact iff $v \in c_0$, i.e., $v_n \to 0$.

3. Let U be a unitary operator on H, i.e., $U : H \to H$ is linear, invertible, and $(Uv, Uw) = (v, w)$ for all $v, w \in H$. Prove the *mean ergodic theorem* of von Neumann: for every $v \in H$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k v = P(v), \tag{+}$$

where P is the orthogonal projection from H onto the (closed) subspace of U-invariant vectors $I = \{v \in H : Uv = v\}$.

(Hint: Show that $W = \{Uv - v : v \in H\}$ is orthogonal to I. Show that (+) holds for any $v \in I \oplus \overline{W}$. Show that $H = I \oplus \overline{W}$.)

4. Let U be unitary operator on H. Show that $\sigma(U) \subset S^1$.

5. Let V be a Banach space and $T \in \mathcal{B}(V)$ with $\|T\| < 1$. Show that then $1 - T$ has a square root, i.e., there exists $S \in \mathcal{B}(V)$ with $S^2 = 1 - T$.

6. Let $\{e_n\}_{n\in\mathbb{N}} \subset H$ be a Hilbert basis for H. For $T \in \mathcal{B}(H)$, the *Hilbert–Schmidt norm* is defined by

$$\|T\|_{\mathrm{HS}} = \left(\sum_{n\in\mathbb{N}} \|Te_n\|^2 \right)^{\frac{1}{2}}.$$

Show that $\|T\|_{\mathrm{HS}} < \infty$ implies that T is compact.

7. For $K \subset \mathbb{C}$ nonempty and compact, find a Hilbert space H and $T \in \mathcal{B}(H)$ such that $\sigma(T) = K$.

8. For $T \in \mathcal{B}(H)$ *normal*, i.e., $TT^* = T^*T$, show that $\|Tv\| = \|T^*v\|$ for all $v \in H$, and conclude that $\ker(T) = \ker(T^*) = \mathrm{im}(T)^\perp = \mathrm{im}(T^*)^\perp$.

9. For $T \in \mathcal{B}(H)$ normal, show that $\sigma(T) = \sigma_{ap}(T) = \sigma_p(T) \cup \sigma_c(T)$.

10. Let $(e_n)_{n\in\mathbb{N}} \subset H$ be a Hilbert basis for H. Define $T : H \to H$ by $T(e_n) = \frac{1}{n} e_{n+1}$. Show that T is compact and that T has no eigenvalues.

11. Let $T \in \mathcal{B}(H)$ be a compact self-adjoint linear operator. For any $\lambda \in \mathbb{R} \setminus \{0\}$, show that the *Fredholm alternative* holds:

(a) Either the only solution to $Tv = \lambda v$ is $v = 0$ and given any $v_0 \in H$ there is a unique solution $v \in H$ to $Tv = \lambda v + v_0$,

(b) or there is a finite-dimensional subspace $N_\lambda \neq \{0\}$ of solutions to $Tv = \lambda v$, and given any $v_0 \in H$ the equation $Tv = \lambda v + v_0$ has a solution $v \in H$ iff v_0 is orthogonal to N_λ. Moreover, the dimension of the space of solutions is equal to that of N_λ.

12. Let V be a Banach space, $U \subseteq \mathbb{C}$ be open, and $f : U \to V$ an analytic V-valued function, in the sense for any $z_0 \in U$ there exists an open neighbourhood $N \subset U$ of z_0 such that f can be represented on N as an absolutely convergent power series: there are $f_n \in V$ such that, for $z \in N$,

$$f(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^n, \qquad \sum_{n=0}^{\infty} \|f_n\| \, |z - z_0|^n < \infty.$$

Prove *Liouville's Theorem*: if $U = \mathbb{C}$ and $\sup_{z\in\mathbb{C}} \|f(z)\| < \infty$, then f is constant.

13. Let X, Y be Banach spaces and $D \subset X$ a dense linear subspace. Show that a bounded linear operator $T : D \to Y$ can be extended uniquely to a bounded linear operator $T : X \to Y$ with the same operator norm (BLT Theorem). Show further that if $T : D \to Y$ is compact then so is its extension.