LINEAR ANALYSIS

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Throughout the following exercises, K is a compact Hausdorff space, and C(K) the space of continuous functions on K with the supremum norm.

1. Given $f \in C(K)$, find explicitly $\varphi \in C(K)^*$ such that $\|\varphi\| = 1$ and $\varphi(f) = \|f\|$.

2. Let $\mu : C(K) \to \mathbb{K}$ be a *positive* linear functional, i.e., linear and $\mu(f) \ge 0$ if $f \ge 0$. Prove that $|\mu(f)| \le \mu(1) ||f||_{\infty}$ for any $f \in C(K)$. In particular, any positive linear functional on C(K) is continuous.

3. Show that $\mu : C[0, 1] \to \mathbb{K}$ defined by the Riemann integral $\mu(f) = \int_0^1 f(x) dx$ is a positive linear functional on C[0, 1]. For $x \in [0, 1]$, show that $\delta_x : C[0, 1] \to \mathbb{K}$ defined by $\delta_x(f) = f(x)$ is a positive linear function on C[0, 1].

4. Prove Dini's Theorem: Let $(f_n) \subset C[0, 1]$ be a monotonously increasing sequence of functions, i.e., $f_{n+1}(x) \ge f_n(x)$ for all x. Suppose that $f_n(x) \to f(x)$ for all x and a continuous function $f \in C[0, 1]$. Show that then $f_n \to f$ uniformly.

5. Let $\mu \in C(K)^*$ be a positive linear functional, $(f_n) \subset C(K)$ be an increasing sequence of functions, and $f \in C(K)$. Show that if $f_n(x) \to f(x)$ for all $x \in K$, then

$$\mu(\lim_{n \to \infty} f_n) = \lim_{n \to \infty} \mu(f_n) = \sup_n \mu(f_n).$$

6. Show that C(K) is finite-dimensional iff K is a finite set.

7. Let $g : \mathbb{R} \to [0, \infty)$ be a continuous nonnegative function with $g(x) \to 0$ as $|x| \to \infty$, and let $f_n : \mathbb{R} \to \mathbb{R}$ be equicontinuous functions such that $|f_n(x)| \le g(x)$ for all $x \in \mathbb{R}$. Show that there exists a subsequence such that f_n converges uniformly along that subsequence.

8. Let *A* be a subalgebra of $C(K, \mathbb{R})$ that separates points but that is not everywhere nonvanishing. Show that there exists $x_0 \in K$ such that $\overline{A} = \{f \in C(K, \mathbb{R}) : f(x_0) = 0\}$.

9. For $f, g \in \mathbb{C}(\mathbb{T}, \mathbb{R})$, where \mathbb{T} is [0, 1] with endpoints identified, the convolution of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(x - y)g(y) \, dy.$$

Show that $C(\mathbb{T}, \mathbb{R})$ is a Banach algebra with product given by * (and the usual $\|\cdot\|_{\infty}$ norm). Prove that it is commutative and that it is not unital.

10. Show that C(K) is separable iff K is metrisable.

11. For any cover of K by open sets U_1, \ldots, U_n , show that there exists a *partition of unity* subordinate to the cover $\{U_i\}$, i.e., continuous functions $\varphi_i : K \to [0, 1]$ such that $\varphi_i(x) = 0$ for $x \notin U_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$ for every $x \in K$.

12. Let *V* be a Euclidean vector space and $T : V \to V$ a linear map. Show that (Tv, Tw) = (v, w) for all $v, w \in V$ iff ||Tv|| = ||v|| for all $v \in V$.

13. Show that a normed vector space V is Euclidean iff the parallelogram identity holds:

$$||v + w||^2 + ||v - w||^2 = 2||v||^2 + 2||w||^2$$
 for all $v, w \in V$.

14. Let *H* be a Hilbert space and $C \subset H$ a nonempty closed convex subset. Show that for any $h \in H$, there exists a unique element $h_C \in C$ such that $||h - h_C|| = \inf_{f \in C} ||f - h||$. Is this true in a general Banach space?

15. Is there a continuous surjective map $\mathbb{R} \to \ell^2$?