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1. Using the Hahn–Banach Theorem for real vector spaces proved in class, prove the following complex analogue. Let V be normed vector space over $\mathbb{C}$. For any (complex) subspace $W \subset V$, any $g \in W^*$ has an extension $f \in V^*$ such that $f|_W = g$ and $\|f\| \leq \|g\|$.

2. For $p \in [1, \infty)$, given $x \in \ell^p$, find explicitly a support functional for x, i.e., $f \in (\ell^p)^*$ with $\|f\| = 1$ and $f(x) = \|x\|_p$.

3. Let V be normed vector space and $f : V \to \mathbb{K}$ linear. Show that f is bounded iff $\ker(f)$ is closed.

4. Let X be a metric space and $A \subset Y \subset X$ be subsets. Show that if A is nowhere dense in Y then it is also nowhere dense in X.

5. For $p, q \in [1, \infty)$, $p < q$, show that ℓ^p is meagre in ℓ^q.

6. Let $f_n : [0, 1] \to \mathbb{R}$ be continuous and assume $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in [0, 1]$. Show that then f has a point of continuity (so that in fact that the set of points of continuity of f is dense in $[0, 1]$).

(Hint: Step 1. Let $P_{n,m} = \{x : |f_n(x) - f(x)| \leq 1/m\}$ and $R_m = \bigcup_n \operatorname{int}(P_{n,m})$. Show that $R = \bigcap_m R_m$ is the set of continuity points of f. Step 2. Show that R is residual, i.e., the complement of a meagre set.)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for any $x > 0$, we have $f(nx) \to 0$ as $n \to \infty$ with n in the integers. Show that then $f(x) \to 0$ as $x \to \infty$.

8. Let V be a vector space with norms $\|\cdot\|$ and $|\cdot|$ such that $|v| \leq C\|v\|$ for all $v \in V$. Show that if V is complete with respect to both norms then the norms are equivalent.

9. Let V be a Banach space and W a normed vector space. Let (T_n) be bounded linear maps $T_n : V \to W$ and $T : V \to W$ a map such that $T_n v \to Tv$ as $n \to \infty$ for every $v \in V$. Show that T is linear and bounded.

10. Let $V = \{v : [0, 1] \to \mathbb{R} \text{ continuous}\}$ with norm $\|v\| = \int_{[0,1]} |v(x)|\, dx$. Define $T_n, T : V \to \mathbb{R}$ by

$$T_n v = n \int_{[1-1/n,1]} v(x)\, dx, \qquad Tv = v(1).$$

Show that the T_n are bounded and that $T_n v \to Tv$ for every $v \in V$. Is T bounded?

11. (challenging) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function such that for every $x \in \mathbb{R}$ there exists n such that $f^{(m)}(x) = 0$ for all $m \geq n$. Prove that f is then a polynomial.

Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, the Fourier coefficients of f are defined by

$$\hat{f}_k = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(x) e^{-ikx}\, dx.$$

The n-th partial sum of the Fourier series of f is defined by

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}$$

Denote the space of (real-valued) continuous 2π-periodic functions by $C(\mathbb{T})$.

12. For any $f \in C(\mathbb{T})$, show that $\hat{f}_k \to 0$ as $|k| \to \infty$.

13. Show that

$$S_n f(x) = \frac{1}{2\pi} \int_{[-\pi,\pi]} D_n(x - y) f(y)\, dy,$$

where $D_n(x)$ is the Dirichlet kernel

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n + \frac{1}{2})x)}{\sin(\frac{1}{2}x)}.$$

14. Define $T_n : C(\mathbb{T}) \to \mathbb{R}$ by $T_n f = [S_n(f)](0)$. Show that T_n is linear and that $\|T_n\| < \infty$ for every n but that $\sup_n \|T_n\| = \infty$. Deduce that there is $f \in C(\mathbb{T})$ such that $[S_n(f)](0)$ does not have a finite limit.

15. Assume that $\sum_k |\hat{f}_k| < \infty$. Does $[S_n(f)](0)$ have a limit as $n \to \infty$ then?