Roland Bauerschmidt ⟨rb812@cam.ac.uk⟩ *Michaelmas 2017*

For a sequence $x = (x_n) \subset \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, recall the definitions of the p-norms,

$$\|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p} \quad \text{for } p \in [1, \infty), \qquad \|x\|_\infty = \sup_n |x_n|,$$

and the sequence spaces

$$\begin{aligned} \ell^p &= \{x = (x_n) \subset \mathbb{K} : \|x\|_p < \infty\}, && \text{with } \|\cdot\|_p\text{-norm, for } p \in [1, \infty], \\ c_0 &= \{x = (x_n) \subset \mathbb{K} : x_n \to 0 \text{ as } n \to \infty\}, && \text{with } \|\cdot\|_\infty\text{-norm.} \end{aligned}$$

For $p \in [1, \infty]$, we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

1. For $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, first show that $|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$. Deduce Hölder's inequality $\|xy\|_1 \leq \|x\|_p \|y\|_q$. Note that the inequality also holds for for $p, q \in [1, \infty]$.

(Hint: use that log is concave and first assume $\|x\|_p = \|y\|_q = 1$.)

2. For $p \in [1, \infty]$, prove Minkowski's inequality $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

(Hint: for $p \in (1, \infty)$, use $|x + y|^p \leq |x + y|^{p-1}|x| + |x + y|^{p-1}|y|$ and Hölder's inquality.)

3. For $p, q \in (1, \infty)$, $q > p$, show that the following inequalities hold on $\mathbb{K}^n$ and cannot be improved:

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q.$$

In particular, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on $\mathbb{K}^n$, but the constants depend on n.

4. Show that the space ℓ^p is complete for every $p \in [1, \infty]$.

5. For $p, q \in [1, \infty]$, show that $\ell^p \subset \ell^q$ if and only if $p \leq q$.

6. For $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, show that $(\ell^p)^* = \ell^q$ (i.e., that the spaces are isometrically isomorphic in a natural way).

7. Show that $c_0^* = \ell^1$ and that $(\ell^1)^* = \ell^\infty$ (i.e., that the spaces are isometrically isomorphic).

8. Show that $(\ell^\infty)^* \neq \ell^1$ (in the sense that the natural map from ℓ^1 to $(\ell^\infty)^*$ is not a bijection).

(Hint: use the Hahn–Banach theorem to construct a bounded linear function $f : \ell^\infty \to \mathbb{R}$ that is not of the form $f(x) = \sum_n x_n y_n$ for some sequence (y_n).)

9. Show that a normed vector space X is complete if and only if every absolutely convergent series is convergent. The latter means that $\sup_N \sum_{n=1}^N \|x_n\| < \infty$ implies that $\sum_{n=1}^N x_n$ converges as $N \to \infty$.

(Hint: to show that a Cauchy sequence (x_n) converges if every absolutely convergent series is convergent, first show that one may assume that $\|x_n - x_m\| \leq 2^{-\min\{n,m\}}$.)

10. For a normed vector space X and bounded linear maps $T : X \to X$ and $S : X \to X$, show that TS is bounded and that $\|TS\| \leq \|T\|\|S\|$. (Here TS is the composition of T and S.)

11. Let X be a normed vector space and define $\pi(x) = x/\|x\|$ for $x \in X \setminus \{0\}$. Either prove that then $\|\pi(x) - \pi(y)\| \leq \|x - y\|$ whenever $\|x\|, \|y\| \geq 1$, or give an example in which this inequality is violated.

12. Let $x \in c_0$ and define $X = \{y \in c_0 : |y_n| \leq |x_n|\}$. Show that X is compact in c_0.

13. Show that that space $C^1[0, 1]$ of continuously differentiable functions on $[0, 1]$ is complete in the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ but incomplete in the norm $\|f\|_\infty$.

In applications, it is often useful to consider spaces with weights. Let $(\mu_n) \subset [0, \infty)$ be a nonnegative sequence of weights. Then define

$$\|x\|_{p,\mu} = \left(\sum_n |x_n|^p \mu_n \right)^{1/p} \qquad \text{for } p \in [1, \infty),$$

and $\ell^p(\mu) = \{x = (x_n) \subset \mathbb{K} : \|x\|_{p,\mu} < \infty\}$.

14. For any sequence of weights μ, prove the Hölder inequality $\|xy\|_{1,\mu} \leq \|x\|_{p,\mu} \|y\|_{q,\mu}$ if $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

15. If $\sum_n \mu_n < \infty$, show that $\ell^p(\mu) \supset \ell^q(\mu)$ if $p \leq q$. Compare this with the case $\mu_n = 1$ in Exercise 5.

16. Let V be a topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Assume that there exists a bounded and convex neighbourhood of 0. Show that then there is also is a balanced bounded convex neighbourhood of 0. Recall that C is balanced if $\lambda C \subset C$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$.