Lemma. *Let V be a topological vector space and $C \subset V$ be a bounded convex neighbourhood of* 0*. There there exists a balanced bounded convex neighbourhood $\tilde{C} \subset V$ of* 0*.*

Proof. Let $\mathbb{D} = \{z \in \mathbb{K} : |z| < 1\}$ be the open unit disk (or interval). Let $S : \mathbb{K} \times V \to V$ be the map of scalar multiplication $S(t, v) = tv$. Since S is continuous, $S^{-1}(C)$ contains an open neighbourhood of 0 in $\mathbb{K} \times V$. Thus it contains the set $t\mathbb{D} \times U$ for some small $t > 0$ and some neighbourhood U of 0 (in V). In other words, for some small $t > 0$, $t\mathbb{D}U$ is a subset of C. You can check that it is a balanced neighbourhood of 0 (remember that $\mathbb{D} \subset \mathbb{K}$ is the unit disk).

Now take $\tilde{C}$ to be the convex hull of $t\mathbb{D}U$ (i.e., the smallest convex subset containing $t\mathbb{D}U$). Since C is convex, and since $t\mathbb{D}U$ is a subset of C, then $\tilde{C}$ is a subset of C and thus bounded. The operation of taking the convex hull also preserves balancedness and convexity, so $\tilde{C}$ is a balanced convex neighbourhood of 0. □