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Throughout the following exercises, K is always a compact Hausdorff space.

1. Using the Hahn–Banach Theorem for real vector spaces proved in class, prove the following complex analogue. Let V be normed vector space over $\mathbb{C}$. For any (complex) subspace $W \subset V$, any $g \in W^*$ has an extension $f \in V^*$ such that $f|_W = g$ and $\|f\| \leq \|g\|$.

2. Given $f \in C(K)$, find explicitly $\varphi \in C(K)^*$ such that $\|\varphi\| = 1$ and $\varphi(f) = \|f\|$.

3. Let $\mu : C(K) \to \mathbb{K}$ be a *positive* linear functional, i.e., linear and $\mu(f) \geq 0$ if $f \geq 0$. Prove that $|\mu(f)| \leq \mu(1)\|f\|_\infty$ for any $f \in C(K)$. In particular, any positive linear functional on $C(K)$ is continuous.

4. Show that $\mu : C[0,1] \to \mathbb{K}$ defined by the Riemann integral $\mu(f) = \int_0^1 f(x)\,dx$ is a positive linear functional on $C[0,1]$. For $x \in [0,1]$, show that $\delta_x : C[0,1] \to \mathbb{K}$ defined by $\delta_x(f) = f(x)$ is a positive linear function on $C[0,1]$.

5. Let $\mu \in C(K)^*$ be a positive linear functional, $(f_n) \subset C(K)$ be an increasing sequence of functions, and $f \in C(K)$. Show that if $f_n(x) \to f(x)$ for all $x \in K$, then

$$\mu(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} \mu(f_n) = \sup_n \mu(f_n).$$

6. Show that $C(K)$ is finite-dimensional iff K is a finite set.

7. Let $g : \mathbb{R} \to [0,\infty)$ be a continuous nonnegative function with $g(x) \to 0$ as $|x| \to \infty$, and let $f_n : \mathbb{R} \to \mathbb{R}$ be equicontinuous functions such that $|f_n(x)| \leq g(x)$ for all $x \in \mathbb{R}$. Show that there exists a subsequence such that f_n converges uniformly along that subsequence.

8. Let A be a subalgebra of $C(K,\mathbb{R})$ that separates points but that is not everywhere nonvanishing. Show that there exists $x_0 \in K$ such that $\bar{A} = \{f \in C(K,\mathbb{R}) : f(x_0) = 0\}$.

9. For $f, g \in \mathbb{C}(\mathbb{T},\mathbb{R})$, where $\mathbb{T}$ is $[0,1]$ with endpoints identified, the convolution of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(x-y)g(y)\,dy.$$

Show that $C(\mathbb{T},\mathbb{R})$ is a Banach algebra with product given by $*$ (and the usual $\|\cdot\|_\infty$ norm). Prove that it is commutative and that it is not unital.

10. Show that if $C(K)$ is separable then K is metrizable.

11. For any cover of K by open sets $U_1, \dots U_n$, show that there exists a *partition of unity* subordinate to the cover $\{U_i\}$, i.e., continuous functions $\varphi_i : K \to [0,1]$ such that $\varphi_i(x) = 0$ for $x \notin U_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$ for every $x \in K$.

12. Let V be a Euclidean vector space and $T : V \to V$ a linear map. Show that $(Tv, Tw) = (v, w)$ for all $v, w \in V$ iff $\|Tv\| = \|v\|$ for all $v \in V$.

13. Show that a normed vector space V is Euclidean iff the parallelogram identity holds:

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2 \quad \text{for all } v, w \in V.$$

14. Let H be a Hilbert space and $C \subset H$ a nonempty closed convex subset. Show that for any $h \in H$, there exists a unique element $h_C \in C$ such that $\|h - h_C\| = \inf_{f\in C} \|f - h\|$. Is this true in a general Banach space?

15. Is there a continuous surjective map $\mathbb{R} \to \ell^2$?