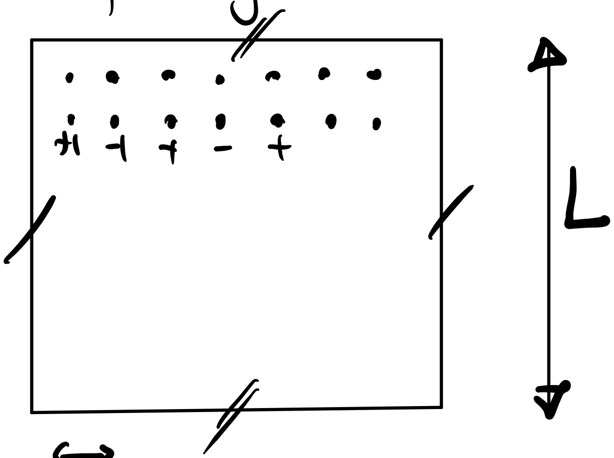
Minicourse Analysis Summer School Imperial

1. Spin systems



$\overleftrightarrow{\varepsilon} = 1$ for now

$\Lambda = \Lambda_{\varepsilon,L}$ finite
periodic boundary

Spin $\sigma_x \in \mathbb{R}^n$, $x \in \Lambda$

$$\nu(d\sigma) = \frac{1}{Z} e^{-\frac{\beta}{2}(\sigma, -\tilde{\Delta}\sigma)} \prod_{x \in \Lambda} \mu(d\sigma_x)$$

$\frac{1}{2} \sum_{x,y \in \Lambda} J_{xy} |\sigma_x - \sigma_y|^2$

single spin measure on $\mathbb{R}^n$

$J_{xy} = 1\{|x-y| = 1\}$

O(n) model
- Ising: $n=1$, $\mu = \frac{1}{2}(\delta_{+1} + \delta_{-1})$ discrete
- XY model: $n=2$, μ uniform on S^1
- Heisenberg model: $n=3$, μ uniform on S^2

Also natural to consider unbounded spins:

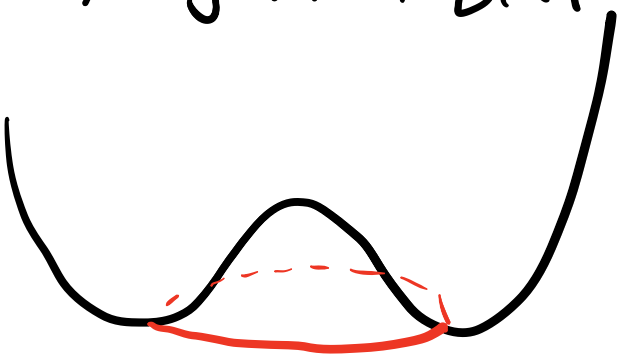
$$\nu(d\sigma) = \frac{1}{Z} e^{-H(\varphi)} \, d\varphi$$

Lebesgue on $\mathbb{R}^{n\Lambda}$

$$H(\varphi) = \frac{1}{2}(\varphi, -\Delta\varphi) + \sum_{x\in\Lambda} V(\varphi_x)$$

Ginzburg-Landau or $|\varphi|^4$ model:

$$V(\varphi) = g|\varphi|^4 + \nu|\varphi|^2 \qquad \begin{array}{l} g>0 \\ \nu<0 \end{array}$$



<u>Phase transition</u>. E.g. Ising $d \geq 2$

$$\underbrace{\sum_{y\in\Lambda} \langle \sigma_x \cdot \sigma_y \rangle}_{\parallel \; E^\nu(\sigma_x \cdot \sigma_y)} \leq C$$

uniformly in L for $\beta < \beta_c$

$$\langle \sigma_x \cdot \sigma_y \rangle \geq c > 0 \quad \text{when } \beta > \beta_c$$

$x, y \in \Lambda$

Much more subtle for XY and Heisenberg models.

<u>Glauber dynamics</u>. Markov process with 'canonical' Dirichlet form:

$$D(F) = \mathbb{E}^\nu |\nabla F|^2 = \sum_{x\in\Lambda} \mathbb{E}^\nu |\nabla_{\sigma_x} F|^2$$

E.g. Ising $\nabla_{\sigma_x} F(\sigma) = F(\sigma^x) - F(\sigma)$

<u>Kawasaki dynamics</u>. exchange spins at x and y

Very good understanding for $\beta \ll 1$ (Zegarlinski, Stroock, Martinelli, Yau, ..)

Major problems: what happens near β_c (and also for $\beta \gg 1$)

spins become strongly correlated

Log-Sobolev constant:

$$\mathrm{Ent}_\nu F \le \frac{2}{\gamma} \mathbb{E}^\nu |\nabla F|^2$$

$\mathbb{E}^\nu \Phi(F) - \Phi(\mathbb{E}^\nu F)$, $\Phi = x \log x$.

$= H(F\nu | \nu)$

Thm. For $O(n)$ model, if $\|\underbrace{\beta\Delta}_{A}\| < n$ then $\gamma \geq \gamma_0 > 0$.

Proof. Recall the measure is

$$\nu(d\sigma) = \frac{1}{Z} e^{-\frac{1}{2}(\sigma, A\sigma)} \prod_{x\in\Lambda} d\sigma_x$$

uniform on S^{n-1}

Since $|\sigma_x| = 1$, may replace A by $A + \varepsilon\,\mathrm{id}$.

Since $\|A\| < c < n$ there is a pos.-def. B s.t.

$$A^{-1} = c^{-1}\mathrm{id} + B^{-1}$$

$$\Rightarrow e^{-\frac{1}{2}(\sigma, A\sigma)} = C \int_{\mathbb{R}^{n\Lambda}} e^{-\frac{c}{2}(\varphi-\sigma, \varphi-\sigma)} e^{-\frac{1}{2}(\varphi, B\varphi)} d\varphi$$

Exercise: sum of ind. Gaussians are Gaussian

Define:

$$e^{-V(\psi)} = \int_{S^{n-1}} e^{-\frac{c}{2}|\psi-\sigma|^2} d\sigma, \quad \psi \in \mathbb{R}^n$$

$$\mu_\psi(d\sigma) = e^{+V(\psi)} e^{-\frac{c}{2}|\psi-\sigma|^2}, \quad \sigma \in S^{n-1}$$

$$\nu_r(d\varphi) = e^{-\frac{1}{2}(\varphi, B\varphi) - \sum_{x\in\Lambda} V(\varphi_x)} d\varphi, \quad \varphi \in \mathbb{R}^{n\Lambda}$$

$$\Rightarrow \mathbb{E}^{\nu} F = \mathbb{E}^{\nu_r} \mathbb{E}^{\mu_\varphi} F$$

field φ
convex for $c<n$

$\mu_\varphi = \bigotimes_{x\in\Lambda} \mu_{\varphi_x}$ product!

① μ_φ is product, so LSI for each μ_{φ_x} with general φ_x implies uniform LSI for μ_φ:

$$\mathrm{Ent}_{\mu_{\varphi_x}} F \le \frac{2}{\gamma_0} \mathbb{E}^{\mu_{\varphi_x}} |\nabla_{\sigma_x} \sqrt{F}|^2$$

indep. of φ_x gradient on S^{n-1}

$$\Rightarrow \mathrm{Ent}_{\mu_\varphi} F \le \frac{2}{\gamma_0} \mathbb{E}^{\mu_\varphi} |\nabla \sqrt{F}|^2$$

② For $c<n$, V is convex:

$c<n$

$$(x, \mathrm{Hess}\, V(\varphi)\, x) = c|x|^2 - c^2 \underbrace{\mathrm{Var}_{\mu_\varphi}(x\cdot\sigma)}_{\le |x|^2/n} \ge \lambda |x|^2$$

Bakry-Émery applies:

$$\mathrm{Ent}_{\nu_r} G \le \frac{2}{\lambda} \mathbb{E} |\nabla \sqrt{G}|^2$$

gradient on $\mathbb{R}^{n\Lambda}$

Combining both gives

$$\mathrm{Ent}_\nu F = \mathbb{E}^{\nu_r} \underbrace{\mathrm{Ent}_{\mu_\varphi}}_{(1)} F(\sigma) + \underbrace{\mathrm{Ent}_{\nu_r}}_{(2)} G(\varphi)$$

$G = \mathbb{E}_{\mu_\varphi} F(\sigma)$

$$\leq \frac{2}{\gamma_0} \underbrace{\mathbb{E}^{\nu_r} \mathbb{E}^{\mu_\varphi} |\nabla \sqrt{F}|^2}_{\mathbb{E}^{\nu} |\nabla \sqrt{F}|^2} + \frac{2}{\lambda} \mathbb{E}^{\nu_r} |\nabla \sqrt{G}|^2$$

$$|\nabla_{\varphi_x} \sqrt{G}|^2 = \left|\frac{\nabla_{\varphi_x} G}{2\sqrt{G}}\right|^2 = \left|\frac{c}{2} \frac{\mathrm{Cov}_{\mu_{\varphi_x}}(F, \sigma_x)}{\sqrt{G}}\right|^2$$

$|\sigma| \leq 1$

C.S.

$$\leq \frac{c^2}{4} \, 8 \, \mathrm{Var}_{\mu_\varphi} \sqrt{F} \leq \frac{2c^2}{\gamma_0} \mathbb{E}^{\nu} |\nabla \sqrt{F}|^2$$

spectral gap for μ_φ (implied by LSI).

<u>Remarks</u>. • Proof involves two scales: microscopic & macroscopic

- Mean-field theory:

  Λ completo graph on $\{1, \dots, N\}$

  $J_{xy} = \frac{1}{N}$

  Condition $\beta \|\Delta^J\| < n$ holds up to $\beta_c = \frac{1}{n}$ and one can also analyse the low temperature phase using this strategy.

- Condition $\beta \|\Delta^J\| < n$ does not use positivity of J and is effective in spin glass situation:

  SK model: $J_{xy} = \frac{1}{\sqrt{N}} H_{xy}$

  i.i.d. Gaussians

- Proof extends to unbounded potentials.

- Condition does not hold up to β_c when there is interesting geometry: there are more than two scales that are important.

2. Gaussian integration

Let $C_s = \int_0^s \dot{C}_{s'}\, ds'$, $\dot{C}_{s'}$ pos.-def. matrices

P_{C_s} = Gaussian measure with covariance C_s.

Example. $\dot{C}_s = e^{-sA} \Rightarrow C_\infty = A^{-1}$

$$A = -\underset{\substack{\uparrow \\ \text{graph Laplacian}}}{\Delta^\Lambda} \Rightarrow \dot{C}_s = e^{s\Delta^\Lambda} = \text{graph heat kernel}$$

Defn.

$$\underset{\substack{\uparrow \\ \text{operator on } \mathbb{R}^\Lambda}}{\Delta_{\dot{C}_s}} = \sum_{x,y\in\Lambda} \dot{C}_s(x,y) \frac{\partial}{\partial\varphi_x}\frac{\partial}{\partial\varphi_y} = (\nabla, \dot{C}_s \nabla)$$

$$= (\nabla, e^{-sA} \nabla)$$

Rk. Let $g_s^{-1} = \dot{C}_s$. Then $\Delta_{\dot{C}_s} = \Delta_{g_s}$ is the Laplace-Beltrami operator on $\mathbb{R}^\Lambda$ with metric g_s.

$$|f|_{g_s} \le 1 \iff |e^{+sA/2} f|_2 \le 1$$

$$\iff f = \underset{\substack{\uparrow \\ \text{heat kernel}}}{e^{-sA/2}} f_0, \quad |f_0|_2 \le 1$$

The unit ball consists of functions smooth at scale $\sqrt{s}$.

<u>Prop</u>. $\frac{\partial}{\partial s} P_{C_s} = \frac{1}{2}\Delta_{\dot{C}_s} P_{C_s}$

<u>Proof</u>.
$$\frac{\partial}{\partial s} \frac{e^{-\frac{1}{2}(\varphi, C_s^{-1}\varphi)}}{Z_{C_s}} = \frac{1}{2} \overbrace{(C_s^{-1}\varphi, \dot{C}_s C_s^{-1}\varphi)}^{(C_s^{-1}\varphi)^2_{\dot{C}_s}} P_{C_s}(\varphi) - (\text{const.}) P_{C_s}(\varphi)$$

$$\frac{1}{2}\Delta_{\dot{C}_s} P_{C_s} = \frac{1}{2}(C_s^{-1}\varphi)^2_{\dot{C}_s} P_{C_s}(\varphi) - \text{const.}\, P_{C_s}(\varphi)$$

$$\Rightarrow \left(\frac{\partial}{\partial s} - \frac{1}{2}\Delta_{\dot{C}_s}\right) P_{C_s} = (\text{const.}) P_{C_s}$$

Since the integral of the LHS is 0, the constant is 0.

<u>Cor</u>. Let $F_s = P_{C_s} * F_0$, i.e., $F_s(\varphi) = \mathbb{E}_{C_s}(F_0(\varphi + \zeta))$

$$\Rightarrow \partial_s F_s = \frac{1}{2}\Delta_{\dot{C}_s} F_s$$

$$\mathbb{E}_{C_s} F_0 = F_s(0)$$

<u>Exercise</u>. Let Q be a polynomial in φ. Define

$$:Q:_{C_s} = e^{-\frac{1}{2}\Delta_{C_s}} Q \qquad \text{(Wick ordering)}$$

backwards in time!

Then $e^{\frac{1}{2}\Delta_{C_s}} :Q:_{C_s} = Q$ and thus

$$\mathbb{E}_{C_s} :Q:_{C_s} = Q(0)$$

Also, if Q_1 and Q_2 are homogeneous polynomials with $\deg Q_1 \neq \deg Q_2$, then

$$\mathbb{E}_{C_s}\left(:Q_1:_{C_s} \; :Q_2:_{C_s} \right) = 0.$$

3. Renormalised potential

Given some potential $V_0 : \mathbb{R}^\Lambda \to \mathbb{R}$, the renormalised potential is defined by

$$e^{-V_s(\varphi)} = \left(P_{C_s} * e^{-V_0}\right)(\varphi)$$

$$= \mathbb{E}_{C_s}\left(e^{-V_0(\varphi+\zeta)}\right)$$

$$\Leftrightarrow \frac{\partial}{\partial s}\left(e^{-V_s}\right) = \frac{1}{2}\Delta_{\dot{C}_s}\left(e^{-V_s}\right)$$

$(u,v)_{\dot{C}_s} = \sum_{x,y} \dot{C}_s(x,y)\, u_x v_y$

$$\Leftrightarrow \frac{\partial}{\partial s} V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$$

Hamilton-Jacobi

Polchinski equation

<u>Prop.</u> Suppose that

$$\frac{\partial}{\partial s} V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$$

$$\frac{\partial}{\partial s} F_s = \frac{1}{2}\Delta_{\dot{C}_s} F_s - (\nabla V_s, \nabla F_s)_{\dot{C}_s} = L_s F_s$$

Then the following integral is independent of s:

$$\int \underbrace{P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}}_{\nu^s(\varphi)} F_s(\varphi)\, d\varphi$$

Proof. Note that $Z_s(\varphi) = e^{-V_s(\varphi)} F_s(\varphi)$ satisfies

$$\frac{\partial}{\partial s} Z_s = \frac{1}{2} \Delta_{\dot{C}_s} Z_s$$

$$\Rightarrow \frac{\partial}{\partial s} \int P_{C_\infty - C_s}(\varphi) Z_s(\varphi)\, d\varphi$$

$$= \int \left[\left(-\frac{1}{2} \Delta_{\dot{C}_s} P_{C_\infty - C_s} \right) Z_s + P_{C_\infty - C_s} \left(\frac{1}{2} \Delta_{\dot{C}_s} Z_s \right) \right]$$

$$= 0$$

Renormalised measure:

$$\nu^s(d\varphi) = P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}\, d\varphi$$

Polchinski semigroup:

$$P_{s',s} F(\varphi) = e^{+V_s(\varphi)} \mathbb{E}_{C_s - C_{s'}}\left(e^{-V_{s'}(\varphi+\zeta)} F(\varphi+\zeta)\right)$$

$$\Rightarrow \mathbb{E}^{\nu^0} F = \mathbb{E}^{\nu^s} P_{0,s} F$$

<u>Exercise</u>: $\frac{\partial}{\partial s} \mathbb{E}^{\nu^s} F = -\mathbb{E}^{\nu^s} L_s F$

$$\frac{\partial}{\partial s} P_{s',s} F = L_s P_{s',s} F$$

$$\frac{\partial}{\partial s'} P_{s',s} F = -P_{s',s} L_{s'} F$$

<u>Exercise</u>. $\mathrm{Hess}\, V_0 \geq 0 \Rightarrow \mathrm{Hess}\, V_s \geq 0 \;\; \forall s \geq 0$.

<u>Proof 1</u>.

$$e^{-V_s(\varphi)} \propto \int_{\mathbb{R}^\Lambda} \underbrace{e^{-\frac{1}{2}(\zeta, C_s^{-1}\zeta) - V_0(\varphi+\zeta)}}_{\text{log-concave in } (\varphi,\zeta)} d\zeta$$

Brascamp-Lieb ineq.: marginals of log-concave measures are log-concave

<u>Proof 2</u>. $\partial_s V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$

$$\Rightarrow \partial_s \nabla V_s = \frac{1}{2}\Delta_{\dot{C}_s}\nabla V_s - (\text{Hess}\, V_s, \nabla V_s)_{\dot{C}_s}$$

$$\Rightarrow \partial_s \text{Hess}\, V_s = \frac{1}{2}\Delta_{\dot{C}_s}\text{Hess}\, V_s - (\nabla \text{Hess}\, V_s, \nabla V_s)_{\dot{C}_s} - (\text{Hess}\, V_s, \text{Hess}\, V_s)_{\dot{C}_s}$$

$$= L_s \text{Hess}\, V_s - \text{Hess}\, V_s \dot{C}_s \text{Hess}\, V_s$$

Suppose $\Lambda = \{0\}$. Then $f_s = V_s''$ satisfies

$$\partial_s f_s = L_s f_s - f_s^2, \qquad f_s = V_s''$$

Maximum principle $f \geq 0 \Rightarrow L_s f \geq 0$

$\Rightarrow \partial_s f_s \geq -f_s^2$ so if $f_0 \geq 0 \Rightarrow f_s \geq 0$

General Λ: Maximum principle for symmetric tensors.

Summary: two ways to evolve a measure

$d\nu_t = F_t \, d\nu_\infty$ with $\partial_t F_t = \Delta F_t - (\nabla H, \nabla F_t)$

ν_∞: e^{-H} measure in equilibrium

Glauber semigroup tends to invariant measure ν_∞

$F \, d\nu^0 \to F^s \, d\nu^s$ with $d\nu^0 = d\nu_\infty$

$$\partial_s F^s = \Delta_{\dot{C}_s} F^s - (\nabla V_s, \nabla F^s)_{\dot{C}_s} = L_s F^s$$

Polchinski semigroup tends to $\nu^\infty = \delta_0$

We are interested in Glauber dynamics, i.e. in the corresponding Log-Sobolev constant

$$\mathrm{Ent}_\nu F \le \frac{2}{\gamma} \mathbb{E}^\nu |\nabla \sqrt{F}|^2 \qquad \nu = \nu_\infty = \nu^0$$

$$\mathrm{Ent}_\nu F = \mathbb{E}^\nu \Phi(F) - \Phi(\mathbb{E}^\nu F), \qquad \Phi(x) = x \log x$$

$$\nu(d\varphi) = e^{-H(\varphi)}\, d\varphi = e^{-\frac{1}{2}(\varphi, A\varphi) - V_0(\varphi)}\, d\varphi$$

<u>Thm</u>. (Bakry-Émery). Assume $A \geq \lambda\, \mathrm{id}$ ($\lambda > 0$) and
$\mathrm{Hess}\, V_0 \geq 0$.

Then $\gamma \geq \lambda$.

γ: Log-Sob. const.

<u>Thm</u>. Assume $A \geq \lambda\, \mathrm{id}$ ($\lambda > 0$) and

$$Q_s (\mathrm{Hess}\, V_s) Q_s \geq \dot{\mu}_s\, \mathrm{id}, \quad Q_s = e^{-sA}$$

$\dot{\mu}_s$: constants that are allowed to be negative!!

$$\Rightarrow \gamma \geq \left(\int_0^\infty e^{-\lambda s - 2\mu(s)}\, ds \right)^{-1}, \quad \mu(s) = \int_0^s \dot{\mu}_{s'}\, ds'$$

<u>Rk</u>. If $\dot{\mu}_0 \geq 0 \Rightarrow \dot{\mu}_s \geq 0 \Rightarrow \left(\int_0^\infty \cdots \right)^{-1} \geq \lambda$

Proof idea: Like Bakry–Émery but use the Polchinski semigroup instead of Glauber semigroup

$$\frac{\partial}{\partial s} \mathbb{E}^{\nu^s} \Phi(F^s) = \mathbb{E}^{\nu^s}\left(-L_s \Phi(F^s) + \Phi'(F^s)\dot{F}^s\right)$$

unlike BE, ref. measure changes

Pol. semigroup

$$= \mathbb{E}^{\nu^s}\Big(\cancel{-\Phi'(F^s)} L_s \cancel{F^s}$$
$$- \frac{1}{2}\Phi''(F^s)(\nabla F^s)^2_{\dot{C}_s}$$
$$+ \cancel{\Phi'(F^s)\dot{F}^s}\Big)$$

$$= -\frac{1}{2}\mathbb{E}^{\nu^s}\Big(\underbrace{\Phi''(F^s)}_{\frac{1}{F^s}}(\nabla F^s)^2_{\dot{C}_s}\Big)$$

$$= -2\,\mathbb{E}^{\nu^s}\left((\nabla\sqrt{F^s})^2_{\dot{C}_s}\right)$$

Now consider change of $\mathbb{E}^{\nu^s}\left((\nabla\sqrt{F^s})^2_{\dot{C}_s}\right)$:

$$\dot{C}_s = e^{-sA}$$
$$Q_s = e^{-sA/2}$$
$$-A\dot{C}_s = -Q_s A Q_s$$

$$(\partial_s - L_s)(\nabla\sqrt{F^s})^2_{\dot{C}_s} = +(\nabla\sqrt{F^s})^2_{\ddot{C}_s}$$

$$-2(\nabla\sqrt{F^s}, \dot{C}_s \operatorname{Hess} V_s \dot{C}_s \nabla\sqrt{F^s})$$

$$\underbrace{-\frac{1}{4} F^s \left| \dot{C}_s^{1/2} (\operatorname{Hess} \log F^s) \dot{C}_s^{1/2} \right|_2^2}_{\geq 0}$$

$\dot{C}_s = Q_s^2$

$$\leq -(\nabla\sqrt{F^s}, Q_s \underbrace{(A + 2 Q_s \operatorname{Hess} V_s Q_s)}_{\geq \lambda + 2\dot{\mu}_s} Q_s \nabla\sqrt{F^s})$$

assumption.

$$\Rightarrow (\partial_s - L_s)(\nabla\sqrt{F^s})^2_{\dot{C}_s} \leq -(\lambda + 2\dot{\mu}_s)(\nabla\sqrt{F^s})^2_{\dot{C}_s}$$

$\Rightarrow \psi(s) = \mathbb{E}^{\nu^s}(\nabla\sqrt{F^s})^2_{\dot{C}_s}$ satisfies

$$\dot{\psi}(s) \leq -(\lambda + 2\dot{\mu}_s)\, \psi(s)$$

$$\psi(s) \leq e^{-\lambda s - 2\mu(s)}\, \psi(0)$$

$$\leq e^{-\lambda s - 2\mu(s)}\, \mathbb{E}(\nabla\sqrt{F^0})^2_{\dot{C}}$$

Id

$$\Rightarrow \mathrm{Ent}_\nu F \le 2\underbrace{\left(\int_0^\infty e^{-\lambda s - 2\mu(s)}\,ds\right)}_{\frac{1}{\gamma}} \mathbb{E}^\nu(\nabla\sqrt{F})^2$$

$$I_0(F\nu|\nu) = \mathbb{E}^\nu(\nabla\sqrt{F})^2$$

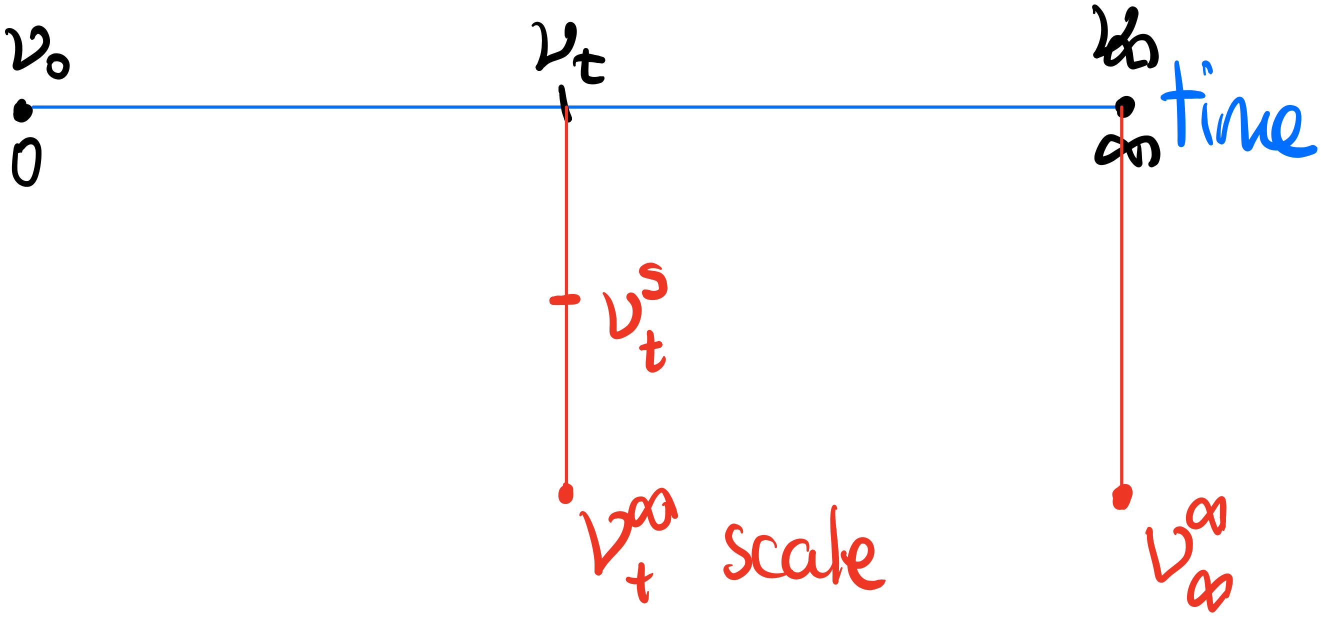
Summary:

$$\frac{\partial}{\partial t} H(\nu_t|\nu_\infty) = -\underbrace{I_0(\nu_t|\nu_\infty)}_{\substack{\text{LSI: lower bound} \\ 2\gamma H(\nu_t|\nu_\infty)}}$$ Glauber

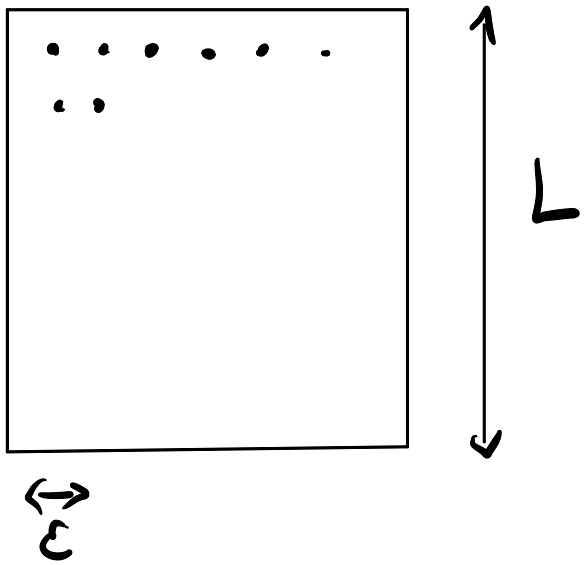
$$\frac{\partial}{\partial s} H(\nu_t^s|\nu_\infty^s) = -I_s(\nu_t^s|\nu_\infty^s)$$ Renormalisation / Polchinski

$$\Rightarrow I_s(\nu_t^s|\nu_\infty^s) \le e^{-\lambda s - 2\mu(s)} I_0(\nu_t^0|\nu_\infty^0)$$

$$\Rightarrow H(\nu_t|\nu_\infty) \le \frac{1}{2}\underbrace{\left(\int_0^\infty e^{-\lambda s - 2\mu(s)}\,ds\right)}_{\frac{1}{\gamma}} I_0(\nu_t|\nu_\infty)$$


ν_0
0
ν_t
ν_∞
∞
time
ν_t^s
ν_t^∞
scale
ν_∞^∞

4. Euclidean field theory



$\Lambda_{\varepsilon,L}$

$e^{-H^\varepsilon(\varphi)}\,d\varphi$

$$H^\varepsilon[\varphi] = \varepsilon^d \sum_{x\in\Lambda_{\varepsilon,L}} \left(\left(\tfrac{1}{2}\varphi(-\Delta^\varepsilon\varphi)\right) + m^2\varphi^2 + V^\varepsilon(\varphi(x)) \right)$$

$$\Delta^\varepsilon\varphi(x) = \varepsilon^{-2}\sum_{y\sim x}\left(\varphi(y)-\varphi(x)\right)$$

$\varphi : \Lambda_{\varepsilon,L} \to \mathbb{R}$ can be viewed as $\varphi \in S'(L\mathbb{T}^d)$

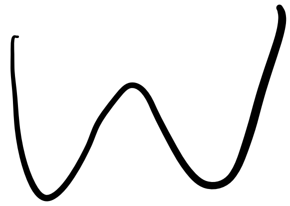
Goal: $\nu^\varepsilon \longrightarrow \nu$ (non-Gaussian) measure on $S'(L\mathbb{T}^d)$

<u>$d=1$</u>.

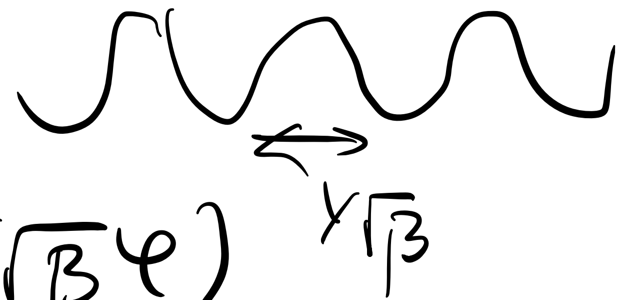
φ^4 model: $V^\varepsilon(\varphi) = g\varphi^4 + \nu_\varepsilon \varphi^2$

$(d=2,3)$

$\nu_\varepsilon \longrightarrow -\infty$



Sine-Gordon model:



$$V^\varepsilon(\varphi) = 2z\,\varepsilon^{-\beta/4\pi}\cos(\sqrt{\beta}\varphi)$$

$(d=2,\ 0<\beta<8\pi)$

In a large body of works (under varying assumptions), it is shown that

(φ^4): For $d \leq 3$ the limits $\varepsilon \to 0$, $L \to \infty$ exist as non-Gaussian measures on $S'(\mathbb{R}^d)$ + properties

(SG): For $d=2$, $\beta < 8\pi$, similar statement.

Glauber dynamics

Dirichlet form: $D^\varepsilon(F) = \frac{1}{\varepsilon^d} \sum_{x \in \Lambda_\varepsilon} \mathbb{E}\left(\frac{\partial F}{\partial \varphi(x)}\right)^2$

SPDE:

$$(\varphi^4) \quad d\varphi = \Delta^\varepsilon \varphi - g\varphi^3 - \nu_\varepsilon \varphi + dW^\varepsilon$$

"space-time white noise"
independent standard BM
with inner product
$\langle f,g \rangle = \varepsilon^d \sum_{x\in\Lambda} f(x) g(x)$.

$$(SG) \quad d\varphi = \Delta^\varepsilon \varphi - m^2 \varphi - 2z\varepsilon^{-\beta/4\pi}\sqrt{\beta}\sin(\sqrt{\beta}\varphi) + dW^\varepsilon$$

Hairer et al: short-time existence as $\varepsilon \to 0$.

Here: long-time behaviour uniformly in $\varepsilon > 0$.

sine-Gordon model.

Thm. Let $0<\beta<6\pi$, $z\in\mathbb{R}$, $m^2>0$, $L>1$. Then there is $\gamma=\gamma(\beta,z,m,L)$ independent of ε s.t.

$$\mathrm{Ent}_{\nu^\varepsilon} F \leq \frac{2}{\gamma} D^\varepsilon(\sqrt{F})$$

Moreover, if $m^{-2+\beta/4\pi}|z| \leq \delta_\beta \ll 1$ then

$$\gamma \geq m^2 + O_\beta(m^{\beta/4\pi}|z|)$$

independent of L.

Rk For φ^4 model, Log-Sobolev inequality is not known. Weber–Tsatsoulis: spectral gap.

Proof relies on estimate for effective potential.

renormalised potential.

$$A^\varepsilon = -\Delta^\varepsilon + m^2$$

$$V_0^\varepsilon(\varphi) = \varepsilon^2 \sum_{x\in\Lambda_{\varepsilon,L}} 2z\,\varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta}\varphi(x))$$

$\langle \cdot ; \cdot \rangle_\varepsilon = \varepsilon^2 \sum_{x \in \Lambda_{\varepsilon,L}} f(x) g(x)$

$\dot{C}_s = e^{-s A^\varepsilon}$ heat kernel

Thm. Let $\beta < 6\pi$, $m^2 > 0$, $z \in \mathbb{R}$, $\varepsilon > 0$. Then

$$\operatorname{Hess} V_s^\varepsilon(\varphi) \gtrsim \underbrace{- e^{-m^2 s} |z| L_s^{-\beta/4\pi}}_{\dot{\mu}_s}$$

where

$$L_s = \left(\sqrt{s} \wedge \frac{1}{m}\right) \vee \varepsilon$$

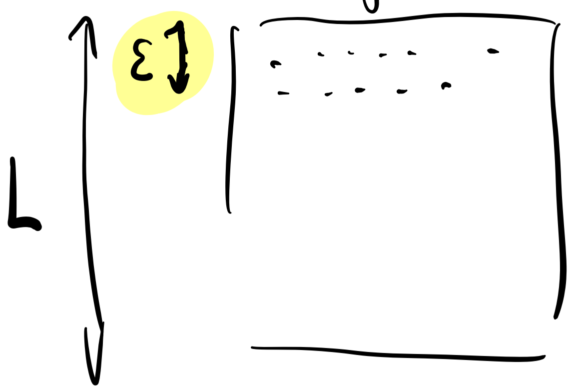
$$\Rightarrow \mu(s) = \int_0^s \dot{\mu}_{s'} \, ds' \approx \int_0^{\text{tm}} s^{-\beta/8\pi} \, ds$$

$$\frac{1}{\gamma} \leq \int_0^\infty e^{-\lambda s} e^{-2\mu(s)} \, ds$$

Recap: $\nu(d\varphi) \propto e^{-\frac{1}{2}(\varphi,A\varphi)-V_0(\varphi)}\,d\varphi$

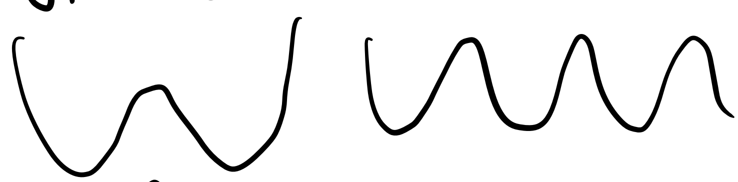
$\varphi\in\mathbb{R}^\Lambda$ — Laplacian — Lebesgue

Λ large finite



$$V_0(\varphi)=\sum_{x\in\Lambda} V(\varphi(x))$$

typically non-convex



Cov. of $e^{-\frac{1}{2}(\varphi,A\varphi)}$ is $\underbrace{A^{-1}}_{C_\infty}=\int_0^\infty \underbrace{e^{-sA}}_{\dot C_s=Q_s^2}\,ds.$

Renormalised pot. $e^{-V_s(\varphi)}=(P_{C_s}*e^{-V_0})(\varphi)$

$$=E_{C_s}\left(e^{-V_0(\varphi+\zeta)}\right)$$

$$\Leftrightarrow\ \partial_s V_s(\varphi)=\frac{1}{2}\Delta_{\dot C_s}V_s-\frac{1}{2}(\nabla V_s)^2_{\dot C_s}$$

Polchinski eqn.

$V_0 =$ 

one possibility

Exercise. $|\Lambda|=1$

$V_0 = \varphi^4 - \varphi^2 \Rightarrow \operatorname{Hess} V_s$ will remain bounded below (by a negative constant).

Thm. $A \geq \lambda\,\mathrm{id}$ $(\lambda>0)$, $Q_s \operatorname{Hess} V_s Q_s \geq \dot{\mu}_s\,\mathrm{id}$

$$\Rightarrow \gamma \geq \left(\int_0^\infty e^{-\lambda s - 2\mu(s)}\,ds\right)^{-1}$$

where $\mu(s) = \int_0^s \dot{\mu}_{s'}\,ds'$.

Sine-Gordon model: $d=2$

$$\frac{1}{2}(\varphi, A\varphi) = \varepsilon^2 \sum_{x\in\Lambda_{\varepsilon,L}} \left(\varphi(x)\,(-\Delta^\varepsilon \varphi(x)) + m^2\varphi(x)^2\right)$$

$$-\Delta^\varepsilon\varphi(x) = \varepsilon^{-2}\sum_{y\sim x}(\varphi(y)-\varphi(x))$$

$$V_0(\varphi) = \varepsilon^2 \sum_{x\in\Lambda_{\varepsilon,L}} 2z\,\varepsilon^{-\frac{\beta}{4\pi}} \cos(\sqrt{\beta}\,\varphi(x))$$

$z\in\mathbb{R}$ $\quad$ $\beta>0$ $\quad$ 'counterterm'

Known: For $\beta<8\pi$, and various assumptions, the limit $\varepsilon\to 0$ exists as a non-Gaussian measure on $S'(L\mathbb{T}^2)$.

Relation to Yukawa gas (sine-Gordon transformation)

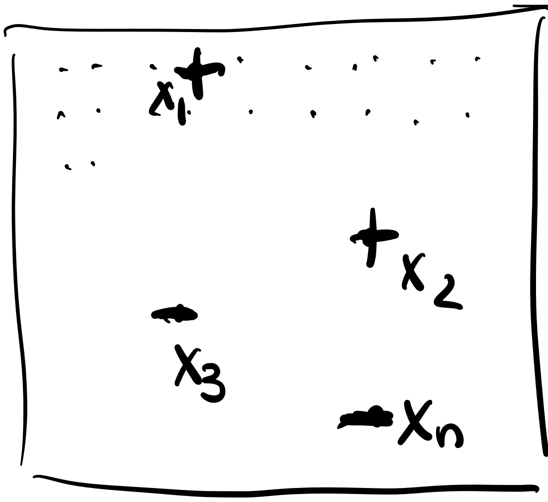
Let φ be a Gaussian field with cov. (GFF)

$$C = (-\Delta^\varepsilon + m^2)^{-1}$$

Yukawa potential $(m^2 > 0)$

$$\Rightarrow \mathbb{E}\left(e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}\right) = e^{-\frac{\beta}{2}\sum_{i,j=1}^{n} C(x_i,x_j)\sigma_i\sigma_j}$$

$$= e^{-\frac{\beta}{2}\sum_{i,j=1}^{n} (-\Delta^\varepsilon + m^2)^{-1}(x_i,x_j)\sigma_i\sigma_j}$$



For the 2D Yukawa potential:

$$(-\Delta^\varepsilon + m^2)^{-1}(0,0) = \frac{1}{2\pi}\log\varepsilon^{-1} + c(m) + o(1)$$

$$e^{\frac{\beta}{2}C(0,0)} = \text{const.}\ \varepsilon^{-\frac{\beta}{4\pi}}$$

Partition function of 2D Yukawa gas:

$$Z^{YG} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\,\varepsilon^{dn}\sum_{\substack{x_1,\dots,x_n \in \Lambda_{\varepsilon,L}\\ \sigma_1,\dots,\sigma_n\in\{\pm 1\}}} e^{-\frac{\beta}{2}\sum_{i\neq j}\sigma_i\sigma_j(-\Delta^\varepsilon+m^2)^{-1}(x_i,x_j)}$$

activity

$$e^{+\frac{\beta}{2}nC(0,0)}\, e^{-\frac{\beta}{2}\sum_{i,j}(\cdots)}$$

$$\sim C\varepsilon^{-\beta/4\pi} \qquad = \mathbb{E}\left(e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}\right)$$

$$= \sum_n \frac{\left(z e^{\frac{\beta}{2}C(0,0)}\right)^n}{n!}\varepsilon^{2n} \underbrace{\underbrace{\sum_{\substack{x_1\cdots x_n\\ \sigma_1\cdots\sigma_n}} \mathbb{E}\left(\underbrace{e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}}\right)}_{\sum_{x_1\cdots x_n}\mathbb{E}\left(2\cos(\sqrt{\beta}\varphi(x_i))\right)}}_{\mathbb{E}\left(\sum_x 2\cos(\sqrt{\beta}\varphi(x))\right)^n}$$

$$= \mathbb{E}\left(\underbrace{\exp\left(\varepsilon^2\sum_x 2z\underbrace{e^{\frac{\beta}{2}C(0,0)}}_{\sim\varepsilon^{-\beta/4\pi}}\cos(\sqrt{\beta}\varphi(x))\right)}_{e^{-V_0(\varphi)}}\right)$$

$$\int e^{-\frac{1}{2}(\varphi,A\varphi)}$$

Normalisation:

$$V(\varphi) = \varepsilon^2 \sum_{x \in \Lambda_\varepsilon} 2z \underbrace{\varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta}\,\varphi(x))}_{\text{singular}}$$ macroscopic

$$= \sum_{x \in \Lambda_\varepsilon} 2z \underbrace{\varepsilon^{2-\beta/4\pi}}_{\text{tiny if } \beta < 8\pi} \cos(\sqrt{\beta}\,\varphi(x))$$ microscopic

$A = -\Delta^\varepsilon + m^2$ w.r.t. $(u,v)_\varepsilon = \varepsilon^2 \sum_x u(x) v(y)$ macro.

$A = -\Delta + \varepsilon^2 m^2$ w.r.t. $(u,v) = \sum_x u(x) u(y)$ micro.

unit lattice Laplacian

Macroscopic p.o.v.: EFT / SPDE $\varepsilon \to 0$

Microscopic p.o.v.: Interacting particle system with weak interaction

Yukawa gas representation of renormalised pot.
(Brydges–Kennedy)

Write

$$V_s(\varphi) = \sum_{n=0}^{\infty} V_s^n(\varphi)$$

$$V_s^n(\varphi) = \frac{1}{n!} \sum_{\substack{x_1 \cdots x_n \in \Lambda \\ \sigma_1 \cdots \sigma_n \in \{\pm 1\}}} \tilde{V}_s^n(\xi_1, \ldots, \xi_n)\, e^{i\sqrt{\beta}\sum_{i=1}^{n} \varphi(x_i)\sigma_i}$$

$(*)$ $\quad \xi_i = (x_i, \sigma_i)$

Initial potential: $\tilde{V}_0(\xi_1) = z_0 = \varepsilon^{2-\beta/4\pi} z$

$$\tilde{V}_0(\xi_1, \ldots, \xi_n) = 0 \quad (n \geq 2).$$

Polchinski eqn. $\partial_s V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - (\nabla V_s)^2_{\dot{C}_s}$

$$\dot{C}_s = e^{-tA}$$

$$\frac{1}{2}\Delta_{\dot{C}_s} V_s = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\xi_1 \cdots \xi_n} \frac{1}{2} \underbrace{\Delta_{\dot{C}_s} V(\xi_1, \ldots, \xi_n)\, e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}}$$

$$= \underbrace{-\frac{1}{2}\sum_{j,k=1}^{n} \beta\sigma_i\sigma_j \dot{C}_s(x_j, x_k)}_{\dot{W}_s(\xi_1, \ldots, \xi_n)}\ \tilde{V}_s(\xi_1, \ldots, \xi_n)$$

$$\widetilde{[\nabla V_s]^2_{\dot C_s}}(\xi_1,\dots,\xi_n) = \sum_{\substack{I_1 \dot\cup I_2 = \{1,\dots,n\} \\ j\in I_1,\, k\in I_2}} \left(-\beta \dot C_s(x_j,x_k)\sigma_j\sigma_k\right) \underbrace{\tilde V(\xi_{I_1})}_{(\xi_{I_1})\,|\,x_1,\dots,x_{|I|})} \tilde V(\xi_{I_2})$$

Upshot: Polchinski in 'Fourier space':

$$\partial_s \tilde V_s = -\dot W_s \tilde V_s - \frac{1}{2}\widetilde{(\nabla V_s)^2}$$

Duhamel formula: $\quad W_s = \int_0^s \dot W_{s'}\, ds'$

$$\underbrace{\tilde V_s(\xi_1,\dots,\xi_n)}_{\text{depends on } n \text{ particles!}} = e^{-W_s(\xi_1,\dots,\xi_n)}\tilde V_0(\xi_1,\dots,\xi_n)$$
$$+\frac{1}{2}\int_0^s e^{-(W_s-W_t)}\sum_{\substack{I_1\dot\cup I_2=[n]\\ j\in I_1,\, k\in I_2}} \dot u_s(\xi_j,\xi_k)\,\underbrace{\tilde V_s(\xi_{I_1})\tilde V_s(\xi_{I_2})}_{\text{depend on at most } n-1 \text{ part.}}$$

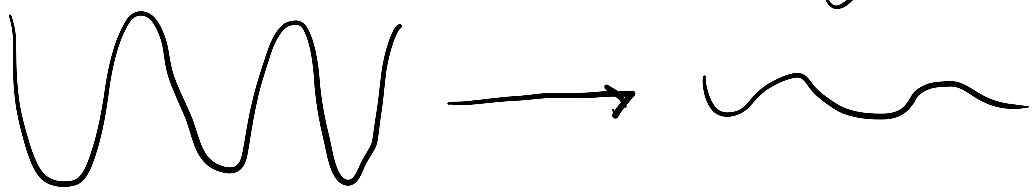
depends on n particles!

depend on at most n-1 part.

$n=1$: $\tilde{V}_s(\xi_1) = e^{-W_s(\xi_1)}\tilde{V}_0(\xi_1)$

$$= e^{-\frac{\beta}{2}C_s(0,0)}\, z_0$$

makes it smaller starts large



$n \geq 2$: $\tilde{V}_s(\xi_1,\dots,\xi_n) = \frac{1}{2}\int_0^s (\cdots)\, \tilde{V}_s(\xi_{I_1})\tilde{V}_s(\xi_{I_2})$

depend only on $n-1$

By induction, $\tilde{V}_s(\xi_1,\dots,\xi_n)$ is well-defined for all t, n.

<u>Fact</u>. If the series $(*)$ converges absolutely then it gives the unique solution to the Polchinski equation.

Thm. (Brydges & Kennedy). Let $\beta < 4\pi$. Then for all $n \geq 2$,

$$\ell_t^2 \sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} \left|\tilde{V}_t(\xi_1,\dots,\xi_n)\right| \leq n^{n-2} C_\beta^{n-1} |z_t|^n$$

are such $C_t(x,x) = \frac{1}{2\pi}\log \ell_t + O(1)$