A short course on spin systems

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Abstract

A short course on spin systems, covering mean-field theory, continuous symmetry breaking, random walk representations, and a glimpse at supersymmetry and the renormalisation group.

Contents

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1 Introduction and mean-field theory

1.1 Spin systems

Classical spin models are collections of many random variables $(\varphi_i)_{i\in\Lambda}$, called spins, whose distribution is specified in terms of the energy of a spin configuration. The spins are indexed by a set Λ, which we initially assume to be finite, but large, and eventually $|\Lambda| \to \infty$. The individual spins φ_i can be discrete or continuous. They take values in a target space T and are described by an *a priori* reference distribution μ on this space. The simplest example is the Ising model, the target space is $T = \mathbb{S}^0 = \{\pm 1\}$, i.e., $\varphi_i \in \mathbb{S}^0 = \{\pm 1\}$, and the reference distribution is uniform on $\mathbb{S}^0$. The energy of a spin configuration is described by a pair potential $U(\varphi, \varphi')$ and a coupling matrix $\beta = (\beta_{ij})_{i,j\in\Lambda}$ as

$$H(\varphi) = \frac{1}{2} \sum_{i,j} \beta_{ij} U(\varphi_i, \varphi_j). \tag{1.1}$$

For the Ising model, the pair potential is $U(\varphi, \varphi') = \frac{1}{2}(\varphi - \varphi')^2 = -\varphi\varphi' + \text{constant}$. Since $|\varphi| = 1$, the constant does not change the measure and is often omitted. By convention, we choose the pair potential U symmetric and such that $U(\varphi, \varphi')$ is smallest when φ and φ' align (in a suitable sense). Then:

- If $\beta_{ij} \geqslant 0$ for $i \neq j$, the energy favours that spins align: the system is ferromagnetic.
- If $\beta_{ij} \leqslant 0$ for $i \neq j$, the energy favours that spins anti-align: the system is anti-ferromagnetic.
- It is also interesting to consider the case where the spin couplings β_{ij} are random themselves, e.g., independent centred Gaussian, independent of the spins. This is called a spin glass.

The Ising model is one example of a large class of interesting models. These include in particular:

- the $O(n)$-model, where $\varphi_i \in \mathbb{S}^{n-1}$, and the a priori distribution is uniform on $\mathbb{S}^{n-1}$;
- the q-state Potts model, where $\varphi_i \in \{1, \dots, q\}$, again with uniform a priori distribution;
- the $|\varphi|^4$ model, where $\varphi \in \mathbb{R}^n$, and the a priori distribution has a "Mexican hat potential;"
- the hyperbolic sigma model, where $\varphi \in \mathbb{H}^n$, and the a priori distribution is uniform;
- Sine–Gordon models, where $\varphi \in \mathbb{R}$, and the a priori distribution is a periodic measure;
- gradient models, where $\varphi \in \mathbb{R}$, and $U(\varphi, \varphi')$ depends only on $\varphi - \varphi'$.

Furthermore, an important example that also arises as a limit of several of the above models is

- the Gaussian Free Field, where $\varphi_i \in \mathbb{R}^n$, and the a priori distribution is the Lebesgue measure.

Symmetry plays an important role in the behaviour of spin models. In the examples above, the single spin distribution and the two-spin interaction is symmetric with respect a group of symmetries.

- Ising and Potts models: permutation group on q elements;
- $O(n)$ model and $|\varphi|^4$ models: the orthogonal group $O(n)$;
- Gaussian Free Field and gradient models: the n-dimensional Euclidean group;
- Sine–Gordon models: the additive integers $\mathbb{Z}$;
- Hyperbolic sigma models: the restricted Lorentz group $SO^+(n, 1)$.

There are key distinctions between continuous and discrete symmetry groups, between compact and non-compact ones, and between abelian and non-abelian symmetry groups. Usually, symmetries imply important constraints (Ward identities) on the measure with often powerful consequences. In these lectures, we focus on the ferromagnetic case, and Λ is often the vertex set of a large graph approximating $\mathbb{Z}^d$ in a suitable way.

The above spin models also have various generalisations, which are interesting in themselves, or as a tool to study other models. These include quantum versions, which are fundamental models in quantum mechanics, as well as supersymmetric models, which arise as effective models in the description of disordered systems such as random matrices and in the description of interacting random walks (e.g., self-avoiding walks). Spin models can also be defined on continuous rather than discrete spaces. Many of the fundamental questions remain the same in this case.

Notation. For $u, v \in (\mathbb{R}^n)^\Lambda$, we write $(u, v) = \sum_{i\in\Lambda} u_i \cdot v_i$. Moreover, $\mathbb{E}$ and $\langle\cdot\rangle$ will denote the expectation of a probability measure, often with subscript to indicate which measure is referred to.

1.2 The free field

1.2.1. Gaussian fields. Let Λ be a finite set, and let $C = (C_{xy})_{x,y\in\Lambda}$ be a symmetric strictly positive definite matrix. Then C has an inverse which we denote by A.

Definition 1.1. *The centred Gaussian measure in $\mathbb{R}^\Lambda$ with covariance C, or equivalently with coupling matrix A, is defined by the density*

$$p_C(\varphi) = (\det 2\pi C)^{-1/2} e^{-(\varphi, A\varphi)/2} \quad (\varphi \in \mathbb{R}^\Lambda) \tag{1.2}$$

with respect to the Lebesgue measure on $\mathbb{R}^\Lambda$. We write $\mathbb{E}_C$ for the expectation of this measure. The Gaussian measure with mean $h \in \mathbb{R}^\Lambda$ is given by the expectation $\mathbb{E}_{C,h}F(\varphi) = \mathbb{E}_C F(\varphi + h)$.

Proposition 1.2. *The centred Gaussian measure p_C is the unique probability measure on $\mathbb{R}^\Lambda$ with Laplace transform (moment generating function)*

$$\mathbb{E}_C(e^{(f,\varphi)}) = e^{(f,Cf)/2}. \tag{1.3}$$

In particular, $\mathbb{E}_C(\varphi_x\varphi_y) = C_{xy}$.

Proposition 1.3. *Let C_1 and C_2 be positive definite matrices on $\mathbb{R}^\Lambda$. Then*

$$p_{C_1} * p_{C_2} = p_{C_1+C_2}, \quad \textit{where } p_1 * p_2(\varphi) = \int_{\mathbb{R}^\Lambda} p_1(\varphi - \zeta) p_2(\zeta)\, d\zeta. \tag{1.4}$$

By (1.3), Gaussian measures can also be defined when the covariance matrix C is positive semi-definite rather than positive definite. They then have support in a subspace of $\mathbb{R}^\Lambda$. The last propositions can be extended to this case. We will return to this later.

1.2.2. The free field. The Gaussian Free Field (GFF) on a set Λ with spin couplings $\beta = (\beta_{ij})_{i,j\in\Lambda}$ and squared mass $m^2 > 0$ and external field $h \in \mathbb{R}^n$ is given by the probability measure on $\mathbb{R}^{n\Lambda}$ with expectation

$$\langle F(\varphi)\rangle_{\Lambda,\beta,m^2,h} \propto \int_{(\mathbb{R}^n)^\Lambda} F(\varphi)\, e^{-H(\varphi)} \prod_{x\in\Lambda} d\varphi, \tag{1.5}$$

where, with $M_{ij} = -\beta_{ij}$ for $i \neq j$ and $M_{ii} = \sum_{j\neq i} \beta_{ij}$,

$$H(\varphi) = \frac{1}{4} \sum_{i,j\in\Lambda} \beta_{ij} |\varphi_i - \varphi_j|^2 + \frac{1}{2} \sum_{i\in\Lambda} m^2 |\varphi_i|^2 - \sum_{i\in\Lambda} h \cdot \varphi_i = \frac{1}{2}(\varphi, (M + m^2\,\mathrm{id})\varphi) - (h, \varphi). \tag{1.6}$$

In other words, the GFF is a Gaussian measure with covariance given by

$$\langle \varphi_x ; \varphi_y \rangle_{\beta,m^2,h} = (M+m^2)^{-1}_{xy}. \tag{1.7}$$

Since the measure is Gaussian the mean is unimportant and we normally set $h = 0$. By Wick's formula for Gaussian integrals, all correlation functions can be computed in terms of the two-point function. We will not need this fact.

Example 1.4. *Let Λ be a discrete torus in dimension $d \geqslant 1$ and side length L. Let $\beta_{ij} = 1_{i\sim j}$ where $i \sim j$ denotes that i and j are neighbours in Λ. Then $M = -\Delta$ is the discrete Laplace operator Λ. Then, for $x, y \in \mathbb{Z}^d$, as $|x-y| \to \infty$,*

$$\lim_{m^2\downarrow 0}\lim_{L\to\infty} (-\Delta + m^2)^{-1}_{xy} \sim \begin{cases} C_d |x-y|^{-(d-2)} & (d \geqslant 3), \\ \infty & (d \leqslant 2), \end{cases} \tag{1.8}$$

$$\lim_{m^2\downarrow 0}\lim_{L\to\infty} ((-\Delta + m^2)^{-1}_{xy} - (-\Delta + m^2)^{-1}_{xx}) \sim \begin{cases} -C_2 \log |x-y| & (d = 2), \\ -C_1 |x| & (d = 1). \end{cases} \tag{1.9}$$

Corollary 1.5. *For $d \geqslant 3$,*

$$\infty = \lim_{\Lambda\uparrow\mathbb{Z}^d}\lim_{m^2\downarrow 0} \langle \varphi_x^2 \rangle_{\Lambda,\beta,m^2,0} \neq \lim_{m^2\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}^d} \langle \varphi_x^2 \rangle_{\Lambda,\beta,m^2,0} < \infty. \tag{1.10}$$

The interpretation of this equation is that the symmetry of translations $\varphi \to \varphi + t$ for $t \in \mathbb{R}$ of the measure, which holds for $m^2 = 0$ in finite volume, is *spontaneously broken*. On the other hand, the rotational symmetry $\varphi \to R\varphi$ for $R \in SO(n)$ is not spontaneously broken; the following exercise shows an instance. The spontaneous breaking of symmetries is one of the mean aspects of interest in the study of spin systems.

Exercise 1.6. *Show that for any bounded continuous function F,*

$$\lim_{m^2\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}^d}\lim_{h\to 0} \langle F(\varphi_x) \rangle_{\Lambda,m^2,h} = \lim_{m^2\downarrow 0}\lim_{h\to 0}\lim_{\Lambda\uparrow\mathbb{Z}^d} \langle F(\varphi_x) \rangle_{\Lambda,m^2,h}. \tag{1.11}$$

Symmetries give rise to *Ward identity.* The following is an example.

Proposition 1.7. *For any $m^2 > 0$,*

$$\sum_{y\in\Lambda} (M+m^2)^{-1}_{xy} = \frac{1}{m^2}. \tag{1.12}$$

This is the simple fact that the constant vector $\mathbf{1} = (1,\dots,1)$ is annihilated by M, i.e., $M\mathbf{1} = 0$, and that $m^2\mathbf{1} = m^2\mathbf{1}$. However, it is also an example of a Ward identity associated to the symmetry of translation. Similar Ward identities exist when explicit Gaussian calculations are not available.

Proof. Write $H_{\beta,m^2}(\varphi)$. Then $H_{\beta,0}(\varphi + t\mathbf{1}) = H_{\beta,0}(\varphi)$ for all $t \in \mathbb{R}$, where $\mathbf{1}_x = 1$ for all $x \in \Lambda$. By translation invariance of the Lebesgue measure and of $H_{\beta,0}$,

$$\langle \varphi_x \rangle_{\beta,m^2} = \frac{1}{Z}\int \varphi_x e^{-H_{\beta,0}(\varphi) - \frac{1}{2}m^2(\varphi,\varphi)}\, d\varphi = \frac{1}{Z}\int (\varphi_x + t) e^{-H_{\beta,0}(\varphi) - \frac{1}{2}m^2(\varphi+t\mathbf{1},\varphi+t\mathbf{1})}\, d\varphi. \tag{1.13}$$

Differentiating with respect to t at $t = 0$ gives

$$0 = \langle 1 - m^2\varphi_x(\varphi,\mathbf{1}) \rangle_{\beta,m^2} = 1 - m^2 \sum_y \langle \varphi_x \varphi_y \rangle_{\beta,m^2}, \tag{1.14}$$

which is the claim. □

1.3 The mean-field $O(n)$ model

The correlation structure of the free field is Gaussian and thus explicit. For general spin models, understanding the detailed behaviour can become extremely difficult. We now consider the $O(n)$ model. For spin couplings $\beta = (\beta_{ij})$, the energy of a spin-configuration is given as for the free field by

$$H(\varphi) = \frac{1}{4}\sum_{i,j}\beta_{ij}|\varphi_i - \varphi_j|^2 - \sum_i h\cdot\varphi_i = \frac{1}{2}(\varphi, M\varphi) - (\varphi, h). \tag{1.15}$$

The reference measure μ is now not the Lebesgue measure, but is still symmetric under rotations and reflections. The simplest such choice of μ is the uniform measure on the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. The expectation of $F : (\mathbb{S}^{n-1})^\Lambda \to \mathbb{R}$ is then given by

$$\langle F\rangle = \mathbb{E}_\nu F \propto \int_{(\mathbb{S}^{n-1})^\Lambda} F(\varphi)\, e^{-H(\varphi)}\, \mu^{\otimes\Lambda}(d\varphi). \tag{1.16}$$

In the following, we show that the $\mathbb{S}^{n-1}$ model can be solved in *mean-field theory*. The methods also apply to general $O(n)$-symmetric models. In mean-field theory, the coupling matrix treats all pairs of spins in the same way, i.e., $\Lambda = \{0, \dots, N-1\}$, and for some constant $\beta > 0$,

$$\beta_{ij} = \frac{\beta}{N} \quad \text{for all } i, j. \tag{1.17}$$

The scaling $1/N$ ensures that the total interaction remains of order 1. Let Q denote the orthogonal projection onto the constant vectors in $\mathbb{R}^N$, i.e., the matrix with all entries equal to $1/N$, and set $P = \mathrm{id} - Q$. Then the matrix M in (1.6) for $\beta_{ij} = \beta/N$ is

$$M = \beta P. \tag{1.18}$$

Since P is a degenerate matrix (it annihilates the constant functions), it can be useful to regularise the spin coupling by considering instead $M + m^2 = M^2 + m^2\,\mathrm{id}$, with $m^2 \downarrow 0$ eventually. Since P and Q are orthogonal projections with $P + Q = \mathrm{id}$, one has

$$M + m^2 = (\beta + m^2)P + m^2 Q. \tag{1.19}$$

By the spectral theorem, hence

$$(M + m^2)^{-1} = \frac{1}{\beta + m^2}P + \frac{1}{m^2}Q = \frac{1}{\beta + m^2}\,\mathrm{id} + \frac{\beta}{m^2(\beta + m^2)}Q. \tag{1.20}$$

The following lemma is a limiting case of Proposition 1.3, when $m^2 \downarrow 0$.

Lemma 1.8. *Let $M = \beta P$ be the mean-field coupling matrix. There is a constant $c > 0$ such that*

$$e^{-\frac{1}{2}(\sigma, M\sigma)} = c\int_{\mathbb{R}^n} e^{-\frac{\beta}{2}(\varphi-\sigma,\varphi-\sigma)}\, d\varphi \quad \textit{for all } \sigma \in (\mathbb{R}^n)^\Lambda, \tag{1.21}$$

where we identify $\varphi \in \mathbb{R}^n$ as a constant vector $(\varphi, \dots, \varphi) \in \mathbb{R}^{n\Lambda}$.

Proof. The proof is a limiting case of Proposition 1.3, where the Gaussian measure with covariance $\beta Q/(m^2(\beta + m^2))$ tends to the Lebesgue measure on the subspace of constant fields φ as $m^2 \downarrow 0$.

Instead of using this, we can also check the claim directly. Let $\bar\sigma = Q\sigma$, i.e., $\bar\sigma_i = \frac{1}{N}\sum_j \sigma_j$ for any $i \in \Lambda$. Since φ and $\bar\sigma$ are constant on Λ, we also write $\varphi = \varphi_i$ and $\bar\sigma = \bar\sigma_i$. Then

$$\begin{aligned}\frac{1}{2}(\varphi-\sigma,\varphi-\sigma) &= \frac{1}{2}((\varphi-\bar\sigma)-(\sigma-\bar\sigma),(\varphi-\bar\sigma)-(\sigma-\bar\sigma))\\ &= \frac{1}{2}(Q(\varphi-\bar\sigma)-P\sigma, Q(\varphi-\bar\sigma)-P\sigma) = \frac{1}{2}N|\varphi-\bar\sigma|^2 + \frac{1}{2}(\sigma, P\sigma)\end{aligned} \tag{1.22}$$

where the last equality holds since the projections P and Q are orthogonal. Take the exponential $\exp(-\beta(\cdot))$ of both sides and integrate over $\varphi \in \mathbb{R}^n$. The right-hand side is

$$e^{-\frac{1}{2}(\sigma,M\sigma)} \int_{\mathbb{R}^n} e^{-\frac{N\beta}{2}|\varphi-\bar{\sigma}|^2}\, d\varphi = e^{-\frac{1}{2}(\sigma,M\sigma)} \int_{\mathbb{R}^n} e^{-\frac{N\beta}{2}|\varphi|^2}\, d\varphi \propto e^{-\frac{1}{2}(\sigma,M\sigma)}. \tag{1.23}$$

The left-hand side already has the claimed form. □

The identity (1.21) allows us to decompose the measure of the $O(n)$ model ν on $(\mathbb{S}^{n-1})^\Lambda$ into two measures, which we call the renormalised measure and the fluctuation measure.

Renormalised measure. The renormalised measure ν_r is a measure on $\mathbb{R}^n$ defined as follows. For $\varphi \in \mathbb{R}^n$, define the renormalised potential by

$$V(\varphi) = -\log \int_{\mathbb{S}^{n-1}} e^{-\frac{\beta}{2}(\varphi-\sigma)^2+(h,\sigma)}\, \mu(d\sigma). \tag{1.24}$$

The renormalised measure is then defined by the expectation

$$\mathbb{E}_{\nu_r}(G(\varphi)) \propto \int_{\mathbb{R}^n} G(\varphi)\, e^{-NV(\varphi)}\, d\varphi. \tag{1.25}$$

Fluctuation measure. The fluctuation measure is a measure on $(\mathbb{S}^{n-1})^\Lambda$ but of simpler form than the original $O(n)$ measure. It is a product measure that depends on the renormalised field $\varphi \in \mathbb{R}^n$, and is defined by

$$\mathbb{E}_{\mu_\varphi}(F(\sigma)) \propto \prod_{x\in\Lambda} \int_{\mathbb{S}^{n-1}} F(\sigma)\, e^{-\frac{\beta}{2}(\varphi-\sigma_x)^2+(h,\sigma_x)}\, \mu(d\sigma_x). \tag{1.26}$$

Lemma 1.9. *For any* $F : (\mathbb{S}^{n-1})^\Lambda \to \mathbb{R}$,

$$\mathbb{E}_\nu(F(\sigma)) = \mathbb{E}_{\nu_r}(\mathbb{E}_{\mu_\varphi}(F(\sigma))). \tag{1.27}$$

Proof. The proof is just a matter of substituting in definitions and using (1.21):

$$\begin{aligned}
\mathbb{E}_\nu(F(\sigma)) &\propto \int_{(\mathbb{S}^{n-1})^\Lambda} F(\sigma)\, e^{-\frac{1}{2}(\sigma,M\sigma)+(h,\sigma)}\, \mu^{\otimes\Lambda}(d\sigma) \\
&\propto \int_{\mathbb{R}^n} \int_{(\mathbb{S}^{n-1})^\Lambda} e^{-\frac{\beta}{2}(\varphi-\sigma,\varphi-\sigma)+(h,\sigma)}\, F(\sigma)\, \mu^{\otimes\Lambda}(d\sigma)\, d\varphi \\
&= \int_{\mathbb{R}^n} e^{-NV(\varphi)} \left(\prod_{x\in\Lambda} e^{V(\varphi)} \int_{(\mathbb{S}^{n-1})^\Lambda} e^{-\frac{\beta}{2}(\varphi-\sigma_x,\varphi-\sigma_x)+(h,\sigma_x)} F(\sigma)\, \mu(d\sigma_x) \right) d\varphi \\
&\propto \mathbb{E}_{\nu_r}(\mathbb{E}_{\mu_\varphi}(F(\sigma))),
\end{aligned} \tag{1.28}$$

and since $\mathbb{E}_\nu(1) = 1 = \mathbb{E}_{\nu_r}(\mathbb{E}_{\mu_\varphi}(1))$, the identity (rather than proportionality) follows. □

To compute the magnetisation, we need the observable $F(\sigma) = \sigma_0$. Let $G(\varphi) = \mathbb{E}_{\mu_\varphi}(\sigma_0)$. Then

$$\mathbb{E}_\nu(\sigma_0) = \mathbb{E}_{\nu_r}(G(\varphi)). \tag{1.29}$$

The right-hand side is a finite-dimensional integral, with dimension n independent of the number of vertices N. Therefore Laplace's Principle can be applied.

Exercise 1.10 (Laplace's Principle)**.** *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be continuous with global minimum at* $\varphi_0 \in \mathbb{R}^n$. *Assume that* $\int_{\mathbb{R}^n} e^{-V}\, d\varphi$ *is finite and that* $\{\varphi \in \mathbb{R}^n : V(\varphi) \leqslant V(\varphi_0) + 1\}$ *is compact. Then for any bounded continuous function* $g : \mathbb{R}^n \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{\int_{\mathbb{R}^n} e^{-NV(\varphi)} g(\varphi)\, d\varphi}{\int_{\mathbb{R}^n} e^{-NV(\varphi)}\, d\varphi} = g(\varphi_0). \tag{1.30}$$

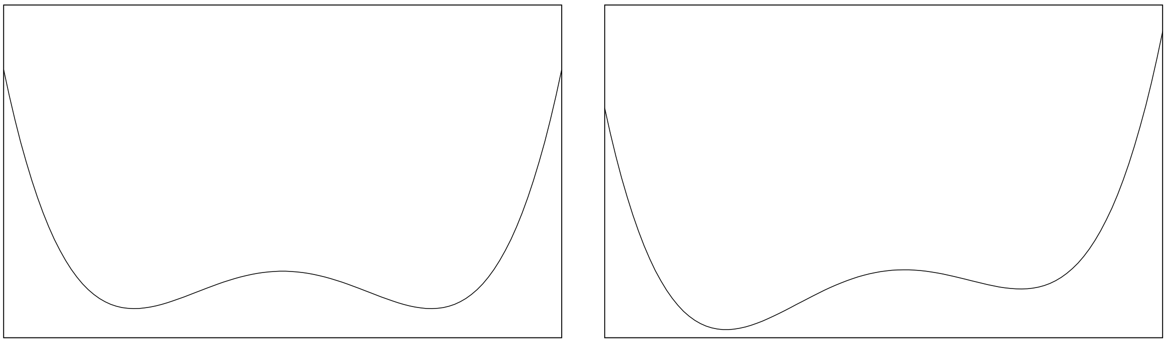

FIGURE 1.1. The renormalised potential for $\beta > \beta_c$ with $h = 0$ (left) and $h \neq 0$ (right). For $h \neq 0$ the minimum is unique, while for $h = 0$ it is assumed on a set with $O(n)$ symmetry.

The critical points φ of the renormalised potential V satisfy (with $G(\varphi) = \mathbb{E}_{\mu_\varphi}(\sigma_0)$ as above)

$$0 = \nabla V(\varphi) = \mathbb{E}_{\mu_\varphi}(\beta(\varphi - \sigma)) = \beta(\varphi - G(\varphi)), \quad \text{i.e., } \varphi = G(\varphi). \tag{1.31}$$

Exercise 1.11. *In the Ising case ($n = 1$), the renormalised potential V and the function G are*

$$V(\varphi) = \frac{\beta}{2}\varphi^2 - \log\cosh(\beta\varphi + h) + \textit{constant}, \qquad G(\varphi) = \tanh(\beta\varphi + h). \tag{1.32}$$

In general, the properties of the solution to $G(\varphi) = \varphi$ are summarised in the following exercise.

Exercise 1.12. *Let $n \geqslant 1$.*

(i) For $h \neq 0$, the effective potential V has a unique minimum $\varphi_{\beta,h}$ parallel to h.

(ii) For $\beta \leqslant n$, the effective potential V is convex and the minimum of V tends to 0 as $h \to 0$. Moreover, $\mathrm{Hess}V(\varphi) \geqslant \beta - \beta^2/n$ for any $h \in \mathbb{R}^n$.

(iii) For $\beta > n$, the minima of the effective potential lie on a sphere $|\varphi| = r$ for some $r = r(\beta) > 0$ if $h = 0$; as $h \downarrow 0$ the unique minimum converges to a point on this sphere.

(You may restrict to $n = 1$. For $n > 1$ parts of the exercise are difficult.)

By Laplace's Principle, for $h \neq 0$ and denoting by $\varphi_{\beta,h}$ the corresponding unique minimum,

$$\lim_{N\to\infty} \mathbb{E}_\nu(\sigma_0) = \lim_{N\to\infty} \mathbb{E}_{\nu_r}(G(\varphi)) = G(\varphi_{\beta,h}) = \varphi_{\beta,h}. \tag{1.33}$$

Taking $h \downarrow 0$, we see that this limit is 0 for $\beta \leqslant n$ and that it is non-vanishing if $\beta > n$.

Exercise 1.13 (Critical exponents)**.** *Let $\chi = \chi(\beta, h) = \lim_{N\to\infty} \sum_y (\langle \varphi_x \cdot \varphi_y \rangle - \langle \varphi_x \rangle \cdot \langle \varphi_y \rangle)$ denote the* susceptibility*, and set $\beta_c = n$. For the Ising case $n = 1$, show that*

(i) Show that the spontaneous magnetisation obeys

$$\varphi_0(\beta, 0_+) \begin{cases} > 0 & (\beta > \beta_c) \\ = 0 & (\beta \leqslant \beta_c), \end{cases} \tag{1.34}$$

and $\varphi_0(\beta, 0_+) \sim \sqrt{3(\beta - \beta_c)}$ as $\beta \downarrow \beta_c$.

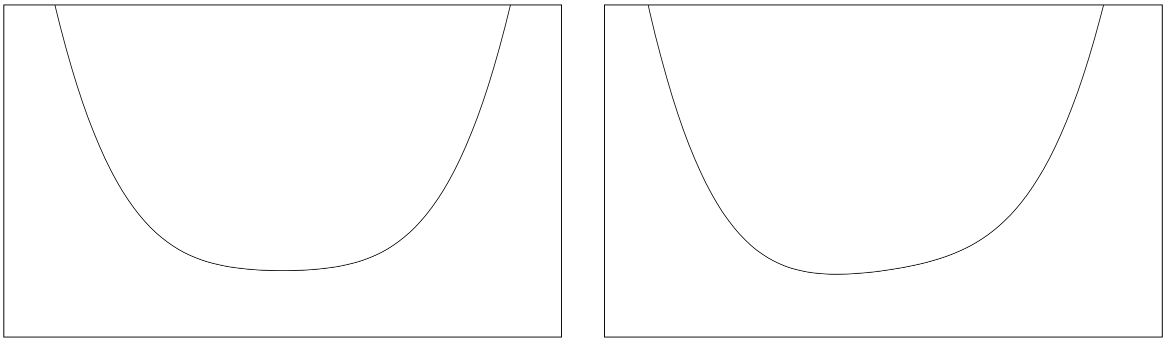

FIGURE 1.2. The renormalised potential for $\beta < \beta_c$ with $h = 0$ (left) and $h \neq 0$ (right). The renormalised potential is convex and the minimum is assumed at a unique point in both cases.

(ii) Show that the susceptibility obeys

$$\chi(\beta, 0) = \frac{1}{\beta_c - \beta} \quad (\beta < \beta_c), \qquad \chi(\beta, 0) \sim \frac{1}{2(\beta - \beta_c)} \quad (\beta \downarrow \beta_c). \tag{1.35}$$

Sketch. (i) $\tanh(x) = x - \frac{1}{3}x^3 + o(x^3)$ and $\varphi_0(\beta, 0_+) \to 0$ as $\beta \to \beta_c$ implies

$$\varphi_0(\beta, 0_+) = \tanh(\beta\varphi_0(\beta, 0_+)) = \beta\varphi_0(\beta, 0_+) - \frac{1}{3}(\beta\varphi_0(\beta, 0_+))^3 + o(\beta\varphi_0(\beta, 0_+))^3 \tag{1.36}$$

and therefore

$$(\beta - 1)\varphi_0 = \frac{1}{3}(\beta\varphi_0)^3 + o(\beta\varphi_0)^3. \tag{1.37}$$

The claim follows by dividing by $\varphi_0/3$ and taking the square root:

$$\varphi_0^2 \sim 3\frac{\beta - 1}{\beta^3} \sim 3(\beta - \beta_c). \tag{1.38}$$

(ii) One can show that

$$\chi = \frac{1}{-\beta + (1 - \varphi_0(\beta, h)^2)^{-1}}. \tag{1.39}$$

This implies

$$\chi = \frac{1}{-\beta + (1 - \varphi_0(\beta, 0_+)^2)^{-1}} = \frac{1}{1 - \beta} = \frac{1}{\beta_c - \beta} \quad (\beta < \beta_c), \tag{1.40}$$

$$\chi \sim \frac{1}{-\beta + (1 - 3(\beta - 1))^{-1}} \sim \frac{1}{1 - \beta + 3(\beta - 1)} = \frac{1}{2(\beta - \beta_c)} \quad (\beta > \beta_c), \tag{1.41}$$

as claimed. □

Exercise 1.14. *For the Ising case, use the self-consistent equation $\varphi_0 = \tanh(\beta\varphi_0 + h)$ to show that $\varphi_0 = \varphi_0(\beta, h)$ satisfies the inviscid Burgers' equation*

$$\frac{\partial}{\partial\beta}\varphi_0 = \varphi_0\frac{\partial}{\partial h}\varphi_0.$$

The inviscid Burger's equation is a prototype for a PDE that develops shocks.

1.4 Brascamp–Lieb inequality

In mean-field theory, the main questions can be reduced to asymptotic analysis of finite-dimensional integrals. For general spin couplings this is not possible. In the presence of *convexity* (excluding the most subtle regions of the phase diagram), the Brascamp–Lieb inequality is a powerful inequality that allows to obtain a bound in terms of the related Gaussian model.

Let Λ be a finite set, and let $\langle\cdot\rangle_H$ denote the expectation of a probability measure on $\mathbb{R}^\Lambda$ with density proportional to e^{-H} where $H : \mathbb{R}^\Lambda \to \mathbb{R}$.

Theorem 1.15 (Brascamp–Lieb inequality)**.** *Assume that H is uniformly convex, i.e., there is $c > 0$ such that $\mathrm{Hess}H(\varphi) \geqslant c\,\mathrm{id}$ for all $\varphi \in \mathbb{R}^\Lambda$. Then for any nice $u : \mathbb{R}^\Lambda \to \mathbb{R}$,*

$$\mathrm{var}_H(u) \leqslant \langle Du(\mathit{Hess}H)^{-1}Du\rangle_H. \tag{1.42}$$

In particular, if $(f, \mathit{Hess}H(\varphi)f) \geqslant (f, Qf)$ for all $f \in \mathbb{R}^\Lambda$, uniformly in $\varphi \in \mathbb{R}^\Lambda$,

$$\langle e^{(\varphi,f)-\langle(\varphi,f)\rangle_H}\rangle_H \leqslant e^{\frac{1}{2}(f,Q^{-1}f)} \quad \textit{for all } f \in \mathbb{R}^\Lambda. \tag{1.43}$$

Proof. We follow the approach of Helffer–Sjöstrand. For a nice function $v : \mathbb{R}^\Lambda \to \mathbb{R}$, define

$$Lv(\varphi) = \sum_{i\in\Lambda}\Big(-D_i^2 v(\varphi) + (D_iH(\varphi))D_iv(\varphi)\Big), \quad D_i = \frac{\partial}{\partial\varphi_i}. \tag{1.44}$$

By integration by parts, then

$$\langle vLv\rangle = \langle (Dv)(Dv)\rangle \equiv \sum_{i\in\Lambda}\langle (D_iv)(D_iv)\rangle. \tag{1.45}$$

For a nice function $g : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$, define the Witten Laplacian

$$\mathcal{L}g(i,\varphi) = \sum_{j\in\Lambda} D_iD_jH(\varphi)g(j,\varphi) + Lg(i,\varphi). \tag{1.46}$$

The operator L is defined by the quadratic form (1.45) and is therefore positive on $L^2(\mu_H)$ and has a self-adjoint extension. The operator $\mathcal{L}$ is then self-adjoint on $\Lambda \otimes L^2(\mu_H)$ and $\mathcal{L} \geqslant \mathrm{Hess}H \geqslant c > 0$ as quadratic forms. In particular, $\mathcal{L}$ is invertible and $\mathcal{L}^{-1} \leqslant (\mathrm{Hess}H)^{-1}$ holds as quadratic forms.

Write $Dv(i,\varphi) = D_iv(\varphi)$. Helffer–Sjöstrand observed the elementary identity $DLv = \mathcal{L}Dv$. By integration by parts, it implies in particular that

$$\langle (Lv)^2\rangle = \langle Dv\mathcal{L}Dv\rangle = \langle DLv\mathcal{L}^{-1}DLv\rangle. \tag{1.47}$$

Now assume $Lv = u - \langle u\rangle$ (this equation can be solved under the above assumptions). Then the left-hand side of the Helffer–Sjöstrand identity (1.47) is $\mathrm{var}_H(u)$ and we obtain

$$\mathrm{var}_H(u) = \langle Du\mathcal{L}^{-1}Du\rangle_H \leqslant \langle Du(\mathrm{Hess}H)^{-1}Du\rangle_H. \tag{1.48}$$

The inequality (1.43) follows from (1.42) by replacing H by $H_t = H - tu$ with $u(\varphi) = (f,\varphi)$. Note that H and H_t have the same Hessian. Let

$$\Phi(t) = \log\langle e^{t((f,\varphi)-\langle(f,\varphi)\rangle)}\rangle_H. \tag{1.49}$$

Then $\Phi(0) = 0$, $\Phi'(0) = 0$, and

$$\frac{\partial^2\Phi}{\partial t^2} = \mathrm{var}_{H_t}((f,\varphi)) \leqslant \langle (f,(\mathrm{Hess}H)^{-1}f)\rangle \leqslant (f,Q^{-1}f), \tag{1.50}$$

which implies the claim $\Phi(1) \leqslant \frac{1}{2}(f,Q^{-1}f)$. □

1.5 High temperature

The Brascamp–Lieb inequality is typically most effective at high temperatures. As an example, the model $O(n)$ has bounded susceptibility whenever the temperature is sufficiently large. Consider the $O(n)$ model with measure

$$\nu_h(d\sigma) = \frac{1}{Z} e^{-\frac{1}{2}(\sigma, M\sigma)+(h,\sigma)} \, \mu^{\otimes\Lambda}(d\sigma) \tag{1.51}$$

for some positive definite matrix M. Denote by $\|M\|$ the largest eigenvalue of M. Write $\langle\cdot\rangle_h = \mathbb{E}_{\nu_h}$.

Exercise 1.16. *Let M be a positive definite matrix with $\|M\| < \beta < n$. There is a universal constant C_β such that, with $h = (h, 0, \dots, 0)$, $h \geqslant 0$,*

$$\frac{1}{|\Lambda|} \sum_{x,y\in\Lambda} (\langle \sigma_x \cdot \sigma_y\rangle - \langle\sigma_x\rangle \cdot \langle\sigma_y\rangle) \leqslant C_\beta, \qquad 0 \leqslant \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \langle \sigma_x^1\rangle_h \leqslant C_\beta h. \tag{1.52}$$

Here is the idea. In the mean-field case, we decomposed the inverse coupling matrix $(M+m^2)^{-1}$ as a sum of two positive definite matrices. In general, if the eigenvalues of M are in $[0,\beta)$, i.e., M is positive definite and $M < \beta\,\mathrm{id}$ as quadratic forms, we can choose m^2 sufficiently small that $m^2 \leqslant M + m^2 \leqslant \beta$ and then decompose $(M+m^2)^{-1}$ as

$$(M+m^2)^{-1} = \frac{1}{\beta}\mathrm{id} + B^{-1} \tag{1.53}$$

with $\beta > 0$ and with B a (strictly) positive definite matrix. By (1.4) and using that $|\sigma| = 1$,

$$e^{-\frac{1}{2}(\sigma, M\sigma)} \propto e^{-\frac{1}{2}(\sigma,(M+m^2)\sigma)} \propto \int_{(\mathbb{R}^n)^\Lambda} e^{-\frac{\beta}{2}(\varphi-\sigma,\varphi-\sigma)} \, e^{-\frac{1}{2}(\varphi, B\varphi)} \, d\varphi. \tag{1.54}$$

This is analogous to (1.21) except that $\varphi \in (\mathbb{R}^n)^\Lambda$ is not constant. Define the renormalised potential as in the mean-field case by (1.24). In particular, by Exercise 1.12, the potential V is strictly convex if $\beta < n$, and in the Ising case, explicitly,

$$V(\varphi) = \frac{\beta}{2}\varphi^2 - \log\cosh(\beta\varphi + h). \tag{1.55}$$

Similarly to the mean-field case, define the renormalised measure ν_r on $(\mathbb{R}^n)^\Lambda$ and and the fluctuation measure on $(\mathbb{S}^{n-1})^\Lambda$ by

$$\mathbb{E}_{\nu_r}(F(\varphi)) \propto \int_{(\mathbb{R}^n)^\Lambda} e^{-\frac{1}{2}(\varphi, B\varphi) - \sum_{x\in\Lambda} V(\varphi_x)} \, d\varphi, \tag{1.56}$$

$$\mathbb{E}_{\mu_\varphi}(F(\sigma)) \propto \prod_{x\in\Lambda} \int_{\mathbb{S}^{n-1}} F(\sigma) \, e^{-\frac{\beta}{2}(\varphi_x - \sigma_x)^2} \, \mu(d\sigma_x), \tag{1.57}$$

and notice that then

$$\mathbb{E}_\nu(F(\sigma)) = \mathbb{E}_{\nu_r}(\mathbb{E}_{\mu_\varphi}(F(\sigma))). \tag{1.58}$$

Since V is strictly convex, provided that $\beta < n$, we can apply the Brascamp–Lieb inequality to the measure ν_r with

$$(\mathrm{Hess} H)^{-1} \leqslant \frac{1}{\beta - \beta^2/n}\,\mathrm{id}. \tag{1.59}$$

Sketch of first bound in (1.52). Let $F(\sigma) = \sum_{x \in \Lambda} \sigma_x$ and $G(\varphi) = \mathbb{E}_{\mu_\varphi}(F(\sigma))$. Then

$$\operatorname{var}_\nu(F) = \mathbb{E}_{\nu_r} \operatorname{var}_{\mu_\varphi}(F(\sigma)) + \operatorname{var}_{\nu_r}(G(\varphi)). \tag{1.60}$$

It is not difficult to check that the first term is of order $|\Lambda|$. The Brascamp–Lieb inequality implies

$$\operatorname{var}_{\nu_r}(G(\varphi)) \leqslant C_\beta \sum_x \mathbb{E}_{\nu_r}(D_x G(\varphi))^2 \leqslant C_\beta |\Lambda|, \tag{1.61}$$

where the last inequality follows from the fact that $DG(\varphi)$ is uniformly bounded. This completes the sketch of the proof. □

Remark 1.17. *Under the same assumption, the measure satisfies a logarithmic Sobolev inequality with constant only depending on β; see [2].*

2 Spontaneous breaking of continuous symmetry

2.1 The infrared bound and its consequences

For simplicity, we again consider the $\mathbb{S}^{n-1}$ model, $n \geqslant 1$, but the results can easily be extended to all models with $O(n)$ symmetry. However, we must again make a special choice of spin couplings: we now assume that Λ is a discrete d-dimensional torus of side length L and that

$$\beta_{xy} = \beta 1_{x\sim y} \quad \text{for some } \beta > 0, \tag{2.1}$$

where $x \sim y$ denotes that x and y are nearest-neighbours in Λ. Thus the corresponding matrix M is $M = -\beta\Delta$ with Δ the nearest-neighbour Laplace operator. The Laplace operator is invertible on the subspace of $\mathbb{R}^\Lambda$ orthogonal to the constant functions, i.e., $\Delta^{-1} f$ exists when $\sum_x f_x = 0$.

The methods used to prove the following results do not work if, e.g., $\beta_{xy} = 1_{|x-y|\leqslant 3}$.

Theorem 2.1 (Infrared bound, Fröhlich–Simon–Spencer). *Let μ be any measure on $\mathbb{R}^n$ with sufficient decay (not necessarily $O(n)$-symmetric), and let $\langle\cdot\rangle$ be the corresponding expectation* (1.16) *with nearest-neighbour interaction as above. Then for any $f : \Lambda \to \mathbb{R}^n$ with $\sum_x f_x = 0$,*

$$\langle e^{(f,\varphi)}\rangle \leqslant e^{\frac{1}{2}(f,(-\beta\Delta)^{-1}f)}. \tag{2.2}$$

In particular, for f with $\sum f = 0$,

$$\operatorname{var}(f,\varphi) = \langle (f,\varphi)^2\rangle \leqslant (f,(-\beta\Delta)^{-1}f). \tag{2.3}$$

Note the formal similarity with the Brascamp–Lieb inequality. If $\mu(d\varphi) = e^{-V(\varphi)}d\varphi$, then the infrared bound is an estimate for the same measure that appears in the Brascamp–Lieb inequality if $H(\varphi)$ is taken to be the specific choice

$$\frac{1}{4}\sum_{i,j} \beta_{ij}|\varphi_i - \varphi_j|^2 + \sum_j V(\varphi_j), \quad \beta_{ij} = \beta 1_{i\sim j}. \tag{2.4}$$

On the other hand, while the Brascamp–Lieb inequality involves the Hessian of the full Hamiltonian, the infrared bound only involves that of the (nearest-neighbour) interaction and requires the restriction $\sum f = 0$. As a result, unlike the Brascamp–Lieb inequality, it applies in particular when V is very *non-convex*. It is a very remarkable estimate and its proof is deceivingly simple. The strength of the infrared bound comes at the cost that the spin coupling part is restricted to the nearest neighbour interaction on the torus, or more generally reflection positive interactions.

Remark 2.2. *The name infrared bound comes from the following formulation in Fourier space. Let Λ^* be the Fourier dual of Λ, and $(\hat\varphi_p)_{p\in\Lambda^*}$ the Fourier transform of $(\varphi_x)_{x\in\Lambda}$:*

$$\Lambda^* = \{\frac{2\pi}{L}n : n \in \Lambda\}, \qquad \hat\varphi_p = \frac{1}{|\Lambda|^{1/2}}\sum_{x\in\Lambda} e^{ip\cdot x}\varphi_x = (\varphi, e_p). \tag{2.5}$$

Note that the Laplace operator acts by

$$(\widehat{\Delta f})_p = \hat\Delta(p) f_p, \quad \hat\Delta(p) = 2\sum_{i=1}^d (\cos p_i - 1). \tag{2.6}$$

The infrared bound (2.3) *implies that, for $p \in \Lambda^* \setminus \{0\}$,*

$$\langle|\hat\varphi_p|^2\rangle = \langle |(e_p,\varphi)|^2\rangle \leqslant (\bar e_p, (-\beta\Delta)^{-1}e_p) = \frac{1}{-\beta\hat\Delta(p)}. \tag{2.7}$$

Note that, by translation invariance, the left-hand side can also be written as

$$\langle|\hat{\varphi}_p|^2\rangle = \frac{1}{|\Lambda|}\sum_{x,y} e^{ip\cdot(x-y)}\langle\varphi_x\cdot\varphi_y\rangle = \frac{1}{|\Lambda|}\sum_{x,y} e^{ip\cdot(x-y)}\langle\varphi_0\cdot\varphi_{x-y}\rangle = \sum_x e^{ip\cdot x}\langle\varphi_0\cdot\varphi_x\rangle. \tag{2.8}$$

This bound is especially useful for p small, giving the name infrared bound.

Exercise 2.3. *Show that, if $d \geqslant 3$ then*

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{-\hat{\Delta}(p)} < \infty. \tag{2.9}$$

The following corollary of the infrared bound shows that the rotational symmetry of the $\mathbb{S}^{n-1}$ model is spontaneously broken if β is large. The proof is easy to extend to any $O(n)$ model.

Corollary 2.4. *Let $d \geqslant 3$. Let $\langle\cdot\rangle_h$ be denote the expectation of the $\mathbb{S}^{n-1}$ model on Λ, with external field he where $h > 0$ and $e \in \mathbb{S}^{n-1}$. Then*

$$\frac{1}{|\Lambda|}\sum_y \langle\varphi_x\cdot\varphi_y\rangle_0 \geqslant 1 - O(\frac{1}{\beta}), \qquad \langle e\cdot\varphi_x\rangle_h \geqslant 1 - O(\frac{1}{\beta}) + O_\beta(\frac{1}{h|\Lambda|}). \tag{2.10}$$

Exercise 2.5. *Use translation invariance and a Ward identity to show that $\langle e'\cdot\varphi_x\rangle_h = 0$ for any e' orthogonal to e. (This was shown in Magaritha Disertori's talk.)*

Proof of Corollary 2.4. Let $M = \frac{1}{|\Lambda|}\sum_x \varphi_x$. Then we aim to estimate

$$\langle|M|^2\rangle = \frac{1}{|\Lambda|}\sum_y \langle\varphi_x\cdot\varphi_y\rangle, \tag{2.11}$$

where the equality follows from translation invariance. This term can also be expressed as

$$\langle|M|^2\rangle = \frac{1}{|\Lambda|^2}\Big\langle\Big|\sum_x \varphi_x\Big|^2\Big\rangle = \frac{1}{|\Lambda|}\langle|\hat{\varphi}_0|^2\rangle. \tag{2.12}$$

By Parseval's identity and since $|\varphi_x| = 1$ for all x,

$$1 = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle|\varphi_x|^2\rangle = \frac{1}{|\Lambda|}\sum_{p\in\Lambda^*}\langle|\hat{\varphi}_p|^2\rangle = \frac{1}{|\Lambda|}\langle|\hat{\varphi}_0|^2\rangle + \frac{1}{|\Lambda|}\sum_{p\neq 0}\langle|\hat{\varphi}_p|^2\rangle. \tag{2.13}$$

Thus the infrared bound (2.7) implies

$$\langle|M|^2\rangle = \frac{1}{|\Lambda|}\langle|\hat{\varphi}_0|^2\rangle = 1 - \frac{1}{|\Lambda|}\sum_{p\neq 0}\langle|\hat{\varphi}_p|^2\rangle \geqslant 1 - \frac{1}{\beta|\Lambda|}\sum_{p\neq 0}(-\hat{\Delta})^{-1}(p) \geqslant 1 - \frac{\beta_0}{\beta}. \tag{2.14}$$

Since the second term on the right-hand side is $O(1/\beta)$ in $d \geqslant 3$ this implies the first bound.

To show that the spontaneous magnetisation is positive, first consider the measure with $h = 0$. Then the distribution of M is rotationally invariant, and hence the distribution of $M/|M|$ is uniform on $\mathbb{S}^{n-1}$ and independent of $|M|$. For any $e \in \mathbb{S}^{n-1}$, define

$$p = \mathbb{P}(|M| \geqslant 1-\delta), \qquad q = \mathbb{P}(\frac{M}{|M|}\cdot e \geqslant 1-\varepsilon). \tag{2.15}$$

Since $M/|M|$ is uniform, for any $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that

$$q = \mathbb{P}(\frac{M}{|M|}\cdot e \geqslant 1-\varepsilon) \geqslant c_\varepsilon. \tag{2.16}$$

By (2.14) and since $|M| \leqslant 1$, i.e., $1_{|M|\geqslant 1-\delta} \geqslant |M|1_{|M|\geqslant 1-\delta} = |M|(1-1_{|M|\leqslant 1-\delta}) \geqslant |M|^2 - (1-\delta)$,

$$p = \mathbb{P}(|M| \geqslant 1-\delta) \geqslant \langle |M|^2 \rangle - (1-\delta) \geqslant 1 - \beta_0/\beta - (1-\delta) \geqslant \delta - \beta_0/\beta. \tag{2.17}$$

Together, hence

$$\begin{aligned}\Phi(h) = \frac{1}{|\Lambda|}\log\langle e^{|\Lambda| h e\cdot M}\rangle_0 \geqslant \frac{1}{|\Lambda|}\log(pqe^{(1-\delta)(1-\varepsilon)h|\Lambda|}) &= (1-\delta)(1-\varepsilon)h + \frac{1}{|\Lambda|}\log(pq)\\ &\geqslant (1-2\delta)h - O_\delta(\frac{1}{|\Lambda|}),\end{aligned} \tag{2.18}$$

where the last inequality holds when choosing $\varepsilon = \delta$ and $\beta \geqslant 2\beta_0/\delta$.

Finally, Φ is convex and $\Phi(0) = 0$. Thus

$$\langle e \cdot M \rangle_h = \Phi'(h) \geqslant \frac{\Phi(h) - \Phi(0)}{h} \geqslant 1 - 2\delta + O_\delta(\frac{1}{h|\Lambda|}) \tag{2.19}$$

as claimed. □

2.2 Reflection positivity and proof of the infrared bound

By rescaling and absorbing the term with h in the reference measure μ, it suffices to prove (2.2) for $\beta = 1$ and $h = 0$ and any reference measure μ. Let $H(\varphi) = \frac{1}{2}(\varphi, -\Delta\varphi)$. For any f with $\sum f = 0$,

$$H(\varphi) - (f, \varphi) = \frac{1}{2}(\varphi, -\Delta\varphi) - (f, \varphi) = H(\varphi - \Delta^{-1} f) + \frac{1}{2}(f, (-\Delta)^{-1} f), \tag{2.20}$$

and hence

$$\int e^{-H(\varphi)+(f,\varphi)}\mu^{\otimes\Lambda}(d\varphi) = e^{\frac{1}{2}(f,(-\Delta)^{-1}f)} \int e^{-H(\varphi-\Delta^{-1}f)}\mu^{\otimes\Lambda}(d\varphi) = e^{\frac{1}{2}(f,(-\Delta)^{-1}f)} Z(-\Delta^{-1}f) \tag{2.21}$$

where

$$Z(f) = \int e^{-H(\varphi+f)}\mu^{\otimes\Lambda}(d\varphi). \tag{2.22}$$

To prove Theorem 2.1, it therefore suffices to show that $Z(f) \leqslant Z(0)$ for any f.

Consider a plane going through the midpoints of edges (an edge plane) splitting the torus into two halves. Such a plane gives a decomposition $\Lambda = \Lambda_+ \cup \Lambda_-$. Let $\theta : \Lambda_\pm \to \Lambda_\mp$ be the reflection about this plane, and

$$(\theta\varphi)_x = \varphi_{\theta(x)}, \quad (\theta F)(\varphi) = F(\theta\varphi). \tag{2.23}$$

Definition 2.6. *A probability measure on* $(\mathbb{R}^n)^\Lambda$ *with expectation* $\langle\cdot\rangle$ *is* reflection positive *if*

$$\langle F\theta G \rangle = \langle G\theta F \rangle, \quad \langle F\theta F \rangle \geqslant 0, \qquad \textit{for all } F, G : (\mathbb{R}^n)^\Lambda \to \mathbb{R}. \tag{2.24}$$

Lemma 2.7. *Any product measure* $\mu^{\otimes\Lambda}$ *is reflection positive.*

Proof. Clearly, $\varphi|_{\Lambda_+}$ and $\varphi|_{\Lambda_-}$ are independent, so

$$\langle F\theta G \rangle = \langle F \rangle \langle \theta G \rangle = \langle F \rangle \langle G \rangle, \tag{2.25}$$

and both conditions for reflection positivity are obvious from this. □

By definition, reflection positivity of $\langle\cdot\rangle$ means that $(F, G) \mapsto \langle F\theta G \rangle$ defines a symmetric positive semi-definite bilinear form. The importance of reflection positivity results from the Cauchy–Schwarz inequality

$$\langle F\theta G \rangle^2 \leqslant \langle F\theta F \rangle \langle G\theta G \rangle. \tag{2.26}$$

Lemma 2.8. *Let θ be a reflection and $\langle\cdot\rangle$ reflection positive. Then for any $A, B, C, D : (\mathbb{R}^n)^{\Lambda_+} \to \mathbb{R}$,*

$$\langle e^{A+\theta B+C\theta D}\rangle^2 \leqslant \langle e^{A+\theta A+C\theta C}\rangle\langle e^{B+\theta B+D\theta D}\rangle, \tag{2.27}$$

and the measures $\langle(\cdot)e^{A+\theta A+C\theta C}\rangle$ and $\langle(\cdot)e^{B+\theta B+D\theta D}\rangle$ are reflection positive. The same holds with $C\theta D$, $C\theta C$, and $D\theta D$ replaced by sums of such terms.

Proof. Expand the exponential as

$$e^{A+\theta B+C\theta D} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{(e^A C^k)}_{X_k} \theta \underbrace{(e^B D^k)}_{Y_k}. \tag{2.28}$$

Reflection positivity and the Cauchy-Schwarz inequality (twice) imply

$$\langle e^{A+\theta B+C\theta D}\rangle^2 \leqslant \left[\sum_{k=0}^{\infty} \frac{1}{k!}\langle X_k\theta X_k\rangle^{1/2}\langle Y_k\theta Y_k\rangle^{1/2}\right]^2 \leqslant \sum_{k=0}^{\infty}\frac{1}{k!}\langle X_k\theta X_k\rangle \sum_{k=0}^{\infty}\frac{1}{k!}\langle Y_k\theta Y_k\rangle. \tag{2.29}$$

By (2.28) and reflection positivity of $\langle\cdot\rangle$, we also have

$$\langle (F\theta F)e^{A+\theta A+C\theta C}\rangle = \sum_{k=0}^{\infty}\frac{1}{k!}\langle (FX_k)\theta(FX_k)\rangle \geqslant 0. \tag{2.30}$$

This completes the proof. $\square$

Lemma 2.9. *Fix a reflection θ. Let $f_+ = f$ on Λ_+ and $f_+ = \theta f$ on Λ_- and similarly for f_-. Then*

$$Z(f)^2 \leqslant Z(f_+)Z(f_-). \tag{2.31}$$

Proof. Denote by $E_\pm$ the set of edges such that both endpoints are contained in $\Lambda_\pm$ and by E_0 the edges crossing from Λ_+ to Λ_-. Then

$$H(\varphi) = \frac{1}{2}\sum_{E_+}|\varphi_x-\varphi_y|^2 + \frac{1}{2}\sum_{E_-}|\varphi_x-\varphi_y|^2 + \frac{1}{2}\sum_{E_0}|\varphi_x-\varphi_y|^2 = H_+(\varphi)+H_-(\varphi)+H_0(\varphi). \tag{2.32}$$

Since $H_-(\varphi) = \theta H_+(\varphi)$, and since

$$H_0(\varphi) = \frac{1}{2}\sum_{xy\in E_0}|\varphi_x-\varphi_y|^2 = \frac{1}{2}\sum_{x\in\Lambda_+\cap E_0}(|\varphi_x|^2 + \theta|\varphi_x|^2 + 2\varphi_x\cdot\theta\varphi_x), \tag{2.33}$$

we see that $H(\varphi)$ is of the form

$$H(\varphi) = A + \theta A + \sum C\theta C, \tag{2.34}$$

with $A, C : (\mathbb{R}^n)^{\Lambda_+} \to \mathbb{R}$. It follows that $H(\varphi+f)$ is of the form

$$H(\varphi+f) = A_{f_+} + \theta A_{f_-} + \sum C_{f_+}\theta C_{f_-}, \tag{2.35}$$

with $A_{f_\pm}, C_{f_\pm} : (\mathbb{R}^n)^{\Lambda_+} \to \mathbb{R}$. Hence Lemma 2.8 implies (2.31). $\square$

Proof of Theorem 2.1. It suffices to prove that, for any $f : \Lambda \to \mathbb{R}$,

$$Z(f) \leqslant Z(0). \tag{2.36}$$

For an edge $e = xy$ write $\nabla_e f = f_x - f_y$. Note that $\nabla_e f_\pm = 0$ for $e \in E_0$ and that $\nabla_e f = 0$ implies $\nabla_e f_\pm = 0$ for any $e \in E$. Therefore, by iteration of the previous lemma, it follows that

$$Z(f) \leqslant \sup_{g : \nabla g = 0} Z(g) = Z(0). \tag{2.37}$$

The last equality follows since $\nabla g = 0$ means $\nabla_e g = 0$ for all $e \in E$ and hence g is constant.

We finally prove (2.3). Since $\sum_x f_x = 0$ and since $\langle \varphi_x \rangle$ is constant in x, we have $\langle (f, \varphi) \rangle = \sum_x f_x \langle \varphi_0 \rangle = 0$. It follows that

$$\langle e^{t(f,\varphi)} \rangle = 1 + \frac{t^2}{2} \langle (f, \varphi)^2 \rangle + O(t^3), \quad e^{t^2 (f, (-\beta\Delta)^{-1} f)/2} = 1 + \frac{t^2}{2} (f, (-\beta\Delta)^{-1} f) + O(t^3). \tag{2.38}$$

We obtain (2.3) by subtracting 1, dividing by t^2, and then taking $t \to 0$. □

2.3 Hyperbolic sigma model

We consider hyperbolic sigma models, which are defined like the $O(n)$ models with the sphere $\mathbb{S}^{n-1}$ replaced by the hyperbolic space $\mathbb{H}^{n-1}$. These models are interesting for several reasons. It terms of phenomena, they are related to random matrix models and to linearly reinforced random walks. Here we consider them as interesting examples where the spontaneous breaking of a continuous symmetry can be shown with robust methods (not involving reflection positivity).

As previously, we start with a finite set Λ, but we now denote its elements by $i, j \in \Lambda$ because the letters x and y will be used to denote components of the spins. The spins of the $\mathbb{H}^n$ hyperbolic sigma model are points $u_i \in \mathbb{H}^n$ where $\mathbb{H}^n$ is n-dimensional hyperbolic space. For simpler notation, we take $n = 2$. Let $\mathbb{R}^{2,1}$ denote $(2+1)$-dimensional Minkowski space. Thus its elements are vectors $u = (x, y, z)$, and it is equipped with the indefinite inner product $u \cdot u = x^2 + y^2 - z^2$. The hyperbolic plane $\mathbb{H}^2$ can be realized as

$$\mathbb{H}^2 = \{u \in \mathbb{R}^{2,1} \mid u \cdot u = -1, z > 0\}. \tag{2.39}$$

Suppose Λ is finite and $h > 0$. To each vertex $i \in \Lambda$ we associate a spin $u_i \in \mathbb{H}^2$. The energy of a spin configuration $u = (u_i)_{i \in \Lambda} \in (\mathbb{H}^2)^\Lambda$ is

$$H(u) = H_{\beta,h}(u) = \frac{1}{2} \sum_{i,j} \beta_{ij} (-u_i \cdot u_j - 1) + h \sum_i (z_i - 1) \tag{2.40}$$

Exercise 2.10. *For $u, v \in \mathbb{H}^2$ verify that $\frac{1}{2}(u - v) \cdot (u - v) = -1 - u \cdot v \geqslant 0$ with equality if and only if $u = v$. Further verify that*

$$z_i - 1 = (-u_i \cdot e - 1), \quad \text{where } e = (0, 0, 1). \tag{2.41}$$

The energy (2.40) favours spin alignment because $-u \cdot v \geqslant 1$ for $u, v \in \mathbb{H}^2$ with equality if and only if $u = v$. The $\mathbb{H}^2$ sigma model is the measure with expectation

$$\langle F(u) \rangle_{\mathbb{H}^2} = \frac{1}{Z} \int_{(\mathbb{R}^2)^\Lambda} F(u)\, e^{-H(u)} \prod_{i \in \Lambda} \frac{dx_i\, dy_i}{z_i}, \quad z_i = \sqrt{1 + x_i^2 + y_i^2}. \tag{2.42}$$

This is completely analogous to the $O(n)$ model, replacing $\mathbb{S}^{n-1}$ by the hyperbolic space $\mathbb{H}^{n-1}$. Indeed, recall that the energy of a configuration $\sigma \in (\mathbb{S}^{n-1})^\Lambda$ of the $O(n)$ model can be written as

$$H(\sigma) = \frac{1}{4} \sum_{i,j} \beta_{ij} (\sigma_i - \sigma_j) \cdot (\sigma_i - \sigma_j) - h \sum_i e \cdot \sigma_i = \frac{1}{2} \sum_{i,j} \beta_{ij} (-\sigma_i \cdot \sigma_j + 1) - h \sum_i e \cdot \sigma_i. \tag{2.43}$$

However, an important difference is that hyperbolic space has infinite volume, and as a consequence the notions of what it means for the symmetry to be spontaneously broken differs slightly.

Exercise 2.11. *Show that*

$$\int_{\mathbb{R}^2} y^2 \,\frac{dx\,dy}{z} = \infty. \tag{2.44}$$

More generally, for any finite Λ and any $i \in \Lambda$, one has $\langle y_i^2\rangle_{\Lambda,h} \to \infty$ as $h \downarrow 0$.

Our goal is to prove the following theorem, showing that symmetry breaking always occurs for the $\mathbb{H}^2$ model, in $d \geqslant 3$. The statement of the theorem involves the coupling matrix M as in (1.6).

(Disertori–Spencer–Zirnbauer [11] proved the much more subtle result that symmetry breaking also occurs for a supersymmetric version of the model at large β.)

Theorem 2.12 (Spencer–Zirnbauer)**.** *For the $\mathbb{H}^2$ model with spin coupling β. Assume that $g = \max_i[(M+h)^{-1}]_{ii}$ is bounded. Then for all p there are constants C_p (depending on g) such that*

$$\langle y_i^p\rangle_h \leqslant C_p. \tag{2.45}$$

In particular, in $d \geqslant 3$, $\langle y_i^2\rangle_{\Lambda,h}$ remains bounded if first $|\Lambda| \uparrow \infty$ and then $h \downarrow 0$.

Remark 2.13. *On the other hand, in $d = 2$, $\langle y_i^2\rangle_{\Lambda,h}$ diverges in the above limit [4].*

An important ingredient of the proof of the theorem is that hyperbolic space can be parametrised by horospherical coordinates. For $\mathbb{H}^2$, these are global coordinates $t \in \mathbb{R}$, $s \in \mathbb{R}$, in terms of which

$$x = \sinh t - \frac{1}{2}s^2e^t, \quad y = e^ts, \quad z = \cosh t + \frac{1}{2}s^2e^t. \tag{2.46}$$

Exercise 2.14. *In horospherical coordinates the inner product can be expressed as*

$$-\,u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2e^{t_i+t_j}. \tag{2.47}$$

Exercise 2.15. *In horospherical coordinates, the following change of variable formula holds:*

$$\int_{(\mathbb{R}^2)^\Lambda} F(u(x,y)) \prod_i \frac{dx_i\,dy_i}{z_i} = \int_{(\mathbb{R}^2)^\Lambda} F(u(s,t)) \prod_i e^{t_i}\,dt_i\,ds_i. \tag{2.48}$$

Lemma 2.16. *For any function of the $t_i = \log(x_i + z_i)$ only,*

$$\langle F(t)\rangle = \frac{1}{Z}\int_{\mathbb{R}^\Lambda} F(t)e^{-\hat H(t)} \prod_i dt_i \tag{2.49}$$

where

$$\hat H(t) = \frac{1}{2}\sum_{i,j}\beta_{ij}\cosh(t_i - t_j) + h\sum_i \cosh(t_i) + \frac{1}{2}\log\det(D(t)/2\pi) - \sum_i t_i, \tag{2.50}$$

and where $D(t)$ is the symmetric matrix defined by the quadratic form

$$(f, D(t)f) = \frac{1}{2}\sum_{i,j}\beta_{ij}e^{t_i+t_j}(f_i - f_j)^2 + h\sum_i e^{t_i}f_i^2 \qquad (f \in \mathbb{R}^\Lambda). \tag{2.51}$$

Proof. By (2.46) and (2.47), the energy can be expressed in horospherical coordinates as

$$H(u(s,t)) = \frac{1}{2}\sum_{i,j}\beta_{ij}(\cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2e^{t_i+t_j}) + h\sum_i(\cosh(t_i) + \frac{1}{2}s_i^2e^{t_i}), \tag{2.52}$$

and by (2.48),

$$\langle F(t)\rangle \propto \int_{(\mathbb{R}^2)^\Lambda} F(t)e^{-H(u(s,t))+\sum_i t_i} \prod dt_i\,ds_i. \tag{2.53}$$

For every fixed $t \in \mathbb{R}^\Lambda$, the integral of the s-variables is Gaussian:

$$\int_{\mathbb{R}^\Lambda} \exp\left(-\frac{1}{4}\sum_{i,j}\beta_{ij}(s_i-s_j)^2 e^{t_i+t_j}) + hs_i^2 e^{t_i}\right)\prod_i ds_i = \frac{1}{\sqrt{\det(D(t)/2\pi)}}, \tag{2.54}$$

where the equality holds since (1.2) defines a probability measure. $\square$

Lemma 2.17. *The function $t \mapsto \log\det D(t)$ is convex on $\mathbb{R}^\Lambda$.*

Proof. The determinant $\det D(t)$ can be written as a convex combination of exponentials [17], i.e., there is a finite set $A \subset \mathbb{R}^\Lambda$ and weights $p(a) > 0$ such that

$$\det D(t) = \sum_{a\in A} p(a)e^{(t,a)}. \tag{2.55}$$

Any such function is log-convex. Indeed, given $t \in \mathbb{R}^\Lambda$ define a probability measure on A by

$$\mathbb{E}_t(g(a)) \propto \sum_{a\in A} g(a)p(a)e^{(t,a)}. \tag{2.56}$$

Then for any $b \in \mathbb{R}^\Lambda$,

$$(b, \mathrm{Hess}(\log\det D(t))b) = \mathbb{E}_t((b,a)^2) - (\mathbb{E}_t(b,a))^2 = \mathrm{var}_t(b,a) \geqslant 0, \tag{2.57}$$

where var_t denotes the variance of the probability measure on A with expectation $\mathbb{E}_t$. $\square$

Proof of Theorem 2.12. By convexity of $\log\det D(t)$ and explicit computation for the other terms,

$$\mathrm{Hess}\hat{H}(t) \geqslant D(0) = M + h. \tag{2.58}$$

By the Brascamp–Lieb inequality,

$$\langle e^{\alpha t_0 - \alpha\langle t_0\rangle}\rangle \leqslant e^{g\alpha^2/2}. \tag{2.59}$$

This shows that the t-field concentrates strongly near its mean. We need bounds on $\langle t_0\rangle$ to complete the proof. Since $\langle \sinh t_0 - \frac{1}{2}s_0^2 e^{t_0}\rangle = \langle x_0\rangle = 0$ by symmetry, and using the Jensen inequality,

$$\langle e^{t_0} - e^{-t_0}\rangle = 2\langle\sinh(t_0)\rangle = \langle s_0^2 e^{t_0}\rangle \geqslant 0, \qquad \langle e^{-t_0}\rangle \geqslant e^{-\langle t_0\rangle}, \tag{2.60}$$

which together with the Brascamp–Lieb inequality gives

$$e^{\langle t_0\rangle + g/2} \geqslant \langle e^{t_0}\rangle \geqslant e^{-\langle t_0\rangle}, \quad \text{i.e., } \langle t_0\rangle \geqslant -g/4. \tag{2.61}$$

Below the proof, we show that $\langle\sinh(t_0)\rangle$ is bounded. Together with the bound on $\langle e^{-t_0}\rangle$ and the Jensen inequality, this implies that $\langle t_0\rangle$ is bounded above:

$$e^{\langle t_0\rangle} \leqslant \langle e^{t_0}\rangle = 2\langle\sinh t_0\rangle + \langle e^{-t_0}\rangle \leqslant 2\langle\sinh t_0\rangle + e^{-\langle t_0\rangle + g/2} \leqslant 2\langle\sinh t_0\rangle + e^{3g/4}. \tag{2.62}$$

Finally, combining the above ingredients, it follows that

$$\langle y_0^p\rangle = \langle x_0^p\rangle \leqslant \langle (x_0+z_0)^p\rangle = \langle e^{pt_0}\rangle \leqslant e^{p\langle t_0\rangle + p^2 g/2} \tag{2.63}$$

which implies the claim. $\square$

Lemma 2.18. *Let $g = \max_i[(M+h)^{-1}]_{ii}$. Then*

$$\langle e^{t_0}s_0^2\rangle \leqslant e^{O(g)}g. \tag{2.64}$$

Proof. Let $G_t = D_t^{-1}$. Then integrating over s gives

$$\langle e^{t_0} s_0^2 \rangle = \langle e^{t_0} G_t(0,0) \rangle. \tag{2.65}$$

Using Cauchy–Schwarz, we will show

$$G_t(0,0) \leqslant \frac{1}{2} \sum_{i,j} \beta_{ij} e^{-t_i - t_j} (G_0(0,i) - G_0(0,j))^2 + h \sum_j e^{-t_j} G_0(0,j)^2. \tag{2.66}$$

By Brascamp–Lieb and the lower bound on $\langle t_0 \rangle \geqslant -O(g)$ we have $\langle e^{-t_i - t_j + t_0} \rangle = e^{-\langle t_0 \rangle + O(g)} = e^{O(g)}$ and $\langle e^{-t_i + t_0} \rangle = e^{O(g)}$. Setting $f(i) = G_0(0,i)$, we get

$$\begin{aligned}
\langle e^{t_0} s_0^2 \rangle = \langle e^{t_0} G_t(0,0) \rangle &\leqslant \frac{1}{2} \sum_{i,j} \beta_{ij} \langle e^{-t_i - t_j + t_0} \rangle (f(i) - f(j))^2 + \sum_i h \langle e^{-t_i + t_0} \rangle f(i)^2 && (2.67)\\
&\leqslant e^{O(g)} \left[\frac{1}{2} \sum_{i,j} \beta_{ij} (f(i) - f(j))^2 + h \sum_i f(i)^2 \right] && (2.68)\\
&= e^{O(g)} (D_0 f, f) = e^{O(g)} G_0(0,0) = e^{O(g)} g. && (2.69)
\end{aligned}$$

This completes the proof, given (2.66). In the remainder of the proof, we show (2.66).

Let $\beta_{ij}(t) = \beta_{ij} e^{t_i + t_j}$ and $h_i(t) = h e^{t_i}$. Let $\nabla_j f_i = f(i) - f(j)$. Then

$$(D_t f, f) = \sum_i (D_t f)_i f_i = \frac{1}{2} \sum_{i,j} \beta_{ij}(t) (\nabla_j f_i)^2 + \sum_i h_i(t) f_i^2, \tag{2.70}$$

i.e.,

$$(D_t f)_i = \sum_j \beta_{ij}(t) \nabla_j f_i + h_i f_i. \tag{2.71}$$

Using that $\beta_{ij} = \beta_{ij}(0) = \sqrt{\beta_{ij}(t)\beta_{ij}(-t)}$ and similarly for h,

$$\begin{aligned}
(f, G_t f) &= (D_0 G_0 f, G_t f) && (2.72)\\
&= \frac{1}{2} \sum_{i,j} \beta_{ij}(-t)^{1/2} (\nabla_j G_0 f(i)) \beta_{ij}(t)^{1/2} (\nabla_j G_t f(i)) + \sum_i h_i(-t)^{1/2} G_0 f(i) h_i(t)^{1/2} G_t f(i).
\end{aligned}$$

By Cauchy–Schwarz,

$$(f, G_t f)^2 \leqslant \frac{1}{2} (D_t G_t f, G_t f) \left[\sum_{i,j} \beta_{ij} e^{-t_i - t_j} (\nabla_j G_0 f(i))^2 + \sum_i h e^{-t_i} (G_0 f(i))^2 \right] \tag{2.73}$$

and using that $D_t G_t = \mathrm{id}$ thus

$$(f, G_t f) \leqslant \frac{1}{2} \sum_{i,j} \beta_{ij} e^{-t_i - t_j} (\nabla_j G_0 f(i))^2 + \sum_i h e^{-t_i} (G_0 f(i))^2. \tag{2.74}$$

This implies (2.66). □

3 Spin systems, random walks and supersymmetry

Spin systems are intimately connected to interacting random walks. This section is based on [4]. More on the mathematics of supersymmetry can be found in [7, 10, 11] and references.

3.1 The BFS–Dynkin isomorphism theorem

The continuous-time simple random walk is a Markov process (X_t) with values in Λ. Given $X_t = i$ it jumps to a vertex j with rate β_{ij}:

$$\mathbb{P}(X_{t+\delta t} = j | X_t = i) = \beta_{ij}\delta t + o(\delta t). \tag{3.1}$$

In terms of the theory of Markov processes, this means that X is described by the generator

$$\mathcal{L}g(i) = \sum_j \beta_{ij}(g(j) - g(i)). \tag{3.2}$$

Denoting by $\mathbb{E}_i$ the expectation of this Markov process with $X_0 = i$, this means that

$$g_t(i) = \mathbb{E}_i(g(X_t)) \tag{3.3}$$

satisfies

$$\frac{\partial}{\partial t} g_t(i) = \mathcal{L}g_t(i). \tag{3.4}$$

The local time $L_t = (L_t^i)_{i\in\Lambda}$ of X is defined by

$$L_t^i = \int_0^t 1_{X_s = i}\, ds. \tag{3.5}$$

The expectation of the n-component free field with the same coupling constants β was defined in (1.5), (1.6) (with $h = 0$).

Theorem 3.1 (BFS–Dynkin isomorphism)**.** *Let $\langle\cdot\rangle_{\mathbb{R}^n}$ denote the expectation of the n-component Gaussian free field with spin coupling β and mass m, and let X denote the simple random walk with jump rates given in terms of the same β. Then*

$$\langle \varphi_i^1 \varphi_j^1 g(\tfrac{1}{2}|\varphi|^2)\rangle_{\mathbb{R}^n} = \int_0^\infty \langle \mathbb{E}_i(1_{X_t=j}\, g(L_t + \tfrac{1}{2}|\varphi|^2))\rangle_{\mathbb{R}^n}\, e^{-m^2 t}\, dt. \tag{3.6}$$

Remark 3.2. *This theorem was proved by Brydges–Fröhlich–Spencer in [8]. Dynkin [12] then expressed it as the statement ("isomorphism theorem") that*

$$\frac{1}{2}|\varphi|^2 \quad \text{under the (signed) measure } \varphi_x\varphi_y P_{\mathrm{GFF}} \tag{3.7}$$

$$L_\infty + \frac{1}{2}|\varphi|^2 \quad \text{under the (positive) measure } P_{\mathrm{GFF}} \otimes P_{xy} \tag{3.8}$$

have the same distribution, where P_{GFF} is the measure of the GFF and P_{xy} is that of the simple random walk from x to y with killing rate m^2. See [18] for a review of such isomorphism theorems.

We will not prove this theorem and instead give a proof for a related theorem for the hyperbolic sigma model. The BFS–Dynkin isomorphism theorem can be proved using the same method as the proof below (though the original proofs look very different).

3.2 Reinforced walks and hyperbolic symmetry

We explain that the $\mathbb{H}^n$ sigma models satisfy a relation analogous to the BFS–Dynkin isomorphism for the free field. In this relation, the simple random walk is replaced by an interacting random walk, the vertex-reinforced jump process (VRJP). The VRJP takes steps from i to j with probability

$$\mathbb{P}(X_{t+\delta t} = j | X_t = i, L_t) = \beta_{ij}(1 + L_t^j)\delta t + o(\delta t). \tag{3.9}$$

It is not a Markov process, but the joint process (X_t, L_t) is a Markov process with generator

$$\mathcal{L}^\beta g(i,\ell) = \sum_j \beta_{ij}(1+\ell_j)(g(j,\ell) - g(i,\ell)) + \frac{\partial}{\partial \ell_i} g(i,\ell). \tag{3.10}$$

We denote the expectation of this Markov process with initial condition $(X_0, L_0) = (i, \ell)$ by $\mathbb{E}^\beta_{i,\ell}$ or simply by $\mathbb{E}_{i,\ell}$.

Theorem 3.3. *Let $h > 0$, let $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a bounded smooth function, and let $a, b \in \Lambda$. Then*

$$\sum_b \langle y_a y_b g(b, z-1)\rangle_{\mathbb{H}^2} = \langle z_a \int_0^\infty \mathbb{E}^\beta_{a,z-1}(g(X_t, L_t))\, e^{-ht}\, dt\rangle_{\mathbb{H}^2}. \tag{3.11}$$

The proof can be done in other coordinates, but we again use horospherical coordinates. Recall

$$x = \sinh t - \frac{1}{2}s^2 e^t, \quad y = e^t s, \quad z = \cosh t + \frac{1}{2}s^2 e^t, \tag{3.12}$$

and

$$-u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2 e^{t_i + t_j} \tag{3.13}$$

and write

$$H_{\beta,h}(u) = \frac{1}{2}\sum_{i,j}\beta_{ij}(-u_i\cdot u_j - 1) + h\sum_i (z_i - 1), \qquad \int_{\mathbb{H}^2} F(u) = \int_{\mathbb{R}^2} F(u(s,t))\, e^t\, dt\, ds. \tag{3.14}$$

Exercise 3.4.

$$\frac{\partial z_i}{\partial s_i} = y_i, \qquad \frac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \frac{\partial (u_i\cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j). \tag{3.15}$$

Lemma 3.5. *Let $a \in \Lambda$, and let $g\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay. Then*

$$-\sum_b \int_{(\mathbb{H}^2)^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = \int_{(\mathbb{H}^2)^\Lambda} e^{-H_{\beta,0}} z_a g(a, z-1). \tag{3.16}$$

Proof. The integral stands for $\int_{(\mathbb{H}^2)^\Lambda}$ and we abbreviate $H = H_{\beta,0}$. By (3.15) we have

$$y_b \frac{\partial}{\partial \ell_b} g(b, z-1) = \frac{\partial}{\partial s_b} g(b, z-1) \tag{3.17}$$

where $\frac{\partial}{\partial \ell_b}$ denotes the derivative with respect to the b-th component of the second argument. Thus

$$\begin{aligned}&\sum_b \int e^{-H} y_a y_b \mathcal{L} g(b, z-1)\\ &\qquad = \int e^{-H} y_a \Big(\sum_{b,c}\beta_{bc} y_b z_c (g(c,z-1) - g(b,z-1)) + \sum_b \frac{\partial}{\partial s_b} g(b, z-1)\Big).\end{aligned} \tag{3.18}$$

Recall (2.48) and integrate the second term in the equation above by parts. This produces two terms; by the rapid decay of g there are no boundary terms. For the first term produced by the integration by parts, using (3.15) again,

$$\begin{aligned}
\sum_b \int e^{-H} y_a \left(-\frac{\partial H}{\partial s_b}\right) g(b, z-1) &= \sum_b \int e^{-H} y_a \left(\sum_c \beta_{bc} \frac{\partial (u_b \cdot u_c)}{\partial s_b}\right) g(b, z-1) \\
&= \int e^{-H} y_a \sum_{b,c} \beta_{bc} (y_c(x_b + z_b) - y_b(x_c + z_c)) g(b, z-1) \\
&= \int e^{-H} y_a \sum_{b,c} \beta_{bc} (y_c z_b - y_b z_c) g(b, z-1) \\
&= \int e^{-H} y_a \sum_{b,c} \beta_{bc} y_b z_c (g(c, z-1) - g(b, z-1)). \qquad (3.19)
\end{aligned}$$

This term cancels the first term on the right-hand side of (3.18). For the second term produced by the integration by parts, we use that $\int x_a e^{-H} g(b, z) = 0$ by symmetry, and thus

$$\int e^{-H} \frac{\partial y_a}{\partial s_b} g(b, z-1) = \delta_{ab} \int e^{-H} (x_a + z_a) g(b, z-1) = \delta_{ab} \int e^{-H} z_a g(a, z-1). \qquad (3.20)$$

Altogether, we have shown (3.16). □

Proof of Theorem 3.3. It suffices to show (3.11) with $h = 0$, by replacing $g(b, z-1)$ by $g(b, z-1)e^{-h(z-1)}$. Therefore assume $h = 0$. To get (3.11) from (3.16), we apply (3.16) with $g(i, \ell)$ replaced by $g_t(i, \ell) = \mathbb{E}_{i,\ell}(g(X_t, L_t))$. By the definition of the generator we have $\mathcal{L} g_t(i, \ell) = \frac{\partial}{\partial t} g_t(i, \ell)$, so (3.16) gives

$$\int e^{-H_{\beta,0}} z_a \mathbb{E}_{a,z-1}(g(X_t, L_t)) = -\frac{\partial}{\partial t} \left(\sum_b \int e^{-H_{\beta,0}} y_a y_b g_t(b, z-1)\right). \qquad (3.21)$$

Note that the process (X_t, L_t) is transient even if the marginal (X_t) is recurrent because $\sum_i L_t^i \to \infty$ as $t \to \infty$. Therefore, integrating both sides over t and using that $g_t(x, \ell) \to 0$ as $t \to \infty$, which follows from the transience of (X_t, L_t) and the rapid decay of $g = g_0$, we get

$$\int e^{-H_{\beta,0}} z_a \int_0^\infty \mathbb{E}_{a,z-1}(g(X_t, L_t))\, dt = \sum_b \int e^{-H} y_a y_b g(b, z-1). \qquad (3.22)$$

This completes the proof. □

Exercise 3.6. *Following the proof for the hyperbolic sigma model, prove Theorem 3.1. Hint: Using*

$$\frac{\partial}{\partial \varphi_i^1} \left(\frac{1}{2}|\varphi|^2\right) = \varphi_i^1 \qquad (3.23)$$

show that

$$-\sum_b \int_{(\mathbb{R}^n)^\Lambda} e^{-H_{\beta,0}} \varphi_a^1 \varphi_b^1 \mathcal{L} g(b, \frac{1}{2}|\varphi|^2) = \int_{(\mathbb{R}^n)^\Lambda} e^{-H_{\beta,0}} g(a, \frac{1}{2}|\varphi|^2) \qquad (3.24)$$

where $\mathcal{L}$ is now defined without the factor $1 + L_t^i$ in the jump rates.

3.3 Supersymmetry

3.3.1. Grassmann variables and supersymmetric integration. Below we consider Grassmann variables. These anticommuting variables generate an algebra, and while the particular realisation of this algebra is unimportant, to be concrete, we can always realise it as an algebra of matrices as in the following example.

Exercise 3.7. *For any integer m, find $2^m \times 2^m$ matrices $\theta_1, \dots, \theta_m$ such that, for all i, j,*

$$\theta_i \theta_j = -\theta_j \theta_i. \tag{3.25}$$

Hint: the Clifford–Jordan–Wigner representation of the Grassmann algebra generated by (3.25) *is*

$$\theta_i = \bigotimes_{j=1}^{i-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \bigotimes_{j=i+1}^{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.26}$$

Definition 3.8. *Generators $\theta_1, \dots, \theta_m$ of a unital algebra satisfying the anticommutation relations* (3.25) *are called* Grassmann variables.

Let Λ be a finite set. For each vertex $i \in \Lambda$, let x_i, y_i be real variables and ξ_i, η_i be two Grassmann variables. Thus by definition all of the x_i and y_i commute with each other and with all of the ξ_i and η_i and all of the ξ_i and η_i anticommute. To fix signs in forthcoming expressions, fix an arbitrary order $i_1, \dots, i_{|\Lambda|}$ of the vertices in Λ.

Definition 3.9. *We define the algebra Ω_Λ to be the algebra of smooth functions on $(\mathbb{R}^2)^\Lambda$ with values in the algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices that have the form*

$$F = \sum_{I,J \subset \Lambda} F_{I,J}(x, y)(\eta\xi)_{I,J}, \tag{3.27}$$

where the coefficients $F_{I,J}$ are smooth functions on $(\mathbb{R}^2)^\Lambda$, and $(\eta\xi)_{I,J}$ is the ordered product $\prod_{i \in I \cap J} \eta_i \xi_i \prod_{i \in I \setminus J} \xi_i \prod_{j \in J \setminus I} \eta_j$. (This ordering has been chosen so that $(\eta\xi)_{\Lambda,\Lambda}$ is $\eta_1\xi_1 \dots \eta_\Lambda \xi_\Lambda$.)

We call elements of Ω_Λ forms because the forms of differential geometry are instances [10, 16].

Definition 3.10. *The integral (sometimes called a superintegral) of a form $F \in \Omega_\Lambda$ is defined by*

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^2)^\Lambda} F_{\Lambda,\Lambda}(x, y) \prod_{i \in \Lambda} \frac{dx_i \, dy_i}{2\pi}, \tag{3.28}$$

where $\mathbb{R}^{2|2}$ refers to the number of commuting and anticommuting variables.

The *degree* of a coefficient $F_{I,J}$ is $|I| + |J|$. Thus the integral of a form F is a constant multiple of the usual Lebesgue integral of the top degree part of F.

Example 3.11. *Let $F = f(x, y)\eta_{i_1}\xi_{i_1} \dots \eta_{i_{|\Lambda|}}\xi_{i_{|\Lambda|}}$ be a form of top degree. Then*

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F = \int_{(\mathbb{R}^2)^\Lambda} f(x, y) \prod_{i \in \Lambda} \frac{dx_i \, dy_i}{2\pi}. \tag{3.29}$$

Definition 3.12. *A form $F \in \Omega_\Lambda$ is* even *if the degree of all non-vanishing coefficients $F_{I,J}$ is even. For even forms $F^1, \dots, F^p$ and a smooth function $g \in C^\infty(\mathbb{R}^p)$, define the form $g(F^1, \dots, F^p) \in \Omega_\Lambda$ by formally Taylor expanding g about the degree-0 part $(F^1_{\varnothing,\varnothing}, \dots, F^p_{\varnothing,\varnothing})$.*

The Taylor expansion is well-defined as there is no ambiguity in the ordering if the F^i are all even because even forms commute, and because the anticommutation relations satisfied by the ξ_i and η_i imply the expansion is finite.

Example 3.13. *For $i, j \in \Lambda$, define a form in Ω_Λ by*

$$\tau_{ij} = x_i x_j + y_i y_j + \xi_i \eta_j - \eta_i \xi_j. \tag{3.30}$$

Then τ_{ij} is even and its degree-0 part is $(\tau_{ij})_{\varnothing,\varnothing} = x_i x_j + y_i y_j$. Hence, with $g(t) = e^t$,

$$e^{\tau_{ij}} = e^{x_i x_j + y_i y_j}(1 + \xi_i \eta_j - \eta_i \xi_j). \tag{3.31}$$

Similarly, with $g(t^1, t^2) = e^{t^1 + t^2}$,

$$\begin{aligned} e^{\tau_{ij}+\tau_{jk}} &= e^{x_i x_j + y_i y_j + x_j x_k + y_j y_k}(1 + \xi_i \eta_j - \eta_i \xi_j + \xi_j \eta_k - \eta_j \xi_k + (\xi_i \eta_j - \eta_i \xi_j)(\xi_j \eta_k - \eta_j \xi_k)) && (3.32)\\ &= e^{x_i x_j + y_i y_j + x_j x_k + y_j y_k}(1 + \xi_i \eta_j - \eta_i \xi_j + \xi_j \eta_k - \eta_j \xi_k + \xi_i \eta_j \xi_j \eta_k + \eta_i \xi_j \eta_j \xi_k). && (3.33)\end{aligned}$$

This calculation demonstrates the general fact that $e^{F^1+F^2} = e^{F^1} e^{F^2}$ for forms.

Example 3.14. *Let $\Lambda = \{i\}$ and $a > 0$. Then*

$$\int_{\mathbb{R}^{2|2}} e^{-\frac{1}{2a}\tau_{ii}} = \int_{\mathbb{R}^{2|2}} e^{-\frac{1}{2a}(x_i^2+y_i^2)}(1 + \frac{\eta_i \xi_i}{a}) = \int_{\mathbb{R}^2} e^{-\frac{1}{2a}(x_i^2+y_i^2)} \frac{dx_i dy_i}{2\pi a} = 1. \tag{3.34}$$

Similarly, for general finite Λ and any $a > 0$,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} e^{-\frac{1}{2a}\sum_{i\in\Lambda}\tau_{ii}} = 1. \tag{3.35}$$

3.3.2. Supersymmetric localisation. Temporarily set $x = x_i, y = y_i, \xi = \xi_i$, and $\eta = \eta_i$. Define an operator $\partial_\eta \colon \Omega_\Lambda \to \Omega_\Lambda$ by linearity, $\partial_\eta(\eta F) = F$, and $\partial_\eta F = 0$ if F does not contain a factor η. Define ∂_ξ in the same manner. Define Q_i by its action on forms F by

$$Q_i F \equiv \xi \partial_x F + \eta \partial_y F + x \partial_\eta F - y \partial_\xi F. \tag{3.36}$$

The *supersymmetry generator* Q acts on a form $F \in \Omega_\Lambda$ by $QF \equiv \sum_{i\in\Lambda} Q_i F$.

Definition 3.15. *$F \in \Omega_\Lambda$ is* supersymmetric *if $QF = 0$.*

Example 3.16. *The forms τ_{ij} defined in (3.30) are supersymmetric. Note that*

$$\begin{aligned} Q_i \tau_{ij} &= (\xi_i \partial_{x_i} + \eta_i \partial_{y_i} + x_i \partial_{\eta_i} - y_i \partial_{\xi_i})(x_i x_j + y_i y_j + \xi_i \eta_j - \eta_i \xi_j) \\ &= \xi_i x_j + \eta_i y_j - x_i \xi_j - y_i \eta_j && (3.37)\\ Q_j \tau_{ij} &= (\xi_j \partial_{x_j} + \eta_j \partial_{y_j} + x_j \partial_{\eta_j} - y_j \partial_{\xi_j})(x_i x_j + y_i y_j + \xi_i \eta_j - \eta_i \xi_j) \\ &= \xi_j x_i + \eta_j y_i - x_j \xi_i - y_j \eta_i && (3.38)\end{aligned}$$

so $Q\tau_{ij} = Q_i \tau_{ij} + Q_j \tau_{ij} = 0$.

Exercise 3.17. *Let $F, G \in \Omega_\Lambda$ be even. Show that Q is a derivation on even forms:*

$$Q(FG) = (QF)G + F(QG). \tag{3.39}$$

Show also that Q obeys the chain rule

$$Qg(F^1, \dots, F^n) = \sum_{i=1}^{p} \frac{\partial g}{\partial t_i}(F^1, \dots, F^n) QF^i. \tag{3.40}$$

In particular, any form that is a function of the collection of forms (τ_{ij}) is supersymmetric.

Much of the significance of supersymmetry is a result of the fundamental localisation theorem, closely related to the celebrated Duistermaat–Heckman theorem.

Theorem 3.18 (Localisation theorem)**.** *Let $F \in \Omega_\Lambda$ be a smooth form with sufficient decay that can be written as a function of the $(\tau_{ij})_{i,j\in\Lambda}$. Then*

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F = F_{\varnothing,\varnothing}(0,0). \tag{3.41}$$

Proof. By assumption, there is a smooth function $f : \mathbb{R}^{\Lambda\times\Lambda} \to \mathbb{R}$ with decay at infinity such that $F = f(\tau)$. By taking a limit, we may in fact assume $F = f(\tau)e^{-H(\tau)}$ with $H(\tau) = \varepsilon \sum_i \tau_{ii}$. Let

$$g(t) = \int_{(\mathbb{R}^{2|2})^\Lambda} f(t\tau)e^{-H(\tau)}. \tag{3.42}$$

By (3.35), then

$$g(0) = f(0) \int_{(\mathbb{R}^{2|2})^\Lambda} e^{-H(\tau)} = f(0) = F_{\varnothing,\varnothing}(0,0). \tag{3.43}$$

We will show that $g(t)$ is independent of $t > 0$. Let $\sigma_{ij} = x_i\eta_j - y_i\xi_j$ and notice that $Q\sigma_{ij} = \tau_{ij}$. Writing $f_{ij}(t) \equiv \frac{\partial}{\partial t_{ij}} f(t)$, using that $Qf_{ij}(t\tau) = 0$ and that Q is a derivation on even forms,

$$\frac{\partial}{\partial t} f(t\tau) = \sum_{ij} f_{ij}(t\tau)\tau_{ij} = \sum_{ij} f_{ij}(t\tau)Q\sigma_{ij} = \sum_{ij} Q(f_{ij}(t\tau)\sigma_{ij}). \tag{3.44}$$

Using that $Qe^{-H(\tau)} = 0$ by Exercise 3.17 and that Q is a derivation on even forms,

$$g'(t) = \frac{\partial}{\partial t} \int_{(\mathbb{R}^{2|2})^\Lambda} f(t\tau)e^{-H(\tau)} = \sum_{ij} \int_{(\mathbb{R}^{2|2})^\Lambda} Q(f_{ij}(t\tau)\sigma_{ij}e^{-H(\tau)}) = 0 \tag{3.45}$$

where the last equality follows from the fact that, for any form F with sufficient decay at infinity,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} QF = \sum_i \int_{(\mathbb{R}^{2|2})^\Lambda} (\xi_i\partial_{x_i}F + \eta_i\partial_{y_i}F) = 0, \tag{3.46}$$

where the first equality holds because any form in the image of ∂_η or ∂_ξ has degree at most $2|\Lambda|-1$, and where the last equality holds because the integral of a derivative. □

3.4 Supersymmetric spin models

3.4.1. Supersymmetric free field. The supersymmetric Gaussian free field is defined as follows. For $i \in \Lambda$, we write

$$\varphi_i \equiv (x_i, y_i, \eta_i, \xi_i) \equiv (\varphi_i^1, \varphi_i^2, \eta_i, \xi_i). \tag{3.47}$$

The first two components are coordinates in $\mathbb{R}^2$ while the second two components are matrices, but we can view each component as an element of Ω_Λ. The coordinate x_i and y_i are the coordinate functions on $(\mathbb{R}^2)^\Lambda$ with values proportional to the identity matrix, while η_i and ξ_i are the constant functions given by the respective matrices. Then we define the forms

$$\varphi_i \cdot \varphi_j = x_ix_j + y_iy_j + \xi_i\eta_j - \eta_i\xi_j = \tau_{ij}, \qquad |\varphi_i|^2 = \varphi_i \cdot \varphi_i = \tau_{ii}. \tag{3.48}$$

With this notation, we define the *supersymmetric Gaussian free field* as the form

$$e^{-H} \in \Omega_\Lambda, \tag{3.49}$$

where

$$H = \frac{1}{4}\sum_{i,j}\beta_{ij}|\varphi_i - \varphi_j|^2 + \frac{1}{2}m^2\sum_i |\varphi_i|^2. \tag{3.50}$$

By definition, H can be written as a function of the (τ_{ij}) and when $m^2 > 0$ it has exponential decay. The localisation theorem implies that

$$\int_{(\mathbb{R}^{2|2})^\Lambda} e^{-H} = 1. \tag{3.51}$$

For a form $F \in \Omega_\Lambda$, we define the *super-expectation*

$$\langle F\rangle = \int_{(\mathbb{R}^{2|2})^\Lambda} F e^{-H}. \tag{3.52}$$

This is in general not the expectation of a probability measure (and it more generally takes forms instead of random variables as input).

Exercise 3.19. *Let F be a degree-0 form, i.e., $F = f(x,y)$ for a smooth function $f : (\mathbb{R}^2)^\Lambda \to \mathbb{R}$. Then the super-expectation of F is equal to the ordinary expectation of the f with respect to the two-component Gaussian free field, i.e., with M as in* (1.6),

$$\begin{aligned}\langle F\rangle_{\mathbb{R}^{2|2}} &= \langle f\rangle_{\mathbb{R}^2} \\ &= \det\Big(\frac{M+m^2}{2\pi}\Big)\int_{(\mathbb{R}^2)^\Lambda} f(x,y)e^{-\frac{1}{2}(x,(M+m^2)x)-\frac{1}{2}(y,(M+m^2)y)}\prod_i dx_i\,dy_i.\end{aligned} \tag{3.53}$$

Sketch. Write M instead of $M+m^2$. In the super-expectation, the exponent is

$$\frac{1}{2}\sum_{i,j} M_{ij}(x_ix_j + y_iy_j + \xi_i\eta_j - \eta_i\xi_j). \tag{3.54}$$

By symmetry of M,

$$\frac{1}{2}\sum_{i,j} M_{ij}(\xi_i\eta_j - \eta_i\xi_j) = \sum_{i,j} M_{ij}\xi_i\eta_j. \tag{3.55}$$

Let $N = |\Lambda|$. The degree $2N$ part of $\exp(-\sum_{i,j} M_{ij}\xi\eta_j)$ is

$$\frac{1}{N!}\left(\sum_{i,j} M_{ij}\xi_i\eta_j\right)^N = \det(M) \tag{3.56}$$

since the ξ and η anticommute. □

Theorem 3.20 (supersymmetric BFS–Dynkin isomorphism)**.** *Let $\langle\cdot\rangle$ denote the expectation of the supersymmetric Gaussian free field with spin coupling β and mass m, and let X denote the simple random walk with jump rates given in terms of the same β. Then*

$$\langle x_ix_jg(\frac{1}{2}|\varphi|^2)\rangle = \int_0^\infty \mathbb{E}_x(1_{X_t=y}g(L_t))e^{-m^2t}\,dt. \tag{3.57}$$

Sketch. The proof is identical to that of Theorem 3.1 and we again arrive at

$$\langle x_ix_jg(\frac{1}{2}|\varphi|^2)\rangle_{\mathbb{R}^{2|2}} = \int_0^\infty \langle\mathbb{E}_i(1_{X_t=j}g(L_t+\frac{1}{2}|\varphi|^2))\rangle_{\mathbb{R}^{2|2}}e^{-m^2t}\,dt, \tag{3.58}$$

with the only difference that the expectation $\langle\cdot\rangle_{\mathbb{R}^n}$ is replaced by $\langle\cdot\rangle_{\mathbb{R}^{2|2}}$ on both sides. Since the right-hand side is a function of the τ_{ij}, the localisation theorem simplifies the right-hand side to

$$\int_0^\infty \langle\mathbb{E}_i(1_{X_t=j}g(L_t+\frac{1}{2}|\varphi|^2))\rangle_{\mathbb{R}^{2|2}}e^{-m^2t}\,dt = \int_0^\infty \mathbb{E}_i(1_{X_t=j}g(L_t))e^{-m^2t}\,dt \tag{3.59}$$

which gives the claim. □

3.4.2. Supersymmetric hyperbolic sigma model. The $\mathbb{H}^{2|2}$ model is defined analogously to the $\mathbb{H}^2$ model, with x^2+y^2 replaced by $x^2+y^2+2\xi\eta$. More precisely, with (x,y,ξ,η) as above, set

$$z = \sqrt{x^2+y^2+2\xi\eta}. \tag{3.60}$$

We then write $u=(x,y,z,\xi,\eta)$,

$$u\cdot u' = xx'+yy'-zz'+\xi\eta'-\eta\xi', \qquad \int_{(\mathbb{H}^{2|2})^\Lambda} F = \int_{(\mathbb{R}^{2|2})^\Lambda} F \prod_{i\in\Lambda}\frac{1}{z_i}. \tag{3.61}$$

Exercise 3.21. *Following the proof for the $\mathbb{H}^2$ model and using in addition the localisation theorem to show that, for the $\mathbb{H}^{2|2}$ model,*

$$\sum_b \langle y_a y_b g(b, z-1)\rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}^\beta_{a,0}(g(X_t,L_t))\, e^{-ht}\, dt. \tag{3.62}$$

The left-hand side is the two-point function of the $\mathbb{H}^{2|2}$ model. The right-hand side the two-point function of the VRJP.

3.5 Self-avoiding walk

Example 3.22. *Let $G=(V,E)$ be a finite graph. Let*

$$G_z^E(a,b) = \sum_{w:a\to b} z^{|w|} 1(w \text{ is edge self-avoiding}), \tag{3.63}$$

where the sum runs over all walks w from a to b with where $|w|$ edges, be the generating function of edge self-avoiding walks on G. In particular, if G has degree bounded by three, then G_z is also the generating function of vertex self-avoiding walks. Then

$$G_z(i,j) = \int_{(\mathbb{R}^{2|2})^V} x_a x_b \prod_{ij\in E}(1+z\varphi_i\cdot\varphi_j)\prod_{i\in V} e^{-\frac12\varphi_i\cdot\varphi_i}. \tag{3.64}$$

Proof. Note that

$$x_i e^{-\frac12\varphi_i\cdot\varphi_i} = -\frac{\partial}{\partial x_i} e^{-\frac12\varphi_i\cdot\varphi_i}. \tag{3.65}$$

Integrating by parts,

$$\begin{aligned} G_z^E(a,b) &= \int_{(\mathbb{R}^{2|2})^V} x_a x_b \prod_{ij\in E}(1+z\varphi_i\cdot\varphi_j)\prod_{i\in V} e^{-\frac12\varphi_i\cdot\varphi_i} \\ &= \delta_{ab}\int_{(\mathbb{R}^{2|2})^V} \prod_{ij\in E}(1+z\varphi_i\cdot\varphi_j)\prod_{i\in V} e^{-\frac12\varphi_i\cdot\varphi_i} + \int_{(\mathbb{R}^{2|2})^V} x_b\left(\frac{\partial}{\partial x_a}\prod_{ij\in E}(1+z\varphi_i\cdot\varphi_j)\right)\prod_{i\in V} e^{-\frac12\varphi_i\cdot\varphi_i} \end{aligned} \tag{3.66}$$

The first integral is a function of (τ_{ij}) and thus localises to 1. The second integral is

$$\begin{aligned} &\int_{(\mathbb{R}^{2|2})^V} x_b\left(\frac{\partial}{\partial x_a}\prod_{ij\in E}(1+z\varphi_i\cdot\varphi_j)\right)\prod_{i\in V} e^{-\frac12\varphi_i\cdot\varphi_i} \\ &= \sum_{c\sim a} z\int_{(\mathbb{R}^{2|2})^V} x_c x_b \prod_{ij\in E\setminus\{ac\}}(1+z\varphi_i\cdot\varphi_j)\prod_{i\in V} e^{-\frac12\varphi_i\cdot\varphi_i} = \sum_{c\sim a} z G_z^{E\setminus\{ac\}}(a,b). \end{aligned} \tag{3.67}$$

Thus we have shown

$$G_z^E(a,b) = \delta_{ab} + \sum_{c\sim a} z G_z^{E\setminus\{ac\}}(a,b). \tag{3.68}$$

This recursion characterises the generating function for edge self-avoiding walks. ☐

4 A glimpse at the renormalisation group

The content of this section is based on [3, 9].

4.1 Decomposition of free field on $\mathbb{Z}^d$

In Section 1, we saw that it is useful to decompose the Gaussian interaction factor in the measure of $O(n)$-invariant spin systems. In mean-field theory, this decomposition is very simple and involves only two scales:

$$(M+m^2)^{-1} = \frac{1}{1+m^2}\,\mathrm{id} + \frac{1}{m^2(1+m^2)}Q, \quad \text{where } Q_{ij} = \frac{1}{N}. \tag{4.1}$$

We now consider the case of $\mathbb{Z}^d$ (or more precisely that of a large torus) with $M = -\Delta$ the nearest neighbour Laplacian. Recall the long-distance behaviour of the Green function from (1.8)–(1.9). From these asymptotics, the following result is plausible.

Theorem 4.1. *Let Λ be a finite discrete torus of side length $D < L^N$, and let $-\Delta$ be the discrete Laplace operator on Λ. There exist positive definite matrices $C_j = C_j(m^2)$ such that*

$$(-\Delta + m^2)^{-1} = C_0 + \cdots + C_N + C_{\hat{N}} \tag{4.2}$$

where C_j has range L^j, i.e.,

$$C_j(x,y) = 0 \quad \textit{if } |x-y| > L^j, \tag{4.3}$$

and C_j is smooth on scale L^j, i.e.,

$$(L^j\nabla)^\alpha C_j(x,y) = O(L^{-(d-2)(j-1)}). \tag{4.4}$$

The construction of such a finite range decomposition is non-obvious because of the competing constraints of positive definiteness and the finite range property. The following example gives such a decomposition in the continuum.

Example 4.2. *Given $L > 1$ and $\alpha > 0$, there exists $u : \mathbb{R}^d \to \mathbb{R}$ which is smooth, positive definite, with support in $[-\frac{1}{2}, \frac{1}{2}]^d$, such that*

$$|x|^{-\alpha} = \sum_{j\in\mathbb{Z}} L^{-\alpha j} u(L^{-j}x) \qquad (x \neq 0). \tag{4.5}$$

Proof. Choose a function $w \in C_c(\mathbb{R})$ which is not the zero function. By the change of variables $t \mapsto |x|t$,

$$\int_0^\infty t^{-\alpha} w(|x|/t)\frac{dt}{t} = c|x|^{-\alpha}, \tag{4.6}$$

with $c = \int_0^\infty t^{-\alpha} w(1/t)\frac{dt}{t}$. After normalising w by multiplication by a constant so that $c = 1$, we obtain

$$|x|^{-\alpha} = \int_0^\infty t^{-\alpha} w(|x|/t)\frac{dt}{t}. \tag{4.7}$$

Now choose w with support in $[-\frac{1}{2}, \frac{1}{2}]$ such that $x \mapsto w(|x|)$ is a smooth, positive definite function on $\mathbb{R}^d$. (Exercise: a function w with these properties exists.) Given $L > 1$, set

$$u(x) = \int_{1/L}^1 t^{-\alpha} w(|x|/t)\frac{dt}{t}. \tag{4.8}$$

It is not hard to check that this is a positive definite function. By change of variables, (4.5) holds, and the proof is complete. ∎

Exercise 4.3. *Find a similar decomposition for the two-dimensional Green function $-\log|x|$.*

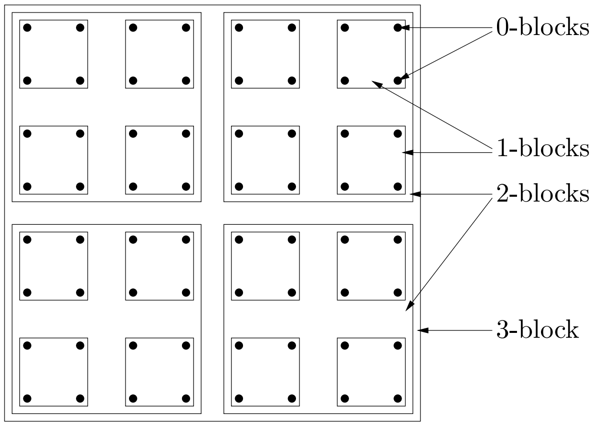

FIGURE 4.1. Blocks in $\mathcal{B}_j$ for $j = 0, 1, 2, 3$ when $d = 2$, $N = 3$, $L = 2$.

4.2 Hierarchical spin coupling

Many of the essential features for the long-distance behaviour of spin systems on $\mathbb{Z}^d$ are contained in their hierarchical approximation. This is a special choice of M mimicking the important features of $-\Delta$ on $\mathbb{Z}^d$. The study of hierarchical models has a long history in statistical mechanics going back to [6, 13]; recent studies include [1, 5, 7, 14, 19] and references.

Let $\Lambda = \Lambda_N$ be a cube of side length L^N in $\mathbb{Z}^d$, $d \geqslant 1$, for some fixed integer $L > 1$, where N is eventually chosen large. For scale $0 \leqslant j \leqslant N$, we decompose Λ as the union of disjoint blocks of side lengths L^j denoted $B \in \mathcal{B}_j$; see Figure 4.1. In particular, $\mathcal{B}_0 = \Lambda$ and the unique block in $\mathcal{B}_N$ is Λ_N itself. The blocks have the structure of a K-ary tree with $K = L^d$, height N and the leaves are indexed by the sites $x \in \Lambda_N$.

Definition 4.4. *For scale j and $x \in \Lambda$, let $B_j(x)$ denote the block in $\mathcal{B}_j$ containing x. Define the block averaging operators by*

$$(Q_j f)_x = \frac{1}{|B_j(x)|} \sum_{y \in B_j(x)} f_y, \quad \textit{for } f \in \mathbb{R}^\Lambda. \tag{4.9}$$

Let $P_j = Q_{j-1} - Q_j$.

Lemma 4.5. *The operators $P_1, \dots, P_N, Q_N$ are orthogonal projections whose ranges are disjoint and provide a direct sum decomposition of $\mathbb{R}^\Lambda$:*

$$P_j P_k = P_k P_j = \begin{cases} P_j & (j = k) \\ 0 & (j \neq k), \end{cases} \qquad \sum_{j=1}^N P_j + Q_N = \mathrm{id}. \tag{4.10}$$

Note that the mean-field model is the special case $L^N = L$, i.e., $N = 1$.

Proof. The second equation is an immediate consequence of the definition of P_j, together with the fact that $Q_0 = \mathrm{id}$. For the other properties, we claim that

$$Q_j Q_k = Q_{j \vee k} = Q_k Q_j. \tag{4.11}$$

In particular, the case $j = k$ shows that Q_j is an orthogonal projection. To prove (4.11), it suffices to consider $j \leq k$. We use primes to denote blocks in the larger scale $\mathcal{B}_k$, and unprimed blocks are in $\mathcal{B}_j$. Then the x,y matrix element of the product is given by

$$\begin{aligned}\sum_z Q_{j;xz}Q_{k;zy} &= L^{-d(j+k)}\sum_z \mathbf{1}_{B_x=B_z}\mathbf{1}_{B'_z=B'_y} = L^{-d(j+k)}\sum_{z\in B_x}\mathbf{1}_{B_x\subset B'_y} \\ &= L^{-dk}\mathbf{1}_{B_x\subset B'_y} = L^{-dk}\mathbf{1}_{B'_x=B'_y} = Q_{k;xy},\end{aligned} \tag{4.12}$$

as claimed. Thus $\{Q_j\}_{j=0,\dots,N}$ is a sequence of commuting decreasing projections that starts with $Q_0 = \mathrm{id}$. By (4.11) it readily follows that $P_1,\dots,P_N,Q_N$ are orthogonal projections that obey (4.10). □

An operator on $\mathbb{R}^\Lambda$ is *hierarchical* if it is diagonal with respect to this decomposition. To obtain a hierarchical Green function with the scaling of the Green function of the usual Laplace operator, we choose the hierarchical Laplace operator on Λ to be

$$-\Delta_H = \sum_{j=1}^N L^{-2(j-1)}P_j. \tag{4.13}$$

Lemma 4.6. *Let* $\gamma_j = (L^{-2(j-1)}+m^2)^{-1}$. *Then*

$$(-\Delta_H+m^2)^{-1} = \sum_{j=1}^N \gamma_j P_j + \frac{1}{m^2}Q_N. \tag{4.14}$$

Proof. The operators $P_1,..,P_N,Q_N$ are spectral projections for $-\Delta_{H,N}$. In fact, let $f(t) = (t+m^2)^{-1}$ and $\lambda_j = L^{-2(j-1)}$. By the spectral calculus,

$$\begin{aligned}(-\Delta_{H,N}+m^2)^{-1} = f(-\Delta_{H,N}) &= f\left(\sum_{j=1}^N \lambda_j P_j + 0\; Q_N\right) \\ &= \sum_{j=1}^N f(\lambda_j)P_j + f(0)Q_N\end{aligned} \tag{4.15}$$

because $f(\lambda_j)=\gamma_j$ and $f(0)=m^{-2}$. □

Exercise 4.7. *(i) Verify that*

$$(-\Delta_H+m^2)^{-1} = \gamma_1 Q_0 + \sum_{j=1}^{N-1}(\gamma_{j+1}-\gamma_j)Q_j + (m^{-2}-\gamma_N)Q_N. \tag{4.16}$$

(ii) Using the result of part (i), prove that as $|x|\to\infty$ *the hierarchical covariance obeys*

$$\lim_{m^2\downarrow 0}\lim_{N\to\infty} C_{0x}(m^2)\begin{cases}\asymp |x|^{-(d-2)} & (d>2)\\ =\infty & (d\leqslant 2)\end{cases} \tag{4.17}$$

and that there is a constant $\sigma>0$ *(depending on* L*) such that*

$$\lim_{m^2\downarrow 0}\lim_{N\to\infty}[C_{0x}(m^2)-C_{00}(m^2)]\begin{cases}\asymp -|x| & (d=1)\\ = -\sigma\log_L|x| + O(1) & (d=2).\end{cases} \tag{4.18}$$

4.3 Degenerate Gaussian measures

We start from the decomposition (4.16), which we write as

$$(-\Delta + m^2)^{-1} = \sum_j C_j, \quad \text{where } C_j = \lambda_j(m^2)Q_j. \tag{4.19}$$

The matrices C_j are positive semi-definite, but not positive definite, and the corresponding Gaussian measures are therefore degenerate (supported on subspaces). Concretely, we can realise the Gaussian measure with covariance C_j as follows. For each block $B \in \mathcal{B}_j$, choose $\varphi_{j,B}$ to be an independent Gaussian random variable (independent of everything else) with variance $\lambda_j L^{-dj}$. For $x \in B$, then set $\varphi_{j,x} = \varphi_{j,B}$.

Exercise 4.8. *Check that $\mathbb{E}(\varphi_{j,x}\varphi_{j,y}) = [C_j]_{xy}$. As a consequence, with the φ_i as above, the Gaussian measure with covariance $(-\Delta_H + m^2)^{-1}$ is given in distribution as $\varphi_1 + \cdots + \varphi_N$.*

4.4 The Sine–Gordon model

The following treatment is adapted from [9, Chapter 3].

4.4.1. Model. We consider spins $\varphi_x \in \mathbb{R}$ with energy

$$H(\varphi) = \frac{1}{4}\sum_{x,y} \beta_{xy}(\varphi_x - \varphi_y)^2 + \sum_x V(\varphi_x) = \frac{1}{2}(\varphi, M\varphi) + \sum_x V(\varphi_x), \tag{4.20}$$

where V is a non-constant periodic function on $\mathbb{R}$. In particular, V is *not convex*. A representative example is $V(t) = z\cos(t)$ with $z \in \mathbb{R}$. Note that H has the symmetry $\varphi \to \varphi + n\mathbf{1}$ for $n \in \mathbb{Z}$. To break this symmetry, we can add an external field, and consider for some $m^2 > 0$, $\bar{\varphi} \in \mathbb{R}$,

$$H(\varphi) + \frac{1}{2}m^2 \sum_x (\varphi_x - \bar{\varphi})^2. \tag{4.21}$$

For simplicity, we assume that $\bar{\varphi} = 0$. This choice amounts to replacing M by $M + m^2$ in (4.20). As before, we are interested in the limit $|\Lambda| \uparrow \infty$ and then $m^2 \downarrow 0$.

4.4.2. Result. For a periodic function $F : S^1 \to \mathbb{R}$, we use the norm

$$\|F\| = \sum_q w(q)|\hat{F}(q)|, \quad w(q) = (1 + |q|)^2, \tag{4.22}$$

where

$$\hat{F}(q) = \int_0^{2\pi} F(\varphi)e^{iq\varphi}\, d\varphi \tag{4.23}$$

are the Fourier coefficients of F. In particular, the potential $V(\varphi) = z\cos(\varphi)$ has norm $O(z)$.

Theorem 4.9. *Let $d = 2$ and $M = -\beta\Delta_H + m^2$. Assume that $\|V - \hat{V}(0)\|$ is sufficiently small. Assume that $0 < \beta < \beta_0$. Then there is $\kappa > 0$ such that for $m^2 = L^{-2N}$,*

$$\sum_{y\in\Lambda} \langle \varphi_x \varphi_y \rangle_h = \frac{1}{m^2}\left(1 - O(\kappa^N)\right). \tag{4.24}$$

Remark 4.10.

- *If the potential V was convex such a result would follow from the Brascamp–Lieb inequality. But the non-convexity of H is significant.*
- *The theorem shows that, at high temperature, the behaviour of a periodic potential differs from that of a double well potential. For a double well potential, the susceptibility remains bounded for β small. For a periodic potential, on the other hand, the susceptibility is unbounded.*
- *The methods that prove this result can be refined to other observables; and there are also much more complicated version for the non-hierarchical lattice $\mathbb{Z}^2$.*

4.4.3. Proof of Theorem 4.9. We define a sequence of effective potentials V_j by

$$e^{-V_{j+1}(\varphi)} = \mathbb{E}_{C_{j+1}} e^{-V_j(\varphi+\zeta)}. \tag{4.25}$$

Due to the hierarchical structure, the potential has the form

$$V_j(\varphi) = \sum_{B\in\mathcal{B}_j} V_j(B,\varphi), \tag{4.26}$$

where each $V_j(B,\varphi)$ depends only on φ inside the block B. Moreover, we restrict $V_j(B,\varphi)$ to fields φ that are constant on B. Thus $V_j(B,\cdot)$ becomes a function on $\mathbb{R}$. Finally, the functions $V_j(B,\cdot)$ are the same for all $B \in \mathcal{B}_j$, up to translation. We take this point of view and use the above norm on this function and will simply denote the above function by $V_j : \mathbb{R} \to \mathbb{R}$.

We study the renormalisation group map $V_j \mapsto V_{j+1}$. Let $\sigma_j = C_j(0,0)$.

Theorem 4.11. *Assume that* $\|V_j - \hat{V}_j(0)\|$ *is sufficiently small. Then*

$$\|V_{j+1} - \hat{V}_{j+1}(0)\| \leqslant L^2 e^{-\sigma_j/2}\|V_j - \hat{V}_j(0)\| + O(\|V_j - \hat{V}_j(0)\|)^2. \tag{4.27}$$

Exercise 4.12. *Assume that* $\max_j L^2 e^{-\sigma_j/2} < \kappa < 1$ *for all* j*. Then*

$$\|V_N - \hat{V}_N(0)\| \leqslant O(\kappa^N)\|V_0 - \hat{V}_0(0)\|. \tag{4.28}$$

For $0 < \beta < \beta_0$ *and* $m = L^{-2N}$ *this condition is satisfied for the hierarchical field.*

Using the proposition, we obtain the claim for the suceptibility.

Proof of Theorem 4.9. Let $C = (-\beta\Delta_H + m^2)^{-1}$ denote the covariance of the hierarchical Gaussian field, and define, for $x \in \Lambda$,

$$g_x = (C\mathbf{1})_x = \sum_y C_{xy} = \frac{1}{m^2}, \qquad \mathbf{1}_x = 1 \text{ for all } x, \tag{4.29}$$

where the last equality is due to (1.12). By completion of the square, using that $Mg = MC\mathbf{1} = \mathbf{1}$,

$$-\frac{1}{2}(\varphi, M\varphi) + t(\varphi,\mathbf{1}) = -\frac{1}{2}(\varphi - tg, M(\varphi - tg)) + \frac{1}{2}t^2(g,\mathbf{1}), \tag{4.30}$$

we get

$$\Gamma(t) = \log \mathbb{E}_C(e^{t(\varphi,\mathbf{1})}e^{-V(\varphi)}) = \frac{1}{2}t^2(g,\mathbf{1}) + \log \mathbb{E}_C(e^{-V(\varphi+tg)}) = \frac{|\Lambda|t^2}{2m^2} - V_N(tg). \tag{4.31}$$

Proposition 4.11 implies that the first and second derivatives of the renormalized potential V_N are uniformly bounded by $O(\kappa^N)$:

$$|V_N''(0)| = |(V_N - \hat{V}_N(0))''| \leqslant \|V_N - \hat{V}_N\| \leqslant \kappa^N\|V_0 - \hat{V}_0(0)\| = O(\kappa^N). \tag{4.32}$$

Thus

$$\mathrm{var}(\varphi,\mathbf{1}) = \frac{\partial^2\Gamma(0)}{\partial t^2} = \frac{|\Lambda|}{m^2} - V_N''(0)g^2 = \frac{|\Lambda|}{m^2}\left(1 - O\left(\frac{\kappa^N}{m^2|\Lambda|}\right)\right). \tag{4.33}$$

This completes the proof. □

4.4.4. Proof of Proposition 4.11. The norm (4.22) has the the following properties. Since $w(p+q) \leqslant w(p)w(q)$, i.e.,

$$
\begin{aligned}
(1+|p+q|)^2 &= 1+p^2+q^2+2|p+q|+2pq \\
&\leqslant 1+p^2+q^2+2|p+q|+4|pq|+2|pq|(|p|+|q|) = (1+|p|)^2(1+|q|)^2,
\end{aligned} \tag{4.34}
$$

the norm satisfies the product property

$$
\|FG\| = \sum_{q,p} w(q)|\hat{F}(q-p)||\hat{G}(p)| \leqslant \sum_{q,p} w(q-p)w(p)|\hat{F}(q-p)||\hat{G}(p)| = \|F\|\|G\|. \tag{4.35}
$$

Exercise 4.13. *For any F in a normed algebra with $\|F\|$ small enough,*

$$
\|e^{-F}-1\| \leqslant \|F\| + O(\|F\|^2), \tag{4.36}
$$

$$
\|\log(1+F)\| \leqslant \|F\| + O(\|F\|^2). \tag{4.37}
$$

To simplify the notation, since we consider a simple renormalisation group step, we now drop the index j and write $+$ in place of $j+1$. Recall that under the Gaussian measure with covariance $C = C_j$ the the random variable ζ_x is Gaussian with covariance $\sigma = C_{xx}$.

Lemma 4.14. *For $F : S^1 \to \mathbb{R}$ with $\hat{F}(0) = 0$ and $\|F\| < \infty$, with $\kappa = e^{-\sigma/2}$,*

$$
\|\mathbb{E}_C\left(F(\cdot+\zeta_x)\right)\| \leqslant \kappa\|F\|. \tag{4.38}
$$

For any $F_x : S^1 \to \mathbb{R}$ with $\|F_x\| < \infty$, where $x \in B$,

$$
\|\mathbb{E}_C\left(\prod_{x\in B} F_x(\cdot+\zeta_x)\right)\| \leqslant \prod_{x\in B}\|F_x\|. \tag{4.39}
$$

Proof. By (4.16), each ζ_x is a Gaussian random variable with variance as above. By (1.3) therefore

$$
\mathbb{E}_C(e^{iq\zeta_x}) = e^{-\sigma q^2/2} = \kappa^{q^2}. \tag{4.40}
$$

This gives

$$
\mathbb{E}_C F(\varphi+\zeta_x) = \mathbb{E}_C \sum_q \hat{F}(q) e^{iq(\varphi+\zeta_x)} = \sum_q \kappa^{q^2}\hat{F}(q)e^{iq\varphi}. \tag{4.41}
$$

Since by assumption $\hat{F}(0) = 0$, we obtain the first bound:

$$
\|\mathbb{E}_C F(\cdot+\zeta_x)\| \leqslant \sum_q \kappa^{q^2}|\hat{F}(q)| \leqslant \kappa\sum_q|\hat{F}(q)| = \kappa\|F\|. \tag{4.42}
$$

The second bound is similar.

$$
\prod_x F_x(\varphi+\zeta_x) = \sum_{q\in\mathbb{Z}} e^{iq\varphi}\left(\sum_{\sum_x q_x = q}\prod_x \hat{F}(q_x)e^{i\sum_x q_x\zeta_x}\right). \tag{4.43}
$$

Using that $w(\sum_x q_x) \leqslant \prod_x w(q_x)$, we obtain

$$
\|\mathbb{E}_C\left(\prod_x F_x(\varphi+\zeta_x)\right)\| \leqslant \sum_q w(q)\sum_{\sum_x q_x=q}\prod_x|\hat{F}_x(q_x)| \leqslant \sum_q\sum_{\sum_x q_x=q}\prod_x|\hat{F}_x(q_x)|w(q_x) = \prod_x\|F_x\| \tag{4.44}
$$

as needed. $\square$

Proof of Proposition 4.11. We may assume that $\hat{V}(0) = 0$. We start from

$$\mathbb{E}_C\left(\prod_{x\in B} e^{-V(\varphi+\zeta_x)}\right) = \mathbb{E}_C\left(\prod_{x\in B}(1+e^{-V(\varphi+\zeta_x)}-1)\right) = \sum_{X\subset B}\mathbb{E}_C\left(\prod_{x\in X}(e^{-V(\varphi+\zeta_x)}-1)\right). \quad (4.45)$$

The term with $|X| = 0$ is simply 1. By (4.38) and (4.36), the terms with $|X| = 1$ are bounded by

$$\left\|\sum_{x\in B}\mathbb{E}_C\Big(e^{-V(\varphi+\zeta_x)}-1\Big)\right\| \leqslant |B|\kappa(\|V\|+O(\|V\|^2)). \quad (4.46)$$

By (4.39), the terms with $|X| > 1$ give

$$\begin{aligned}\left\|\sum_{|X|>1}\mathbb{E}_C\left(\prod_{x\in X}(e^{-V(\varphi+\zeta_x)}-1)\right)\right\| &\leqslant \sum_{|X|>1}\|\prod_{x\in X}(e^{-V(\varphi+\zeta_x)}-1)\| \\ &\leqslant \sum_{|X|>1}(\|V\|+O(\|V\|^2))^{|X|} = O(\|V\|^2). \quad (4.47)\end{aligned}$$

In summary, for $\|V\|$ small enough, we get

$$\left\|\mathbb{E}_C\left(\prod_{x\in B} e^{-V(\varphi+\zeta_x)}\right)-1\right\| \leqslant |B|\kappa\|V\|+O(\|V\|^2) = L^2\kappa(\|V\|+O(\|V\|^2)). \quad (4.48)$$

Finally, by (4.37),

$$\|V_+\| = \left\|\log\left(1+\mathbb{E}_C\left(\prod_{x\in B} e^{-V(\varphi+\zeta_x)}\right)-1\right)\right\| \leqslant L^2\kappa(\|V\|+O(\|V\|^2)), \quad (4.49)$$

as needed. □

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