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Please submit solutions to `columbia2026@bauerschmidt.ca`.

1. Day 1 – Mean-field theory

Problem 1.1. Complete the exercises mentioned in the lecture for the mean-field Ising model and extend the analysis to the mean-field $O(3)$ model.

Problem 1.2. Analyze the mean-field Ising model in a more direct fashion by counting the possible configurations of the magnetization $\frac{1}{N}\sum_{i=1}^N \sigma_i$.

2. Day 2 – Correlation inequalities

Problem 2.1. Adapt high temperature expansion and the Peierls argument from percolation to the Ising model. For example, show that the Ising model on $\mathbb{Z}^2$ with + boundary conditions has positive magnetization.

Use correlation inequalities to extend the conclusion of positive magnetization from $\mathbb{Z}^2$ to $\mathbb{Z}^d$ when $d \geqslant 2$.

Problem 2.2. Let A be a positive definite matrix and $z > 0$. Use Ginibre's method to derive inequalities for the measure proportional to the following density on $\mathbb{R}^N$:

$$e^{-\frac{1}{2}(\varphi, A\varphi) + z\sum_{i=1}^N \cos(\varphi_i)} \prod_i d\varphi_i,$$

where $d\varphi$ is the Lebesgue measure. In particular, show that, for any test vector $f \in \mathbb{R}^N$:

$$\langle e^{i(\varphi, f)} \rangle = \langle \cos((\varphi, f)) \rangle \text{ is monotone increasing in } z,$$

where $(\varphi, f) = \sum_{i=1}^N \varphi_i f_i$. Deduce that

$$\langle (\varphi, f)^2 \rangle \text{ is monotone decreasing in } z.$$

What happens as $z \to \infty$?

Problem 2.3. Show that the XY model has an infinite volume with either free or Dirichlet boundary conditions.