Lattice Models of Statistical Physics (SLMath Summer School)

June 9, 2026

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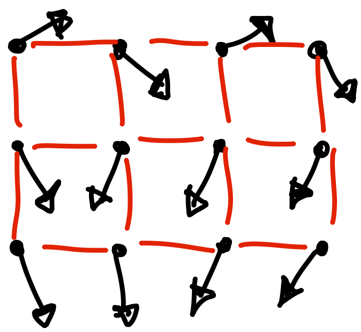
1. Spin models and mean-field theory

- $\Lambda \subset \mathbb{Z}^d$ finite set of points
- $J_{xy} = J_{yx} \geq 0$ ferromagnetic coupling constants
- $\sigma : \Lambda \to S^{n-1} \subset \mathbb{R}^n$ spin configurations



$n=1$: Ising model, $\sigma_x \in \{\pm 1\}$

$n=2$: XY model, $\sigma_x = \begin{pmatrix} \cos\theta_x \\ \sin\theta_x \end{pmatrix}$

$n=3$: classical Heisenberg model

Defn. For $\beta \geq 0$, the $O(n)$ model on Λ with coupling constants J is given by prob. meas.

$$\nu_\beta(d\sigma) \propto e^{\beta \sum_{xy} J_{xy} \sigma_x \cdot \sigma_y} \prod_{x\in\Lambda} d\sigma_x$$

$$\propto e^{-\frac{\beta}{4}\sum_{x,y} J_{xy} (\sigma_x - \sigma_y)^2} \prod_{x\in\Lambda} d\sigma_x$$

Expectation:

$$\langle F \rangle_\beta = \int F(\sigma)\, \nu_\beta(d\sigma)$$

Two-point function:

$$G_\beta(x,y) = \langle \sigma_x \cdot \sigma_y \rangle_\beta .$$

Main question: Do spins align?

- LRO: $G_\beta(x,y) \gtrsim c > 0$
- Exp. decay: $G_\beta(x,y) \lesssim e^{-c|x-y|}$
- Poly. decay: $G_\beta(x,y) \approx |x-y|^{-\alpha}$

General results:

- Exp. decay for β small, any d: high temp. expan.
- Exp. decay for $d=1$, any β: tranfer matrix
- LRO for $n=1$, $d\geq 2$ if β large: Peierls argument.
- LRO for $n\geq 2$, $d\geq 3$ if β large, Λ torus: RP
- no LRO for $n\geq 2$, $d=2$, any $\beta>0$: Mermin-Wagner
- Poly. decay for $n=2$, $d=2$, β large: KT phase

Polyakov conjecture: Exp. decay for $n\geq 3$, $d=2$, $\beta>0$.

Goldstone picture: LRO for $n\geq 3$, $\beta \gg 1$.

Critical behavior: Near $\beta \sim \beta_c$.

Exercise: Similar to percolation arguments:

- High temperature expansion ($\beta \ll 1$).
- Peierls argument ($\beta \gg 1$, $n=1$, $d=2$).

<u>Mean-field model</u>: $\Lambda = [N] = \{1, \dots, N\}$, $J_{ij} = \frac{1}{N}$.

<u>Exercise</u>: Enumerate the configurations of the Ising model on the complete graph and compute

$$m(\beta) = \left\langle \left(\frac{1}{N} \sum \sigma_i \right)^2 \right\rangle^{1/2}.$$

<u>Lemma</u>. For any $\sigma = (\sigma_i)_{i=1}^N \in (\mathbb{R}^n)^N$,

$$\exp\left[\frac{\beta}{2N} \sum_{i,j=1}^N \sigma_i \cdot \sigma_j \right] \propto \int_{\mathbb{R}^n} d\varphi \, \exp\left[-\frac{1}{2} \beta N |\varphi|^2 + \beta \varphi \cdot \sum_{i=1}^N \sigma_i \right]$$

Equivalently,

$$\exp\left[\frac{-\beta}{4N} \sum_{i,j=1}^N |\sigma_i - \sigma_j|^2 \right] \propto \int d\varphi \, \exp\left[-\frac{\beta}{2} \sum_{i=1}^N |\varphi - \sigma_i|^2 \right]$$

N spins   single 'block' spin

<u>Proof</u>. By rescaling, can take $\beta = 1$.

$$\begin{aligned} C_N &= \int d\varphi \, \exp\left[-\frac{N}{2} |\varphi|^2 \right] \\ &= \int d\varphi \, \exp\left[-\frac{N}{2} \left| \varphi - \frac{1}{N} \sum \sigma_i \right|^2 \right] \\ &= \int d\varphi \, \exp\left[-\frac{N}{2} |\varphi|^2 + \varphi \cdot \sum \sigma_i - \frac{1}{2N} \left| \sum \sigma_i \right|^2 \right]. \end{aligned}$$

Rearrange.

Partition function:

$$Z_\beta = \int_{\mathbb{R}^{nN}} e^{-\frac{\beta}{4N}\sum(\sigma_i-\sigma_j)^2 + h\sum_i \underset{\text{unit vector}}{e}\cdot\sigma_i} \prod_{i=1}^{N} d\sigma_i$$

$$= \int_{\mathbb{R}^n} d\varphi \Big(\underbrace{\int e^{-\frac{\beta}{2}|\varphi-\sigma_1|^2 + h e\cdot\sigma_1} d\sigma_1}_{e^{-NV_{\beta,h}(\varphi)}} \Big)^N$$

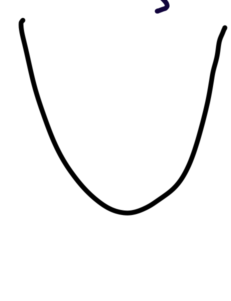
For $n=1$: (Ising) $V_{\beta,h}(\varphi) = -\log \int e^{-\frac{\beta}{2}(\varphi-\sigma)^2 + h\sigma} d\sigma$

$$= -\log\left(e^{-\frac{\beta}{2}(\varphi-1)^2+h} + e^{-\frac{\beta}{2}(\varphi+1)^2-h}\right)$$

$$= \frac{\beta}{2}\varphi^2 - \log\cosh(\beta\varphi+h) + \text{const.}$$

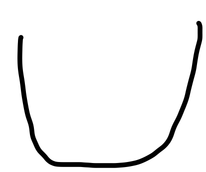
<u>Exercise</u>: Plot the function $V_{\beta,h}$. For $h=0$,

- For $\beta<1$, V_β is strictly convex:



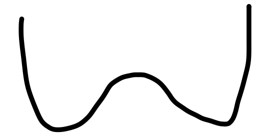
$$V_\beta''(\varphi) \geq \beta-\beta^2$$

- For $\beta=1$, V_β is convex but



$$V_\beta''(0)=0, \quad V_\beta(\varphi) \sim |\varphi|^4$$

- For $\beta>1$, V_β has two minima



<u>Laplace Principle</u>. Let $V: \mathbb{R}^n \to \mathbb{R}$ be

- continuous, bounded below
- $\{\varphi : V(\varphi) \leq \min V + 1\}$ is compact
- $\int e^{-V(\varphi)} d\varphi < \infty$.

Then

$$-\frac{1}{N} \log \int e^{-NV(\varphi)} d\varphi \xrightarrow{N\to\infty} \min V.$$

What are the minima of $V_{\beta,h}$?

$$V_{\beta,h}(\varphi) = \frac{\beta}{2}\varphi^2 - \log\cosh(\beta\varphi + h)$$

$$\Rightarrow \quad V'_{\beta,h}(\varphi) = \beta\varphi - \beta\tanh(\beta\varphi + h)$$

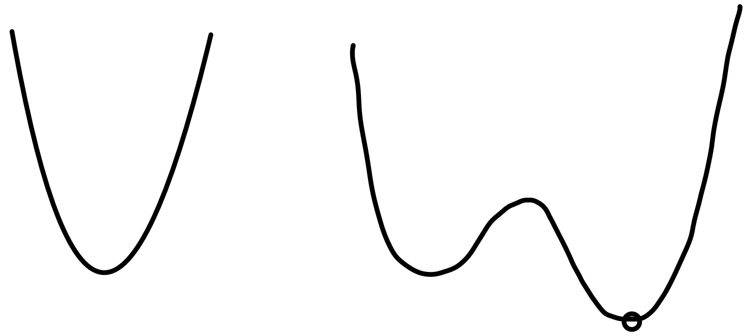
$$V''_{\beta,h}(\varphi) = \beta - \frac{\beta^2}{\cosh^2(\beta\varphi + h)}$$

So $V'_{\beta,h}(\varphi_0) = 0 \iff \varphi_0 = \tanh(\beta\varphi_0 + h)$

self-consistent eqn.

<u>Exercise</u>: For $h > 0$, there is a unique soln $\varphi_0(\beta,h)$.

- If $\beta \leq 1$ then $\varphi_0(\beta,h) \to 0$ as $h \downarrow 0$.
- If $\beta > 1$ then $\varphi_0(\beta,h) \to \varphi_0(\beta,0+) > 0$ as $h \downarrow 0$.



<u>Cor.</u> If $\beta > 1$ then

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \langle \sigma_1 \rangle_{\beta,h} = \lim_{h \downarrow 0} \lim_{N \to \infty} \frac{\partial}{\partial h} \underbrace{\frac{1}{N} \log Z_{\beta,h}}_{-F^N(\beta,h)} > 0 .$$

<u>Proof</u>. $h \mapsto -F^N(\beta,h)$ is convex (check!).
If F^N are diffentiable convex functions, $F^N(h) \to F(h)$ for all h, then F is convex and at every h for which F is differentiable, $\nabla F^N(h) \to \nabla F(h)$.
By Laplace Principle, $F^N_\beta(h) \longrightarrow F_\beta(h) = \min V_{\beta,h}$.

$$\Rightarrow \lim_{h \downarrow 0} \lim_{N \to \infty} \langle \sigma_1 \rangle_{\beta,h} \overset{\text{(above)}}{=} -\lim_{h \downarrow 0} \frac{\partial}{\partial h} \Big(V_{\beta,h}(\varphi_0(\beta,h)) \Big) = -\lim_{h \downarrow 0} V'_{\beta,h}(\varphi_0)$$

where φ_0 is ! soln to $\varphi = \tanh(\beta\varphi + h)$.

$$\Rightarrow \langle \sigma_1 \rangle_{\beta,0+} = \lim_{h \downarrow 0} \beta \tanh(\beta\varphi_0 + h) = \lim_{h \downarrow 0} \beta \varphi_0(\beta,h).$$

This is called <u>spontaneous</u> symmetry breaking. Symmetry implies, for any $\beta \geq 0$,

$$\lim_{h \downarrow 0} \langle \sigma_1 \rangle_{\beta,h} = \langle \sigma_1 \rangle_{\beta,0} = 0$$

Thus the limits do not commute.

<u>Exercise</u>: Compute other observables. For example,

$$\langle \sigma_1 \rangle_{\beta,0+} \sim (3(\beta - \beta_c))^{1/2} \qquad (\beta \downarrow \beta_c = 1)$$

$$\langle \sigma_1 \rangle_{\beta_c,h} \sim (3h)^{1/3} \qquad (h \downarrow 0)$$

The susceptibility $\chi(\beta,h) = \dfrac{\partial \langle \sigma_1 \rangle_{\beta,h}}{\partial h}$ satisf.

$$\chi(\beta,0) = \frac{1}{\beta_c - \beta} \qquad (\beta < \beta_c)$$

$$\chi(\beta,0_+) \sim \frac{1}{2(\beta - \beta_c)}.$$

The above are examples of critical exponents.

<u>Exercise</u>: Generalize this to the O(3) model.

2. Correlation inequalities

O(n) model:

uniform on S^{n-1}

$$\langle F \rangle_{\beta J, h} \propto \int F(\sigma)\, e^{+\beta \sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i e \cdot \sigma_i} \prod d\sigma_i$$

unit vector

<u>1st Griffith inequality</u>. For $n \geq 1$, $J_{ij} \geq 0$, $h_i \geq 0$,

$$\langle \sigma_{x_1}^{m_1} \cdots \sigma_{x_k}^{m_k} \rangle \geq 0, \qquad m_\ell \in [n]. \qquad \text{(GI)}$$

<u>2nd Griffith inequality</u>. For $n = 1, 2$, $J_{ij} \geq 0$, $h_i \geq 0$,

$$\underbrace{\langle \sigma_i \cdot \sigma_j ; \sigma_k \cdot \sigma_\ell \rangle}_{\langle (\sigma_i \cdot \sigma_j)(\sigma_k \cdot \sigma_\ell) \rangle - \langle \sigma_i \cdot \sigma_j \rangle \langle \sigma_k \cdot \sigma_\ell \rangle} \geq 0. \qquad \text{(GII)}$$

Open question: $n \geq 3$?

<u>Cor</u>. $\langle \sigma_0 \cdot \sigma_x \rangle_{\beta, h}$ is increasing in $\beta \geq 0$, $h \geq 0$.

<u>Proof</u>.

$$\frac{\partial}{\partial \beta} \langle F \rangle_{\beta, h} = \frac{\partial}{\partial \beta} \frac{\int e^{+\beta \sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i \sigma_i \cdot e} F}{\int e^{+\beta \sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i \sigma_i \cdot e}}$$

$$= \langle F ; \sum J_{ij} \sigma_i \cdot \sigma_j \rangle_{\beta, h} \geq 0.$$

Proof of (GI). By expanding

$$e^{\sum J_{ij}\sigma_i\cdot\sigma_j + \sum h_i\sigma_i}$$

it suffices to consider $J=0$, $h=0$. Thus the measure is product. For any $O(n)$-invariant measure of a single spin one can check that

$$\int s^{m_1}\cdots s^{m_k}\,\mu(ds)\geq 0.$$

Proof of (GII), $n=1$.

$$\langle \sigma_i\cdot\sigma_j\,;\,\sigma_k\cdot\sigma_\ell\rangle = \frac{1}{2}\langle(\sigma_i\cdot\sigma_j - \sigma_i'\cdot\sigma_j')(\sigma_k\cdot\sigma_\ell - \sigma_k'\cdot\sigma_\ell')\rangle$$

where σ' is an independent copy.

$$H(\sigma)+H(\sigma') = \sum J_{ij}(\sigma_i\cdot\sigma_j + \sigma_i'\cdot\sigma_j')$$

Write

$$\left.\begin{aligned} t_i &= \tfrac{1}{2}(\sigma_i+\sigma_i') \\ q_i &= \tfrac{1}{2}(\sigma_i-\sigma_i') \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \sigma_i\sigma_j + \sigma_i'\sigma_j' &= 2(t_it_j + q_iq_j) \\ \sigma_i\sigma_j - \sigma_i'\sigma_j' &= 2(t_iq_j + t_jq_i) \end{aligned}$$

$$\Rightarrow H(\sigma)+H(\sigma') = -\sum 2J_{ij}(t_it_j + q_iq_j)$$

$$(\sigma_i\sigma_j - \sigma_i'\sigma_j')(\sigma_k\sigma_\ell - \sigma_k'\sigma_\ell') = 4(t_iq_j + t_jq_i)(t_kq_\ell + t_\ell q_k)$$

$\Rightarrow \langle \sigma_i \sigma_j ; \sigma_k \sigma_\ell \rangle = \sum (\text{pos.}) \int e^{+2\sum J_{ij}(t_i t_j + q_i q_j)} \ldots$

Expand.

<u>Exercise</u>. Show $\langle \sigma_{x_1} \cdots \sigma_{x_k} ; \sigma_{y_1} \cdots \sigma_{y_\ell} \rangle \geq 0$.

<u>Proof</u> of (GII), $n=2$. Use polar coordinates.

$$\sigma_i = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}.$$

For the duplicated system, set

$$\varphi = \tfrac{1}{2}(\theta + \theta'), \quad \varphi' = \tfrac{1}{2}(\theta' - \theta)$$

Then (with $m\theta = \sum m_i \theta_i$)

$$\cos(m\theta) + \cos(m\theta') = 2\cos(m\varphi)\cos(m\varphi')$$
$$\cos(m\theta) - \cos(m\theta') = 2\sin(m\varphi)\sin(m\varphi')$$

Thus

$$-H(\theta) - H(\theta') = \sum_{ij} J_{ij}(\cos(\theta_i - \theta_j) + \cos(\theta_i' - \theta_j'))$$
$$= \sum_{ij} 2J_{ij} \cos(\varphi_i - \varphi_j)\cos(\varphi_i' - \varphi_j')$$

$$\Rightarrow e^{-H} = \sum (\text{positive}) \prod \cos(m\varphi)\cos(m\varphi')$$

Thus

$$\langle \sigma_i \cdot \sigma_j \, ; \, \sigma_k \cdot \sigma_\ell \rangle$$

$$= \langle \cos(\theta_i - \theta_j) \, ; \, \cos(\theta_k - \theta_\ell) \rangle$$

$$= \frac{1}{2} \Big\langle \big(\underbrace{\cos(\theta_i - \theta_j) - \cos(\theta_i' - \theta_j')}_{2\sin(\theta_i - \theta_j)\sin(\theta_i' - \theta_j')}\big)\big(\underbrace{\cos(\theta_k - \theta_\ell) - \cdots}_{\cdots}\big) \Big\rangle$$

$$\propto \sum (\text{positive}) \left(\int \prod \sin(m'\theta) \prod \cos(m\theta) \, d\theta \right)^2 \geq 0.$$

Rk. The proof immediately implies the more general inequality

$$\Big\langle \cos\Big(\sum_i m_i \theta_i\Big) \, ; \, \cos\Big(\sum n_i \theta_i\Big) \Big\rangle \geq 0$$

for every $m_i \in \mathbb{Z}$, $n_i \in \mathbb{Z}$.

<u>Example</u>. $\langle \sigma_i \cdot \sigma_j \rangle^{XY}_{2J} \le \langle \sigma_i \sigma_j \rangle^{Ising}_J$.

<u>Proof</u>. Let

$$\langle F \rangle^\lambda_J \propto \int F(\sigma)\, e^{\sum J_{ij} \cos(\theta_i - \theta_j) + \lambda \sum_i \cos(\theta_i)} \prod_i d\theta_i$$

$$\sigma = (\cos\theta, \sin\theta).$$

Then $\langle \cdot \rangle^{XY}_J = \langle \cdot \rangle^0_J$ and

$$\lim_{\lambda \to \infty} \langle \cdot \rangle^\lambda_J \propto \sum_{\sigma \in C_4^\Lambda} F(\sigma)\, e^{\sum J_{ij} \sigma_i \cdot \sigma_j}, \qquad C_4 = \{(\pm 1, 0), (0, \pm 1)\}.$$

Ginibre: $\frac{\partial}{\partial \lambda} \langle \sigma_i \cdot \sigma_j \rangle^\lambda \ge 0$

$\Rightarrow \langle \sigma_i \cdot \sigma_j \rangle^{XY}_J \le \langle \sigma_i \cdot \sigma_j \rangle^\infty_J$

Let $t_i = \sigma_i^1 + \sigma_i^2$, $q_i = \sigma_i^1 - \sigma_i^2$. Then

$$2 \sum_{ij} J_{ij} \sigma_i \cdot \sigma_j = \sum_{ij} J_{ij} (t_i t_j + q_i q_j)$$

and if $\sigma_i \sim C_4$ then $(t_i, q_i) \sim \{(\pm 1), (\pm 1)\}$. Thus

$$\langle \sigma_i \cdot \sigma_j \rangle^\infty_J = \langle t_i t_j \rangle^{Ising}_{\frac{1}{2}J}$$

Example. Consider the XY model on $\Lambda \subset \mathbb{Z}^d$ with

- free boundary conditions:

$$\langle F\rangle^{\Lambda,f} \propto \int_{(S^1)^\Lambda} F \, e^{\sum_{ij\subset\Lambda} J_{ij}\cos(\theta_i-\theta_j)} \prod_i d\theta_i$$

- Dirichlet boundary conditions:

$$\langle F\rangle^{\Lambda,+} \propto \int_{(S^1)^\Lambda} e^{\sum_{ij\subset\Lambda} J_{ij}\cos(\theta_i-\theta_j)+\sum_{\substack{i\in\Lambda\\ j\notin\Lambda}} J_{ij}\cos(\theta_i)} \prod_i d\theta_i.$$

Then if $F=\cos(\sum m_i\theta_i)$, $\Lambda_1 \subset \Lambda_2$,

$$\langle F\rangle^{\Lambda_1,f} \leq \langle F\rangle^{\Lambda_2,f} \qquad \text{(F)}$$

$$\langle F\rangle^{\Lambda_1,+} \geq \langle F\rangle^{\Lambda_2,+}. \qquad \text{(D)}$$

As a consequence, if $\Lambda_i \nearrow \mathbb{Z}^d$, the limits

$$\langle F\rangle^{\#} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle F\rangle^{\Lambda,\#}$$

exist for any trigonometric polynomial F (and by density for all local continuous observables).