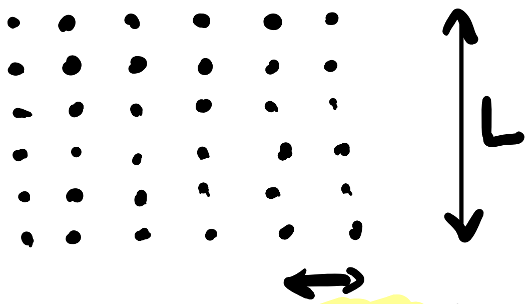
Stochastic dynamics and renormalisation

Beijing 2022

https://www.dpmms.cam.ac.uk/~rb812/teaching/beijing2022/

1. Spin systems



$\varepsilon = 1$ for now

$\Lambda = \Lambda_{\varepsilon, L}$ finite $\subset \mathbb{Z}^d$
(often periodic b.c.)

Spin σ_x, $x \in \Lambda$

continuous, $\sigma_x \in \mathbb{R}$ or $\mathbb{R}^n$
discrete, $\sigma_x \in \{\pm 1\}$

$$\nu(d\sigma) = \frac{1}{Z} e^{-\frac{\beta}{2}(\sigma, -\Delta^J \sigma)} \prod_{x \in \Lambda} \mu(d\sigma_x)$$

$\frac{1}{2} \sum_{x,y \in \Lambda} J_{xy} |\sigma_x - \sigma_y|^2$ single-spin measure

$J_{xy} = 1_{x \sim y}$

O(n) (vector) models:

- Ising $n=1$, $\mu = \frac{1}{2}(\delta_{+1} + \delta_{-1})$
- XY model $n=2$, μ uniform on S^1
- Heisenberg model $n=3$, μ uniform on S^2

Also natural to consider unbounded spins:

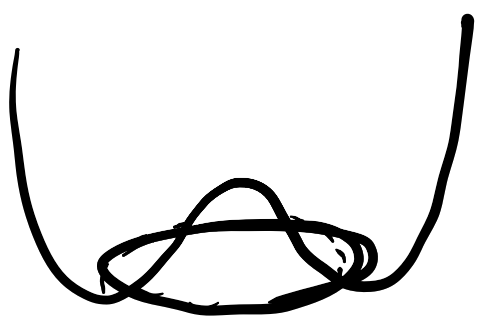
$$\nu(d\varphi) = \frac{1}{Z} e^{-H(\varphi)} d\varphi$$

Lebesgue measure $\mathbb{R}^{n\Lambda}$

$$H(\varphi) = \frac{1}{2}(\varphi, -\Delta\varphi) + \sum_{x\in\Lambda} V(\varphi_x)$$

Ginzburg-Landau-Wilson $|\varphi|^4$ model:

$$V(\varphi) = \frac{1}{4} g |\varphi|^4 + \frac{1}{2}\nu|\varphi|^2, \quad g>0, \ \nu \in \mathbb{R}$$



Phase transition: E.g. Ising $d\geq 2$

$$\sum_{y\in\Lambda} \langle \sigma_x \sigma_y \rangle \leq C \quad \text{uniformly in } L \text{ for } \beta<\beta_c$$

expectation w.r.t. ν

high temperature $T = \frac{1}{\beta}$

$\langle \sigma_x \sigma_y \rangle \geq c > 0$ when $\beta > \beta_c$

low temperature

Euclidean field theory (models defined in cont.)

$$\nu(d\varphi) = \frac{1}{Z} \exp\Big(- \underbrace{\varepsilon^d \sum_{x,y \in \Lambda_{\varepsilon,L}}}_{\sim \int dx\, dy} \frac{(\varphi(x)-\varphi(y))^2}{\varepsilon^2}$$

$$- \varepsilon^d \sum_{x \in \Lambda_{\varepsilon,L}} \Big(\frac{1}{4} g \varphi_x^4 + \frac{1}{2} \nu_\varepsilon \varphi_x^2 \Big) \Big)$$

"φ^4 model"
$d = 2, 3$

$$\nu_\varepsilon = \nu - \begin{cases} c g \log \varepsilon^{-1} & (d=2) \\ c g \frac{1}{\varepsilon} + c' \log \varepsilon^{-1} & (d=3) \end{cases}$$

counter-terms

$$\nu(d\varphi) = \frac{1}{Z} \exp\Big(- \varepsilon^d \sum_{x,y \in \Lambda_{\varepsilon,L}} \frac{(\varphi(x)-\varphi(y))^2}{\varepsilon^2}$$

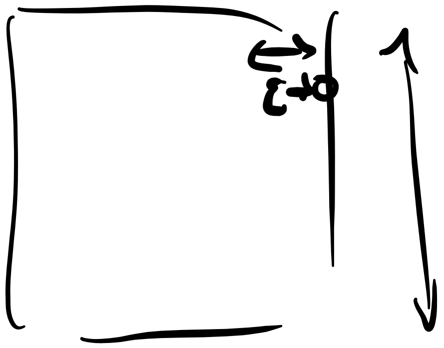
$$- \varepsilon^d \sum_{x \in \Lambda_{\varepsilon,L}} 2 z \varepsilon^{-\frac{\beta}{4\pi}} \cos(\sqrt{\beta} \varphi_x) \Big)$$

"sine-Gordon model"
$d = 2$

Difficulties:

- Critical point ($\varepsilon = 1$) : $\langle \sigma_x \sigma_y \rangle \approx |x-y|^{-\kappa}$
  strong correlations at large distances

- Continuum limit ($\varepsilon \to 0$) : $\langle \varphi_x \varphi_y \rangle \approx |x-y|^{-(d-2)}$
  at short distances



2. Generalities about dynamics – cont. case

$$d\varphi_t = -\nabla H(\varphi_t)\, dt + \sqrt{2}\, dW_t$$

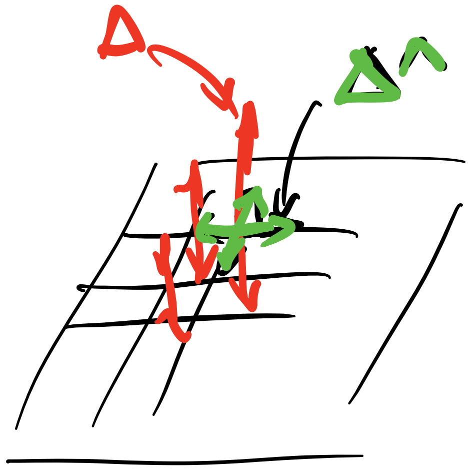
$$= + \Delta \varphi_t\, dt - V'(\varphi_x)\, dt + \sqrt{2}\, dW_t$$

lattice Laplacion

finite-dimensional SDE

Infinitesimal generator $\Delta_H = \Delta - (\nabla H, \nabla)$

$$= e^{+H} (\nabla, e^{-H} \nabla)$$



$$\sum_{x\in\Lambda} \frac{\partial^2}{\partial^2\varphi_x} \qquad \left(\frac{\partial}{\partial\varphi_x}\right)_{x\in\Lambda}$$

ν is reversible: $\mathbb{E}_\nu(F(-\Delta_H G)) = \mathbb{E}_\nu(\nabla F, \nabla G)$

Itô $\Rightarrow$ $F_t(\varphi) = \mathbb{E}_{\varphi_0=\varphi} F(\varphi_t)$ solves

↑ soln.

$$\frac{\partial}{\partial t} F_t(\varphi) = \Delta_H F_t(\varphi)$$

Law ν_t of φ_t satisfies $\mathbb{E}_{\nu_t} F = \mathbb{E}_{\nu_0} F_t$

If $d\nu_t = G_t \, d\nu$ then $\partial_t G_t = \Delta_H^* G_t$

$$= \Delta G_t$$

Ergodicity implies $\nu_t \to \nu$. How fast?

Relative entropy: $H(\nu_t|\nu) = \underbrace{\mathbb{E}_\nu F \log F - (\mathbb{E}_\nu F)(\log \mathbb{E}_\nu F)}_{\mathrm{Ent}_\nu F}$

$$F = \frac{d\nu_t}{d\nu} \geq 0$$

Pinsker's inequality: $\|\nu_t - \nu\|_{TV}^2 \leq 2H(\nu_t|\nu)$

<u>Fact</u>. (de Bruin identity).

$$\frac{\partial}{\partial t} H(\nu_t|\nu) = -I(\nu_t|\nu) < 0$$

where

$$\begin{aligned} I(\nu_t|\nu) &= \mathbb{E}_\nu(\nabla \log F_t, \nabla F_t) \\ &= \mathbb{E}_\nu\left(\frac{(\nabla F_t)^2}{F_t}\right) \\ &= 4\mathbb{E}_\nu(\nabla\sqrt{F_t})^2 \quad \text{(Fisher inf.)} \end{aligned}$$

<u>Proof</u>. $\frac{\partial}{\partial t} \mathbb{E}_\nu \Phi(F_t) = \mathbb{E}_\nu \Phi'(F_t) \dot{F}_t$

$(\Phi(x) = x\log x)$

$$\begin{aligned} &= \mathbb{E}_\nu \Phi'(F_t) \Delta_H F_t \\ &= -\mathbb{E}_\nu(\underbrace{\nabla\Phi'(F_t)}_{\nabla \log F_t}, \nabla F_t) \end{aligned}$$

Upshot: $H(\nu_t|\nu) \leq e^{-2\gamma t} H(\nu_0|\nu)$

$$\Leftrightarrow \frac{\partial}{\partial t} H(\nu_t|\nu) \leq -2\gamma H(\nu_t|\nu)$$

$$\iff I(\nu_t|\nu) \geq 2\gamma\, H(\nu_t|\nu)$$

Log-Sobolev inequality:

$$\mathrm{Ent}_\nu F \leq \frac{2}{\gamma}\, \mathbb{E}_\nu (\nabla\sqrt{F})^2 \qquad \text{(LSI)}$$

Exercises:

- LSI $\Rightarrow$ Spectral gap ineq.
$$\mathrm{Var}_\nu F \leq \frac{1}{\gamma}\, \mathbb{E}(\nabla F)^2$$
- Tensorisation: if ν_1 and ν_2 satisfy LSIs with the same γ then $\nu_1 \otimes \nu_2$ also does.
- LSI $\iff$ Hypercontractivity of the Markov semigroup (Gross)
- Holley-Stroock criterion: if ν satisfies LSI with constant γ and $d\mu/d\nu = F$ then μ satisfies LSI with constant $(\sup F / \inf F)\,\gamma$.

Bakry-Émery theorem.

$$\nu(d\varphi) \propto e^{-H(\varphi)}\,d\varphi \leftarrow \mathbb{R}^{\Lambda}$$

and

$$\operatorname{Hess} H(\varphi) \geq \lambda \,\mathrm{id} \quad (\lambda > 0)$$

$\Rightarrow$ LS constant γ of ν is at least λ, i.e. $\gamma \geq \lambda$.

Proof.

$$H(\nu_0|\nu) = -\int_0^\infty \frac{\partial}{\partial t} H(\nu_t|\nu)\,dt$$

(using $H(\nu_t|\nu) \to 0$)

$$= \int_0^\infty I(\nu_t|\nu)\,dt$$

$$\overset{\text{WANT}}{\leq} \frac{1}{2\gamma} I(\nu_0|\nu)$$

Follows if $I(\nu_t|\nu) \leq e^{-2\gamma t} I(\nu_0|\nu)$.

To see this, differentiate again:

$$\frac{\partial}{\partial t} I(\nu_t|\nu) = \frac{\partial}{\partial t} \mathbb{E}_\nu\left(\frac{(\nabla F_t)^2}{F_t}\right), \quad F_t = \frac{d\nu_t}{d\nu}$$

uses $\frac{\partial}{\partial t} F_t = \Delta_H F_t$ elementary but tedious

$$= -2\,\mathbb{E}_\nu\Big(\underbrace{F_t\,|\text{Hess}\log F_t|_2^2}_{\geq 0} + F_t(\nabla \log F_t, \underbrace{\text{Hess}\,H}_{\geq \lambda\,\text{id}}\nabla \log F_t)\Big)$$

$$\leq -2\lambda\,\mathbb{E}_\nu\left(\frac{(\nabla F_t)^2}{F}\right)$$

$$= -2\lambda\, I(\nu_t|\nu)$$

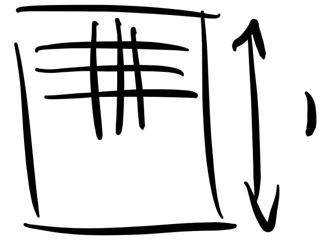
Gronwall's lemma implies claim.

Example (Gaussian free field).

$$H(\varphi) = \frac{1}{2}(\varphi, (-\Delta^\varepsilon + m^2)\varphi)_\varepsilon$$

$$(u,v)_\varepsilon = \varepsilon^d \sum_{x\in\Lambda} u(x)v(x)$$

$$\Delta^\varepsilon f(x) = \varepsilon^{-2} \sum_{x\sim y} \big(f(y) - f(x)\big)$$



Invariant measure is GFF on $\mathbb{T}_\varepsilon^d$

Dynamics is diagonal in Fourier basis:

$$d\hat{\varphi}_t(k) = -\underline{(|k|^2+m^2)}\,\hat{\varphi}(k)\,dt + \sqrt{2}\,d\hat{W}(k)$$

$\Rightarrow$ Different Fourier modes ($\equiv$ length scale) equilibrate at different rates

Small scales ($|k|\gg 1$) converge very quickly.

Macroscopic scales ($|k|\sim 1$) are slowest.

Difficult: $H(\varphi) = \frac{1}{2}(\varphi, (-\Delta^\varepsilon+m^2)\varphi) + V_0^\varepsilon(\varphi)$

$\sum_{x\in\Lambda} V^\varepsilon(\varphi)$ <u>local</u> nonlinear

nonlocal in Fourier sp.

Example: For the Ising model, if $\underbrace{\|\beta\Delta^\nu\|}_{A} < 1$ then the log-Sobolev constant γ is strictly positive:

$$\underset{\text{Ising}}{\mathrm{Ent}_\nu F} \le \frac{2}{\gamma} \sum_{x\in\Lambda} \mathbb{E}_\nu\Big(\big(\sqrt{F(\sigma)} - \sqrt{F(\sigma^x)}\big)^2 \Big)$$

$\sigma \in \{\pm 1\}^\Lambda$

$\sigma^x \in \{\pm 1\}^\Lambda$ obtained by flipping sign at x

Proof. Recall the Ising measure is

$$\nu(d\sigma) = \frac{1}{Z} e^{-\frac{1}{2}(\sigma, A\sigma)} \prod_{x\in\Lambda} d\sigma_x$$

Since $|\sigma_x| = 1$, may replace A by $A + \varepsilon\,\mathrm{id}$.

Since $\|A\| < c < 1$ there is a pos.-def B s.t.

$$A^{-1} = c^{-1}\,\mathrm{id} + B^{-1}$$

$$\Rightarrow e^{-\frac{1}{2}(\sigma, A\sigma)} = \text{const.} \int_{\mathbb{R}^\Lambda} e^{-\frac{c}{2}(\varphi-\sigma, \varphi-\sigma)} e^{-\frac{1}{2}(\varphi, B\varphi)} d\varphi$$

Exercise: sum of ind. Gaussians are Gauss.

Define:

$$e^{-V(\psi)} = \sum_{\sigma\in\{\pm 1\}} e^{-\frac{c}{2}(\psi-\sigma)^2} \qquad (\psi\in\mathbb{R})$$

$$\mu_\psi(d\sigma) = e^{+V(\psi)} e^{-\frac{c}{2}(\psi-\sigma)^2} \qquad (\sigma\in\{\pm 1\})$$

$$\nu_r(d\varphi) = e^{-\frac{1}{2}(\varphi,B\varphi)-\sum_{x\in\Lambda}V(\varphi_x)}\,d\varphi \quad (\varphi\in\mathbb{R}^\Lambda)$$

$$\Rightarrow \mathbb{E}_\nu F(\sigma) = \mathbb{E}_{\nu_r}\mathbb{E}_{\mu_\varphi}F(\sigma)$$

$$\mu_\varphi = \bigotimes_{x\in\Lambda}\mu_{\varphi_x}$$

① μ_φ is product, so LSI for each μ_{φ_x} with general φ_x implies uniform LSI for μ_φ

$$\mathrm{Ent}_{\mu_{\varphi_x}} F \le \frac{2}{\gamma_0}\mathbb{E}_{\mu_{\varphi_x}}\left(\sqrt{F(\sigma)}-\sqrt{F(\sigma^x)}\right)^2$$

ind. of φ_x

$$\Rightarrow \mathrm{Ent}_{\mu_\varphi} F \le \frac{2}{\gamma_0}\sum_{x\in\Lambda}\mathbb{E}_{\mu_\varphi}\left(\sqrt{F(\sigma)}-\sqrt{F(\sigma^x)}\right)^2$$

② For $c<1$, V is convex:

$$V''(\psi) = c - c^2\underbrace{\mathrm{Var}_{\mu_\psi}(\sigma)}_{\le 1} \ge c-c^2=\lambda>0$$

$\Rightarrow$ Bakry-Émery applies

$$\mathrm{Ent}_{\nu_r} G \le \frac{2}{\lambda}\, \mathbb{E}\, |\nabla \sqrt{G}|^2$$

gradient on $\mathbb{R}^\Lambda$

Combining both gives

$$\mathrm{Ent}_\nu F = \mathbb{E}_{\nu_r}\, \mathrm{Ent}_{\mu_\varphi} F(\sigma) + \mathrm{Ent}_{\nu_r} G(\varphi)$$

$G = \mathbb{E}_{\mu_\varphi} F(\sigma)$

$$\le \frac{2}{\gamma_0} \underbrace{\mathbb{E}_{\nu_r} \mathbb{E}_{\mu_\varphi}}_{\mathbb{E}_\nu} \left(\sum_x \left(\sqrt{F(\sigma)} - \sqrt{F(\sigma^x)}\right)^2 \right)$$

$$+ \frac{2}{\lambda}\, \mathbb{E}_{\nu_r} \underbrace{|\nabla \sqrt{G}|^2}_{\sum_x |\nabla_{\varphi_x} \sqrt{G}|^2}$$

$$|\nabla_{\varphi_x} \sqrt{G}|^2 = \frac{|\nabla_{\varphi_x} G|^2}{4G} = \left(\frac{c}{2}\right)^2 \frac{\mathrm{Cov}_{\mu_{\varphi_x}}(F, \sigma_x)^2}{G}$$

Cauchy-Schwarz and $|\sigma_x| \le 1$ $\longrightarrow$ $\le \frac{c^2}{4}\, 8\, \mathrm{Var}_{\mu_{\varphi_x}} \sqrt{F}$

Spectral gap on $\{\pm 1\}$ $\longrightarrow$ $\le \frac{2c^2}{\gamma_0}\, \mathbb{E}_{\nu_\varphi} \left(\sqrt{F(\sigma)} - \sqrt{F(\sigma^x)}\right)^2$

Rks. • Proof involves two scales: microsopic & macroscopic

- On the complete graph, above is sharp.
- Condition $\beta \|\Delta^J\| \leq 1$ does not use positivity of J and also applies to spin glasses:

  SK model: $J_{xy} = \frac{1}{\sqrt{N}} H_{xy}$

  i.i.d. Gaussian

- Proof can also be adapted to other Dirichlet forms (spectral gap, mod. LSI, etc.)
- Also applies to $O(n)$ model!

3. Gaussian integration

Let $C_s = \int_0^s \dot{C}_u \, du$, $\dot{C}_u$ is pos.-def. on $\mathbb{R}^\Lambda$

P_{C_s} be the Gaussian measure with covariance C_s

<u>Example.</u> $\dot{C}_s = e^{-sA} \Rightarrow C_\infty = A^{-1}$

$A = (-\Delta^2 + m^2) \Rightarrow P_{C_\infty}$ is GFF

$C_s = (A + 1/s)^{-1}$, i.e. $\dot{C}_s = (sA+1)^{-2}$

$\Rightarrow C_\infty = A^{-1}$

<u>Prop.</u> $\frac{\partial}{\partial s} P_{C_s}(\varphi) = \frac{1}{2} \Delta_{\dot{C}_s} P_{C_s}(\varphi)$

$$\Delta_{\dot{C}_s} = \sum_{x,y \in \Lambda} \dot{C}_s(x,y) \frac{\partial^2}{\partial \varphi(x) \partial \varphi(y)}$$

<u>Proof.</u> $\frac{\partial}{\partial s} \left(\frac{1}{Z_s} e^{-\frac{1}{2}(\varphi, C_s^{-1}\varphi)} \right)$

$$= \frac{1}{2} \underbrace{(C_s^{-1}\varphi, \dot{C}_s C_s^{-1}\varphi)}_{(C_s^{-1}\varphi)^2_{\dot{C}_s}} P_{C_s}(\varphi) - \text{const.}\, P_{C_s}(\varphi)$$

$$\frac{1}{2}\Delta_{\dot C_s} P_{C_s}(\varphi) = \frac{1}{2}(C_s^{-1}\varphi)^2_{\dot C_s}\, P_{C_s}(\varphi) - \text{const.}\, P_{C_s}(\varphi)$$

$$\Rightarrow (\partial_s - \tfrac{1}{2}\Delta_{\dot C_s}) P_{C_s}(\varphi) = \text{const.}\, \underbrace{P_{C_s}(\varphi)}_{\int (\cdots) = 1}$$

$\Rightarrow$ Integral of LHS is 0, so const. $= 0$.

Cor. Let $F_s = P_{C_s} * F_0$, i.e., $F_s(\varphi) = E_{C_s} F(\varphi + \zeta)$.

$$\Rightarrow \partial_s F_s = \tfrac{1}{2}\Delta_{\dot C_s} F_s$$

$$F_s(0) = E_{C_s} F_0$$

Defn (renormalised potential)

$$e^{-V_s(\varphi)} = (P_{C_s} * e^{-V_0})(\varphi) = E_{C_s}(e^{-V_0(\varphi+\zeta)})$$

$$\Leftrightarrow \frac{\partial}{\partial s} e^{-V_s} = \tfrac{1}{2}\Delta_{\dot C_s} e^{-V_s}$$

Polchinski eqn.

$$\Leftrightarrow \frac{\partial}{\partial s} V_s = \tfrac{1}{2}\Delta_{\dot C_s} V_s - \tfrac{1}{2}(\nabla V_s)^2_{\dot C_s} \qquad (P)$$

diffusion H.-J.

<u>Prop</u>. Suppose V_s satisfies (P) and

$$\frac{\partial}{\partial s} F_s = \frac{1}{2}\Delta_{\dot{C}_s} F_s - (\nabla V_s, \nabla F_s)_{\dot{C}_s}$$
$$= L_s F_s$$

Then the following is indep. of s:

$$\int \underbrace{P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}}_{\nu^s(\varphi)\,:\ \text{renormalised measure}} F_s(\varphi)\, d\varphi$$

$s=0 \Rightarrow$ original measure

<u>Proof</u>. Define $Z_s(\varphi) = e^{-V_s(\varphi)} F_s(\varphi)$.

$$\Rightarrow \frac{\partial}{\partial s} Z_s(\varphi) = \frac{1}{2}\Delta_{\dot{C}_s} Z_s(\varphi)$$

$$\Rightarrow \frac{\partial}{\partial s} \int P_{C_\infty - C_s}(\varphi)\, \underbrace{e^{-V_s(\varphi)} F_s(\varphi)}_{Z_s(\varphi)}\, d\varphi$$

$$= \int \left(-\frac{1}{2}\Delta_{\dot{C}_s} P_{C_\infty - C_s}\right) Z_s + P_{C_\infty - C_s}\left(+\frac{1}{2}\Delta_{\dot{C}_s} Z_s\right)$$

$= 0$ by integration by parts

Define

$$\nu^s(d\varphi) = P_{C_0 - C_s}(\varphi)\, e^{-V_s(\varphi)}\, d\varphi$$

(ren. measure)

$$P_{s',s} F(\varphi) = e^{+V_s(\varphi)}\, \mathbb{E}_{C_s - C_{s'}}\left(e^{-V_{s'}(\varphi+\zeta)} F(\varphi+\zeta)\right)$$

(Polchinski semigroup)

$$\Rightarrow \mathbb{E}_{\nu_0} = \mathbb{E}_{\nu_s} P_{0,s} F \quad \forall s$$

<u>Exercise</u>: $\frac{\partial}{\partial s} \mathbb{E}_{\nu_s} F = -\mathbb{E}_{\nu_s} L_s F$

$$L_s = +\tfrac{1}{2}\Delta_{\dot{C}_s} - (\nabla V_s, \nabla)_{\dot{C}_s}$$

$$\frac{\partial}{\partial s} P_{s',s} F = +L_s P_{s',s} F$$

$$\frac{\partial}{\partial s'} P_{s',s} F = -P_{s',s} L_{s'} F$$

Summary: two ways to evolve measure

Glauber semigroup: $d\nu_t = F_t \, d\nu_\infty$

law of φ_t at time t

$$\frac{\partial}{\partial t} F_t = \Delta F_t - (\nabla H, \nabla F_t)$$
$$= \Delta_H F_t$$

tends to inv. measure $\nu = \nu_\infty$

Polchinski semigroup: $F^0 d\nu^0 \to F^s \, d\nu^s$

renorm. meas

$$\frac{\partial}{\partial s} F^s = \frac{1}{2}\Delta_{\dot{C}_s} F_s - (\nabla V_s, \nabla F_s)_{\dot{C}_s}$$

tends to $\nu^\infty = \delta_0$

Thm. Consider the measure

$$\nu^0(d\varphi) = e^{-\frac{1}{2}(\varphi, A\varphi) - V_0(\varphi)}$$

$\dot{C}_0 = \mathrm{id}$

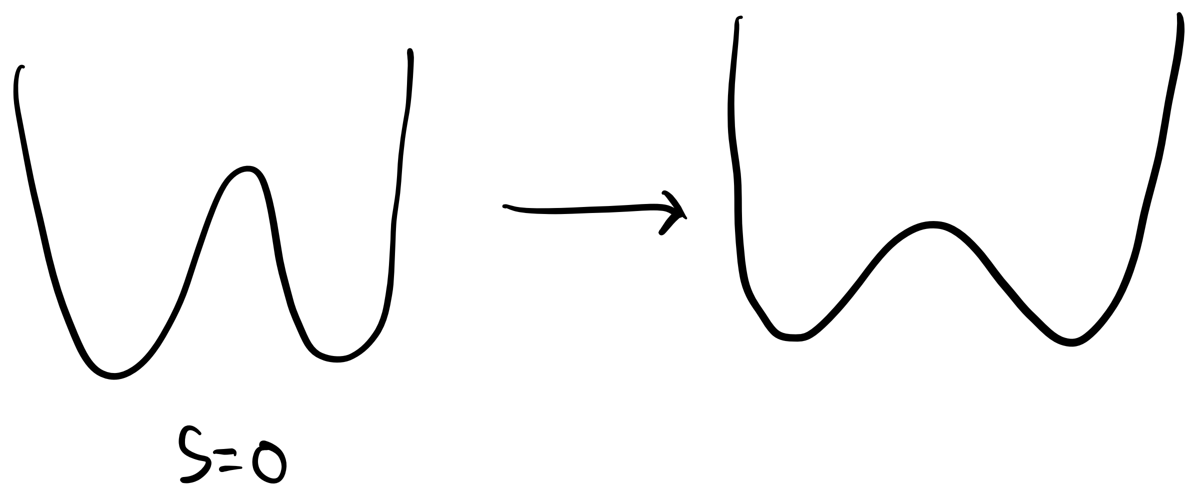
and define the ren. pot. with respect to some decomposition of $C_\infty = A^{-1}$ as above.

$$\dot{C}_s \operatorname{Hess} V_s(\varphi) \dot{C}_s - \frac{1}{2}\ddot{C}_s \geq \dot{\lambda}_s C_s \qquad (*)$$

allowed to be negative

$\Rightarrow \nu_0$ satisfies LSI with

$$\frac{1}{\gamma} \leq \int_0^\infty e^{-2\lambda_s} ds, \quad \lambda_s = \int_0^s \dot{\lambda}_u du.$$



Example. Consider $e^{-H(\varphi)} d\varphi$ with $\operatorname{Hess} H(\varphi) \geq \lambda\, \mathrm{id} \quad (\lambda > 0)$.

Then you can write

$$H(\varphi) = \frac{1}{2}(\varphi, A\varphi) + V_0(\varphi)$$

with

$$A = \lambda \, \mathrm{id}$$

$$V_0 \text{ convex}$$

<u>Fact</u>. If V_t is convex for some $t \geq 0$, then V_s is convex for all $s \geq t$.

Apply theorem with $\dot{C}_t = e^{-tA} \Rightarrow C_\infty = A^{-1}$

$$\ddot{C}_t = -A\dot{C}_t = -\lambda \dot{C}_t$$

$\Rightarrow$ can take $\dot{\lambda}_t = \frac{1}{2}\lambda$

Thus conclusion is LSI with

$$\frac{1}{\gamma} = \int_0^\infty e^{-\lambda t} \, dt = \frac{1}{\lambda}$$

This is exactly the same as Bakry-Emery!

Proof.

$\Phi(x) = x \log x$

$$\frac{\partial}{\partial s} \mathbb{E}_{\nu_s} \Phi(F^s) = \mathbb{E}_{\nu_s}\left(-L_s \Phi(F^s) + \Phi'(F^s)\dot{F}^s\right)$$

$$\frac{\partial}{\partial s} F^s = L_s F^s$$

$$L_s = \frac{1}{2}\Delta_{\dot{C}_s} - (\nabla V_s, \nabla)_{\dot{C}_s}$$

$$= \mathbb{E}_{\nu_s}\Big(-\Phi'(F^s) L_s F^s - \frac{1}{2}\Phi''(F^s)(\nabla F^s)^2_{\dot{C}_s} + \Phi'(F^s)\dot{F}^s\Big)$$

B-Bodineau
Log-Sobolev ineq.
for cont. sine-Gordon

$$= -\frac{1}{2}\mathbb{E}_{\nu_s}\left(\frac{1}{F_s}(\nabla F^s)^2_{\dot{C}_s}\right)$$

$$= -2\, \mathbb{E}_{\nu_s}\left((\nabla \sqrt{F^s})^2_{\dot{C}_s}\right)$$

Now consider the change of $\mathbb{E}_{\nu_s}((\nabla\sqrt{F^s})^2_{\dot{C}_s})$

$$\left(\frac{\partial}{\partial s} - L_s\right)(\nabla\sqrt{F^s})^2_{\dot{C}_s} = +(\nabla\sqrt{F^s})^2_{\ddot{C}_s}$$

like Bakry-Emery
tedious but
elementary

$$-2\left(\nabla\sqrt{F^s}, \dot{C}_s \operatorname{Hess} V_s \dot{C}_s \nabla\sqrt{F^s}\right)$$

$$\underbrace{-\frac{1}{4}F^s\left|\dot{C}_s^{1/2}(\operatorname{Hess}\log F^s)\dot{C}_s^{1/2}\right|_2^2}_{\geq 0}$$

$$\le -\Big(\nabla\sqrt{F^s},\ \underbrace{-\ddot{c}_s + 2\dot{c}_s \operatorname{Hess} V_s\, \dot{c}_s}_{\ge 2\dot{\lambda}_s}\, \nabla\sqrt{F^s}\Big)$$

$$\le -2\dot{\lambda}_s\, (\nabla\sqrt{F^s})_{\dot{c}_s}$$

$\Rightarrow$ $\psi(s) = \mathbb{E}_{\nu_s}(\nabla\sqrt{F^s})^2_{\dot{c}_s}$ satisfies

$$\dot{\psi}(s) \le -2\dot{\lambda}_s\, \psi(s)$$

$$\Rightarrow \psi(s) \le e^{-2\lambda_s}\psi(0) = e^{-2\lambda_s}\,\mathbb{E}_{\nu_0}(\nabla\sqrt{F})^2_{\dot{c}_0}$$

$$\Rightarrow \mathrm{Ent}_{\nu_0} F = \left(\int_0^\infty e^{-2\lambda_s}\, ds\right) 2\,\mathbb{E}_{\nu_0}(\nabla\sqrt{F})^2_{\dot{c}_0}$$

id

(maybe factor 2 lost here)

Example.

$$V_s(\varphi) = -\log \int_{\mathbb{R}^\Lambda} \underbrace{e^{-\frac{1}{2}(\zeta, C_s^{-1}\zeta) - V_0(\varphi+\zeta)}}_{(+\text{const})} d\zeta$$

$$= -\log \int_{\mathbb{R}^\Lambda} e^{-\frac{1}{2}(\zeta-\varphi, C_t^{-1}(\zeta-\varphi)) - V_0(\zeta)} d\zeta$$

$$\Rightarrow \nabla V_s(\varphi) = (P_{0,s} \nabla V_0)(\varphi)$$

$$= \mathbb{E}_{\mu_{0,s}^{\varphi}} \nabla V_0(\varphi+\zeta)$$

$$\nabla V_s(\varphi) = \mathbb{E}_{\mu_{0,s}^{\varphi}} C_t^{-1}(\varphi-\zeta)$$

$$\Rightarrow \operatorname{Hess} V_s(\varphi) = C_t^{-1} - \underbrace{\operatorname{Var}_{\mu_{0,s}^{\varphi}}(C_t^{-1}\zeta)}_{C_t^{-1}\Sigma_t(\varphi) C_t^{-1}}$$

essentially same kind of measure as original one, but with external field φ and Gaussian part $\rightsquigarrow C_s$

$\Sigma_t(\varphi)$: covariance matrix of $\mu_{0,s}^{\varphi}$

For Ising and φ^4 models, recent DSS inequality can be used to check the condition.

DSS = Ding-Song-Sun

→ Papers with Benoit Dagallier.

<u>Example</u>: sine-Gordon → direct series soln. of Polchinski eqn gives control.

→ Paper with Bodineau

<u>Upshot</u>:
- Ising $T > T_c$ (any d)
- Ising $T = T_c$ ($d \geq 5$)
- cont. φ^4 ($d = 2, 3$)
- sine-Gordon ($d = 2$, $\beta < 6\pi$)

Many interesting models open:

E.g. $V_0(\varphi) = \varepsilon^d \sum_{x \in \Lambda} \varepsilon^{\beta/4\pi} \cosh(\sqrt{\beta}\, \varphi_x)$

sinh-Gordon ($d=2$)

<u>convex</u>

4. Stochastic realisation of the Polchinski flow

The Polchinski semigroup acts on measures:

$$P_{s,t} = P_{r,t} P_{s,r} \qquad (0 < s < r < t < \infty)$$

If $\mu^0 = F^0 d\nu^0$ is a prob. measure then

$\mu^s = (P_{0,s} F^0) d\nu^s$ is also one

$P_{s,t}$ can be described in terms of SDE running from time t (backwards) to s.

$$\varphi_s = \varphi_t + \int_s^t \dot{C}_u \nabla V_u(\varphi_u) du + \int_s^t \sqrt{\dot{C}_u} dW_u \qquad (S)$$

$(s<t)$

<u>Prop</u>. $P_{s,t} F(\varphi) = \mathbb{E}_{\varphi_t = \varphi} F(\varphi_s)$

Of course can reverse time direction: $\tilde{\varphi}_r = \varphi_{t-r}$
then

$$d\tilde{\varphi}_r = -\dot{C}_{t-r} \nabla V_{t-r}(\tilde{\varphi}_r) dr + \sqrt{\dot{C}_{t-r}} d\tilde{W}_r$$

Application to coupling:

satisfies (S)

Note: $\mathbb{E}_{\nu_0} F = P_{0,\infty} F(0) = \mathbb{E}_{\varphi_\infty = 0} F(\varphi_0)$

$\Rightarrow$ SDE corresponding to $P_{0,\infty}$ samples from full measure

$$\Rightarrow \underbrace{\varphi_0}_{\sim \nu_0} = \int_0^\infty \dot{C}_u \nabla V_u(\varphi_u)\, du + \underbrace{\int_0^\infty \sqrt{\dot{C}_u}\, dW_u}_{\varphi^{GFF}}$$

Application to extrema of non-Gaussian fields:

Paper with M. Hofstetter (sine-Gordon)

Barashkov - Gunaratnam - Hofstetter (φ^4)

Relation to variational approach (Barashkov - Gubinelli).

Exercise.

satisfies (S)

$$V_t(\varphi) = \mathbb{E}_{\varphi_t=\varphi}\left[V_0(\varphi_0) + \frac{1}{2}\int_0^t (\nabla V_s(\varphi_s))^2_{\dot{C}_s}\, ds\right]$$

$$\varphi + \int_0^t \dot{C}_s \nabla V_s(\varphi_s)\, ds + \int_0^t \sqrt{\dot{C}_u}\, dW_u$$

and $M_s = V_s(\varphi_s) + \frac{1}{2}\int_s^t (\nabla V_u)^2_{\dot{C}_u}\, du$ is a martingale.

(Use: (P) and Ito's formula).

$$V_t(\varphi) \geq \inf_u \mathbb{E}\left[V_0\left(\varphi + \int_0^t \dot{C}_s u_s\, ds + \int_0^t \sqrt{\dot{C}_u}\, dW_u\right) + \frac{1}{2}\int_0^t (u_s)^2_{\dot{C}_s}\, ds\right]$$

Boué-Dupuis (or Borell) formula states $\geq$ is actually $=$.

Gives variational formula for $V_t(\varphi)$.

Barashkov-Gubinelli used this to est. $V_t(\varphi)$.

<u>Relation to Stochastic localisation</u> (Eldan)

Essentially $\tilde{\varphi}$ up to transformation is the same as stochastic localisation, the Föllmer process, ...

<u>Application to transportion of measure</u> (Shenfeld)