# Log-Sobolev inequality and renormalisation

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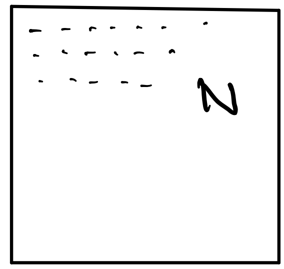
$$d\varphi_t = -\nabla H(\varphi_t) + \sqrt{2}\, dW_t \text{ on } \mathbb{R}^N$$




$\Lambda = \{1, \dots, N\}$

with invariant measure $e^{-H(\varphi)} d\varphi \propto \nu_\infty(d\varphi)$

$$= -A\varphi_t - V'(\varphi_t) + \sqrt{2}\, dW_t$$

$\approx$ Laplacian on $\Lambda \approx \mathbb{R}^d$ or $\mathbb{Z}^d$

Local non-linear potential

(continuous) Glauber dyn.

<u>Examples:</u> $H(\varphi) = \frac{1}{2}(\varphi, A\varphi) + V(\varphi)$

- Lattice models: $A = -\Delta$,

  $(\Delta f)(x) = \sum_{y \sim x} (f(y) - f(x))$

  $\Lambda \uparrow \mathbb{Z}^d$

  $V(\varphi) = \sum_{x \in \Lambda} (g\varphi^4 + \nu\varphi^2)$ $\quad (g > 0,\ \nu < 0)$

  $V(\varphi) = \sum_{x \in \Lambda} z \cos(\sqrt{\beta}\varphi)$ $\quad (z \in \mathbb{R},\ \beta > 0)$

- Continuum models ($\to$ SPDE): $\Omega_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^d$

  $A = -\Delta^\varepsilon$,

  $\Delta^\varepsilon f(x) = \varepsilon^{-2} \sum_{y \sim x} (f(y) - f(x))$

  $(f, g) = \varepsilon^d \sum_{x \in \Omega_\varepsilon} f_x g_x$

  $V(\varphi) = \varepsilon^d \sum_{x \in \Omega_\varepsilon} (g\varphi^4 + \nu_\varepsilon \varphi^2)$

  $V(\varphi) = \varepsilon^d \sum_{x \in \Omega_\varepsilon} (\varepsilon^{-\beta/4\pi} z \cos(\sqrt{\beta}\varphi))$

  <u>counterterms</u>

# Generalities on dynamics

$$d\varphi = -\nabla H(\varphi_t)\,dt + \sqrt{2}\,dW_t$$

Infinitesimal generator $\Delta_H = \Delta - (\nabla H, \nabla) = e^{+H}(\nabla, e^{-H}\nabla)$

$$\sum_{x\in\Lambda} \frac{\partial^2}{\partial\varphi(x)^2} \qquad \sum_{x\in\Lambda} \frac{\partial H}{\partial\varphi(x)}\frac{\partial}{\partial\varphi(x)}$$

- $\nu_\infty$ is reversible $\mathbb{E}_{\nu_\infty} F(-\Delta_H G) = \mathbb{E}_{\nu_\infty}(\nabla F, \nabla G)$.
- Itô $\Rightarrow F_t(\varphi) = \mathbb{E}_{\varphi_0=\varphi} F(\varphi_t)$ solves $\frac{\partial}{\partial t} F_t = \Delta_H F_t$

Law $\nu_t$ of $\varphi_t$ satisfies $\mathbb{E}_{\nu_t} F = \mathbb{E}_{\nu_0} F_t$.

- If $d\nu_t = F_t\, d\nu_\infty$ then $\frac{\partial}{\partial t} F_t = \Delta_H^* F_t = \Delta_H F_t$

  adjoint of $\Delta_H$ w.r.t. $\nu_\infty$ $\qquad$ self-adjoint

- Fokker–Planck equation $\frac{\partial \nu_t}{\partial t} = \Delta \nu_t + (\nabla, \nu_t \nabla H)$
$$= (\nabla, \nu_t \nabla(\log \nu_t + H)).$$

Ergodicity implies $\nu_t \longrightarrow \nu_\infty$. <u>How fast?</u>

Relative entropy: $H(\nu_t | \nu_\infty) = \underbrace{\mathbb{E}_{\nu_\infty} F \log F}_{\mathrm{Ent}_{\nu_\infty} F}$, $F_t = \frac{d\nu_t}{d\nu_\infty}$

Pinsker's inequality: $\| \nu_t - \nu_\infty \|_{TV}^2 \leq 2 H(\nu_t | \nu_\infty)$

Fisher information: $I(\nu_t | \nu_\infty) = \mathbb{E}_{\nu_\infty} \frac{(\nabla F)^2}{F} = 4 \mathbb{E}_{\nu_\infty} (\nabla \sqrt{F})^2$

de Bruin identity: $\frac{\partial}{\partial t} H(\nu_t | \nu_\infty) = - I(\nu_t | \nu_\infty) < 0$

Proof:

$$\begin{aligned}
\frac{\partial}{\partial t} \mathbb{E}_{\nu_\infty} \Phi(F_t) &= \mathbb{E}_{\nu_\infty} \Phi'(F_t) \dot{F}_t \\
&= \mathbb{E}_{\nu_\infty} \Phi'(F_t) \Delta_H F_t \\
&= - \mathbb{E}_{\nu_\infty} (\nabla \underbrace{\Phi'(F_t)}_{\log F_t + 1}, \nabla F_t) \\
&= - \mathbb{E}_{\nu_\infty} (\nabla \log F_t, \nabla F_t) \\
&= - I(\nu_t | \nu_\infty)
\end{aligned}$$

$\Phi(t) = t \log t$

$\Phi'(t) = \log t + 1$

Upshot: $H(\nu_t | \nu_\infty) \leq e^{-2\gamma t} H(\nu_0 | \nu_\infty)$

$\Leftrightarrow \frac{\partial}{\partial t} H(\nu_t | \nu_\infty) \leq -2\gamma H(\nu_t | \nu_\infty)$

$\Leftrightarrow I(\nu | \nu_\infty) \geq 2\gamma H(\nu | \nu_\infty)$

Log-Sobolev inequality: $\underbrace{\mathrm{Ent}_{\nu_\infty} F}_{H(\nu|\nu_\infty)} \leq \frac{2}{\gamma} \underbrace{\mathbb{E}_{\nu_\infty}(\nabla\sqrt{F})^2}_{\frac{1}{4}I(\nu|\nu_\infty)}$

(LSI)

Exercise. (LSI) $\Rightarrow$ Spectral gap: $\mathrm{Var}_{\nu_\infty}(F) \leq \frac{1}{\gamma}\mathbb{E}_{\nu_\infty}(\nabla F)^2$

Proof. $\Phi(1+\varepsilon t) = (1+\varepsilon t)\log(1+\varepsilon t) = 1 + \frac{1}{2}\varepsilon^2 t^2 + O(\varepsilon^3)$

$$\Rightarrow \mathrm{Ent}_{\nu_\infty}(1+\varepsilon F) = \frac{\varepsilon^2}{2}\mathrm{Var}(F) + o(\varepsilon^2)$$

$$\mathbb{E}_{\nu_\infty}\frac{(\nabla(1+\varepsilon F))^2}{1+\varepsilon F} = \varepsilon^2\,\mathbb{E}(\nabla F)^2 + o(\varepsilon^2)$$

Bakry–Émery Theorem.

Hess $H(\varphi) \geq \lambda$ id for all $\varphi$ $\Rightarrow$ Log-Sob. with $\gamma \geq \lambda$
$(\lambda > 0)$

Proof.

$$H(\nu_0|\nu_\infty) = \int_0^\infty I(\nu_t|\nu_\infty)\,dt$$ (de Bruin's identity and $H(\nu_t|\nu_\infty) \to 0$)

$$\leq \frac{1}{2\gamma} I(\nu_0|\nu_\infty)$$

if $I(\nu_t|\nu_\infty) \leq e^{-2\gamma t} I(\nu_0|\nu_\infty)$.

To see this, differentiate again:

$$\frac{\partial}{\partial t} I(\nu_t | \nu_\infty) = \frac{\partial}{\partial t} \mathbb{E}_{\nu_\infty} \frac{(\nabla F_t)^2}{F_t}$$

$$\underset{\substack{\text{elementary but} \\ \text{tedious exercise}}}{=} -2\, \mathbb{E}_{\nu_\infty}\Big( \underbrace{F_t \,|\mathrm{Hess} \log F_t|_2^2}_{>0} + F_t \big( \nabla \log F_t ,\, \underbrace{(\mathrm{Hess}\, H)}_{\geq \lambda} \nabla \log F_t \big) \Big)$$

$$\leq -2\lambda\, \mathbb{E}_{\nu_\infty} \frac{(\nabla F_t)^2}{F_t} = -2\lambda\, I(\nu_t | \nu_\infty)$$

Summary: $\nu_0 \xrightarrow{\quad H(\nu_t | \nu_\infty) \quad} \nu_\infty$

$t \xrightarrow[\quad I(\nu_t | \nu_\infty) \quad]{} \nu_\infty$

## Remarks and exercises.

- Tensorisation: if $\mu$ and $\nu$ satisfy LS inequalities with constant $\gamma$ then $\mu \otimes \nu$ satisfies the LS inequality with constant $\gamma$ on the product space.

- Holley–Stroock perturbation: if $\nu$ satisfies the LS inequality with constant $\gamma$ and $d\mu = F d\nu$, then $\mu$ satisfies the LS inequality with constant
$$\frac{\sup F}{\inf F}\,\gamma$$
In our context, the dependence of this on the dimension is usually exponentially bad.

- Variational formula for the entropy
$$H(\mu|\nu) = \sup\left\{ \mathbb{E}_\mu F - \log \mathbb{E}_\nu e^F \right\}.$$
Entropy inequality: for any $\alpha > 0$,
$$\mathbb{E}_\mu F \leq \frac{1}{\alpha} H(\mu|\nu) + \frac{1}{\alpha} \log \mathbb{E}_\nu e^{\alpha F}.$$

- Herbst argument: if $\nu$ has LS constant $\gamma$ and $F$ is 1-Lipschitz then
$$\mathbb{E}_\nu e^{\alpha F - \alpha \mathbb{E}_\nu F} \leq e^{\alpha^2/2\gamma}.$$

Example (Gaussian free field).

$H(\varphi) = \frac{1}{2}(\varphi, A\varphi)$, $\quad A = -\Delta^\varepsilon + m^2$ on $L\mathbb{T}^d \cap \varepsilon\mathbb{Z}^2$

Invariant measure is (discrete) Gaussian free field.

Dynamics diagonalized in Fourier basis:

$$d\hat{\varphi}(k) = -(|k|^2 + m^2)\hat{\varphi}(k)\,dt + \sqrt{2}\,d\hat{B}(k)$$

$\Rightarrow$ Different Fourier modes = length scales equilibrate at different rates

Small scales ($|k| \gg 1$) converge quickly.

Macroscopic modes ($|k| \approx \frac{1}{L}$) are slowest.

Difficulty: $H(\varphi) = \frac{1}{2}(\varphi, A^\varepsilon\varphi) + V^\varepsilon(\varphi)$

$\sum_{x\in\Lambda}' v^\varepsilon(\varphi_x)$ local, non-linear

$\to$ nonlocal in Fourier space

UV problem: choose $v^\varepsilon$ s.t. measure converges to non-Gaussian measure as $\varepsilon \to 0$.
($\sim$ GFF at small scales, non-G at large scales)

IR problem: choose $v^\varepsilon$ s.t. scaling limit is GFF
($\sim$ GFF at large scales, non-G at small scales)

<u>Exercise</u>: Consider the continuum models in d=1. No counterterms are required to get non-Gaussian limit!

$$\Omega_\varepsilon = [-L,L] \cap \varepsilon\mathbb{Z} \qquad V(\varphi) = \varepsilon \sum_{x\in\Omega_\varepsilon} (g\varphi^4 + \nu\varphi^2)$$

$$A = -\Delta^\varepsilon + m^2 \qquad V(\varphi) = \varepsilon \sum_{x\in\Omega_\varepsilon} z\cos(\sqrt{\beta}\varphi)$$

finite (d=1)

Using Bakry-Emery + Holley-Stroock, show that whenever $L<\infty$ the Log-Sobolev inequality holds uniformly in $\varepsilon$ (with constants depending on $L$)!

<u>Exercise</u>: Consider the lattice models in d=1 (use Dirichlet boundary conditions on the left and free boundary conditions on the right for simplicity). Show that then the Log-Sobolev constant is uniform in L. (Use induction in size of the interval.)

# Gaussian integration

Let $C_s = \int_0^s \dot{C}_{s'}\, ds'$, $\dot{C}_{s'}$ pos.-def. matrix on $\mathbb{R}^N$.

$P_{C_s}$ be the Gaussian measure with cov. $C_s$.

<u>Example.</u> $\dot{C}_s = e^{-sA} \Rightarrow C_\infty = A^{-1}$

$A = -\Delta^\varepsilon + m^2 \Rightarrow P_{C_\infty}$ is GFF

Careful: two very different Laplacians!

<u>Defn.</u> $\Delta_{\dot{C}_s} = \sum_{x,y} \dot{C}_s(x,y) \frac{\partial}{\partial \varphi_x} \frac{\partial}{\partial \varphi_y} = (\nabla, \dot{C}_s \nabla)$

<u>Rk.</u> Let $g_s^{-1} = \dot{C}_s$. Then $\Delta_{\dot{C}_s} = \Delta_{g_s}$ is the Laplace-Beltrami operator on $\mathbb{R}^N$ associated to the metric $g_s$.

$$|f|_{g_s} \leq 1 \Leftrightarrow |e^{+sA/2} f|_2 \leq 1$$

heat kernel if $A = -\Delta + 1$

$$\Leftrightarrow f = e^{-sA/2} f_0, \quad |f_0|_2 \leq 1$$

The unit ball consists of functions smooth at scale $\sqrt{s}$.

<u>Prop.</u> $\frac{\partial}{\partial s} P_{C_s} = \frac{1}{2} \Delta_{\dot{C}_s} P_{C_s}$

<u>Proof.</u> $$\frac{\partial}{\partial s} \frac{e^{-\frac{1}{2}(\varphi, C_s^{-1}\varphi)}}{Z_{C_s}} = \frac{1}{2} (C_s^{-1}\varphi)^2_{\dot{C}_s} P_{C_s}(\varphi) - (\text{const.}) P_{C_s}(\varphi)$$

$$\tfrac{1}{2}\Delta_{\dot{C}_s} P_{C_s}(\varphi) = \tfrac{1}{2}(C_s^{-1}\varphi)^2_{\dot{C}_s} P_{C_s}(\varphi) - (\text{const.}) P_{C_s}(\varphi)$$

$$\Rightarrow \left(\frac{\partial}{\partial s} - \tfrac{1}{2}\Delta_{\dot{C}_s}\right) P_{C_s}(\varphi) = (\text{const.}) P_{C_s}(\varphi)$$

Since the integral of the LHS is 0, the constant on the RHS is 0.

<u>Cor.</u> Let $F_s = P_{C_s} * F_0$, i.e. $F_s(\varphi) = E_{C_s}(F_0(\varphi+\zeta))$. Then

$$\partial_s F_s = \tfrac{1}{2}\Delta_{\dot{C}_s} F_s$$

$$F_s(0) = E_{C_s} F_0 .$$

<u>Exercise.</u> Let $P$ be a polynomial in $\varphi$. Define

$$:P:_{C_s} = e^{-\frac{1}{2}\Delta_{C_s}} P \qquad \text{(Wick ordering).}$$

inverse heat operator

Then $e^{\frac{1}{2}\Delta_C} :P:_{C_s} = P$ and in particular

$$E_{C_s} :P:_{C_s} = P(0)$$

Also, if $P$ and $Q$ are homogeneous with $\deg P \neq \deg Q$,

$$E_{C_s}\left(:P:_{C_s} :Q:_{C_s}\right) = 0.$$

# Renormalised potential

The renormalised potential $V_s$ is defined by

$$e^{-V_s(\varphi)} = (P_{C_s} * e^{-V_0})(\varphi) = \mathbb{E}_{C_s}\left(e^{-V_0(\varphi+\zeta)}\right)$$

$$\Leftrightarrow \quad \frac{\partial}{\partial s} e^{-V_s(\varphi)} = \frac{1}{2}\Delta_{\dot{C}_s} e^{-V_s(\varphi)} \qquad (u,v)_{\dot{C}_s} = \sum \dot{C}_s(x,y)u(x)v(y)$$

$$\Leftrightarrow \quad \frac{\partial}{\partial s} V_s = \underbrace{\frac{1}{2}\Delta_{\dot{C}_s} V_s}_{\text{viscosity}} - \underbrace{\frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}}_{\text{Hamilton-Jacobi}} \qquad \text{Polchinski equation}$$

Prop. Suppose that

$$\frac{\partial}{\partial s} V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$$

$$\frac{\partial}{\partial s} F_s = \frac{1}{2}\Delta_{\dot{C}_s} F_s - (\nabla V_s, \nabla F_s)_{\dot{C}_s} = L_s F_s$$

Then the following integral is independent of $s$:

$$\int \underbrace{P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}}_{\nu^s(\varphi)} F_s(\varphi)\, d\varphi$$

*Proof.* Note that $Z_s(\varphi) = e^{-V_s(\varphi)} F_s(\varphi)$ satisfies

$$\frac{\partial}{\partial s} Z_s = \frac{1}{2} \Delta_{\dot{C}_s} Z_s .$$

$$\Rightarrow \frac{\partial}{\partial s} \int P_{C_\infty - C_s}(\varphi) Z_s(\varphi)\, d\varphi$$

$$= \int \left[ \left( -\frac{1}{2} \Delta_{\dot{C}_s} P_{C_\infty - C_s} \right) Z_s + P_{C_\infty - C_s} \left( \frac{1}{2} \Delta_{\dot{C}_s} Z_s \right) \right] d\varphi = 0.$$

Renormalised measure:

$$\nu^s(d\varphi) = P_{C_\infty - C_s}(\varphi) e^{-V_s(\varphi)} d\varphi .$$

Polchinski semigroup:

$$P_{s',s} F(\varphi) = e^{+V_s(\varphi)} E_{C_s - C_{s'}} \left( e^{-V_{s'}(\varphi + \zeta)} F(\varphi + \zeta) \right).$$

$$\Rightarrow E_{\nu^0} F = \mathbb{E}_{\nu^s} P_{0,s} F$$

Exercise:
$$\frac{\partial}{\partial s} \mathbb{E}_{\nu^s} F = -\mathbb{E}_{\nu^s} L_s F$$

$$\frac{\partial}{\partial s} P_{s',s} F = L_s P_{s',s} F$$

$$\frac{\partial}{\partial s'} P_{s',s} F = -P_{s',s} L_{s'} F$$

<u>Summary</u>: two ways to evolve measure

$\nu_t = F_t \, d\nu_\infty$ with $\partial_t F_t = \Delta F_t - (\nabla H, \nabla F_t)$

$= \Delta_H F_t$

Glauber semigroup

Tends to invariant measure $\nu_\infty$.

$\nu^s = F^s \, d\nu^s$ with $\partial_s F^s = \frac{1}{2}\Delta_{\dot{C}_s} F^s - (\nabla V_s, \nabla F^s)_{\dot{C}_s}$

$= L_s F^s$

reference measure changes

Polchinski semigroup (time-dependent)

Tends to $\nu^\infty = \delta_0$.

**Example.** $\mathrm{Hess}\, V_0 \geq 0 \Rightarrow \mathrm{Hess}\, V_s \geq 0 \quad \forall s > 0.$

**Proof 1.**

$$e^{-V_s(\varphi)} \propto \int \underbrace{e^{-\frac{1}{2}(\zeta, C_s^{-1}\zeta) - V_0(\varphi+\zeta)}}_{\text{log-concave in } (\varphi,\zeta)} d\zeta$$

Brascamp-Lieb inequality: marginals of log-concave measures are log-concave

**Proof 2.** $\partial_s V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$

$$\Rightarrow \partial_s \nabla V_s = \frac{1}{2}\Delta_{\dot{C}_s} \nabla V_s - (\mathrm{Hess}\, V_s, \nabla V_s)_{\dot{C}_s}$$

$$\begin{aligned}\Rightarrow \partial_s \mathrm{Hess}\, V_s &= \frac{1}{2}\Delta_{\dot{C}_s} \mathrm{Hess}\, V_s \\ &\quad - (\nabla \mathrm{Hess}\, V_s, \nabla V_s)_{\dot{C}_s} \\ &\quad - (\mathrm{Hess}\, V_s, \mathrm{Hess}\, V_s)_{\dot{C}_s} \\ &= L_s \mathrm{Hess}\, V_s - \mathrm{Hess}\, V_s \dot{C}_s \mathrm{Hess}\, V_s\end{aligned}$$

Suppose $N=1$. Then this is simply

$$\partial_s f_s = L_s f_s - f_s^2, \qquad f_s = V_s''$$

Maximum principle: $f \geq 0 \Rightarrow L_s f \geq 0$

$$\Rightarrow \partial_s f_s \geq -f_s^2, \; f_0 \geq 0 \Rightarrow f_s \geq 0.$$

General $N$: Maximum principle for symmetric tensors.

Thm (BB '19). Assume $A \geq \lambda\,\mathrm{id}$ $(\lambda > 0)$ and

$$Q_s \,\mathrm{Hess}\, V_s \, Q_s \geq \dot{\mu}_s \,\mathrm{id} \qquad (Q_s = e^{-sA/2})$$

$$\Rightarrow \text{Log-Sob. with } \frac{1}{\gamma} = \int_0^\infty e^{-\lambda s - 2\mu_s}\, ds$$

can be negative!

$$\mu_s = \int_0^s \dot{\mu}_{s'}\, ds'$$

Rk. In terms of the metric $g_s$ the condition is

$$\mathrm{Hess}_{g_s} V_s \geq \dot{\mu}_s\, g_s .$$

Rk. If $V_0$ is convex this recovers Bakry–Émery.

$$A = \lambda\,\mathrm{id}, \quad V_0 = H - \frac{\lambda}{2}(\varphi, \varphi)$$

$$\dot{\mu}_s \geq 0 \;\forall s \;\Rightarrow\; \frac{1}{\gamma} \leq \int_0^\infty e^{-\lambda s} = \frac{1}{\lambda}.$$

Exercises.

- Let $A = \lambda\,\mathrm{id}$ and $V = V^c + V^b$ with

  $$\mathrm{Hess}\, V^c \geq 0 \quad \text{and} \quad \|V^b\|_{C^2} < \infty.$$

  Show that there is $C$ (depending on dimension and the potential) such that

  $$\mathrm{Hess}\, V_s \geq -C \quad \text{for all } s \geq 0.$$

- Prove that always $\mathrm{Hess}\, V_s \leq C_s^{-1}$.

<u>Proof idea</u>: Like Bakry-Émery but use Polchinski semigroup instead of Glauber semigroup.

$$\frac{\partial}{\partial s} \mathbb{E}_{\nu_s} \Phi(F^s) = \mathbb{E}_{\nu_s}\left(-L_s \Phi(F^s) + \Phi'(F_s)\dot{F}^s\right)$$

$$= \mathbb{E}_{\nu_s}\Big(-\Phi'(F^s) L_s F^s - \tfrac{1}{2}\Phi''(F_s)(\nabla F^s)^2_{\dot{C}_s}$$
$$+ \Phi'(F_s)\dot{F}^s\Big)$$

unlike Bakry-Émery measure changes!

$$= -\tfrac{1}{2}\mathbb{E}_{\nu_s}\Big(\underbrace{\Phi''(F^s)}_{=\frac{1}{F_s}}(\nabla F^s)^2_{\dot{C}_s}\Big)$$

'Fisher information' at scale s

$$= -2\,\mathbb{E}_{\nu_s}(\nabla\sqrt{F^s})^2_{\dot{C}_s}$$

Now consider change of $\mathbb{E}_{\nu_s}(\nabla\sqrt{F^s})^2_{\dot{C}_s}$.

$$\left(\frac{\partial}{\partial s} - L_s\right)(\nabla\sqrt{F^s})^2_{\dot{C}_s} = +(\nabla\sqrt{F^s})^2_{\ddot{C}_s}$$

$\ddot{C}_s = -A\dot{C}_s$

$$-2(\nabla\sqrt{F^s}, \dot{C}_s \operatorname{Hess} V_s \dot{C}_s \nabla\sqrt{F^s})$$
$$\underbrace{-\tfrac{1}{4}F^s\left|\dot{C}_s^{1/2}(\operatorname{Hess}\log F^s)\dot{C}_s^{1/2}\right|_2^2}_{\geq 0}$$

$Q_s = \dot{C}_s^{1/2}$

$$\leq -(\nabla\sqrt{F^s}, Q_s(A + 2Q_s \operatorname{Hess} V_s Q_s)Q_s \nabla\sqrt{F^s})$$

$$\Rightarrow \left(\frac{\partial}{\partial s} - L_s\right) (\nabla \sqrt{F^s})^2_{\dot{C}_s} \leq -(\lambda + 2\dot{\mu}_s)\, \mathbb{E}(\nabla \sqrt{F^s})^2_{\dot{C}_s}$$

$\Rightarrow \psi(s) = \mathbb{E}_{\nu_s}(\nabla \sqrt{F^s})^2_{\dot{C}_s}$ satisfies

$$\dot{\psi}(s) \leq -(\lambda + 2\dot{\mu}_s)\, \psi(s)$$
$$\psi(s) \leq e^{-\lambda s - 2\mu_s}\, \psi(0) \leq e^{-\lambda s - 2\mu_s}\, \mathbb{E}(\nabla \sqrt{F^s})^2$$

$$\Rightarrow \mathrm{Ent}_{\nu_0} F \leq 2 \underbrace{\left(\int_0^\infty e^{-\lambda s - 2\mu_s}\, ds\right)}_{\frac{1}{\gamma}} \mathbb{E}_{\nu_0}(\nabla \sqrt{F})^2$$

<u>Rk.</u> • The same proof show that the renormalised measures $\nu^s$ (large scales) satisfy LS inequalities with similar constants.

• Moreover, the 'fluctuation measures' defined by

$$\mathbb{E}_{\mu^{\varphi}_{0,s}} F = P_{0,s} F(\varphi)$$

satisfy LS inequalities with const. $\gamma_s \gg \gamma$:

$$\frac{1}{\gamma_s} = \int_0^s (\dots)\,.$$

Thus the small scales equilibrate quickly.

Summary of argument:

$$\frac{\partial}{\partial t} H(\nu_t | \nu_\infty) = - I_0(\nu_t | \nu_\infty) \qquad \text{Glauber}$$

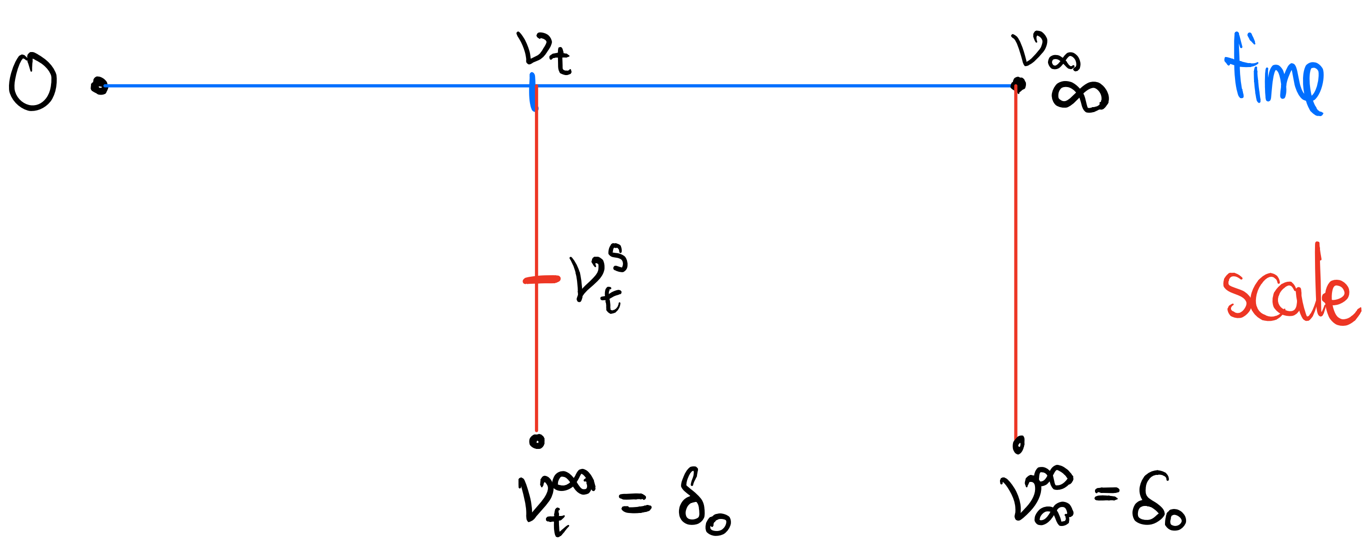
$$\frac{\partial}{\partial s} H(\nu_t^s | \nu_\infty^s) = - I_s(\nu_t^s | \nu_\infty^s) \qquad \text{Polchinski}$$

$$\frac{\partial}{\partial s} I_s(\nu_t^s | \nu_\infty^s) \leq -(\lambda + 2\dot{\mu}_s) I_s(\nu_t^s | \nu_\infty^s)$$

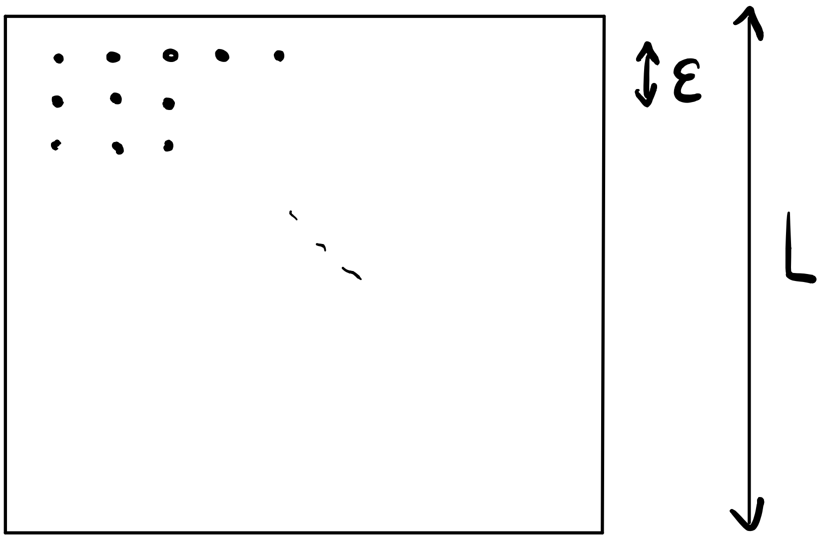
$$\Rightarrow I_s(\nu_t^s | \nu_\infty^s) \leq e^{-\lambda s - 2\mu_s} I_0(\nu_t^0 | \nu_\infty^0)$$

$$\Rightarrow H(\nu_t | \nu_\infty) \leq \frac{1}{2} \Big( \underbrace{\int_0^\infty e^{-\lambda s - 2\mu_s}}_{\frac{1}{\gamma}} \Big) I_0(\nu_t | \nu_\infty)$$

$$\Rightarrow \frac{\partial}{\partial t} H(\nu_t | \nu_\infty) \leq -2\gamma H(\nu_t | \nu_\infty)$$



# Continuum sine-Gordon model




$\Omega_\varepsilon = \Omega \cap \varepsilon \mathbb{Z}^2$

$\Omega = L\,\mathbb{T}^2$

very non-convex!

$$\nu(d\varphi) \propto \exp\Big[-\varepsilon^2 \underbrace{\sum_{x\in\Omega_\varepsilon}\Big(\tfrac{1}{2}\varphi(-\Delta^\varepsilon\varphi) + \tfrac{m^2}{2}\varphi^2 + 2z\,\varepsilon^{-\frac{\beta}{4\pi}}\cos(\sqrt{\beta}\,\varphi)\Big)}_{\frac{1}{2}(\varphi, A^\varepsilon\varphi)_\varepsilon + V^\varepsilon(\varphi)}\Big]\prod_{x\in\Omega_\varepsilon} d\varphi_x$$

$(a,b)_\varepsilon = \varepsilon^2 \sum_{\Omega_\varepsilon} a_x b_x$

In various works (under varying assumptions on $\beta$, $m^2$ and $z$), it is shown that

For $\beta < 8\pi$ the limits $\varepsilon \to 0$, $(L\to\infty)$ exist and define a non-Gaussian measure + properties

[Fröhlich, F-Seiler, Benfatto-Gallavotti-Nicolo, N-Renn-Steinmann, Brydges-Kennedy, Dimock-Hurd, Lacoin-Rhodes-Vargas, ...]

For $\beta \geq 8\pi$ all contiuum limits expected to be Gaussian.

Comparing with the $\varphi^4$ model, $\beta$ plays the role of dimension in the scaling with the rough correspondence

$$\beta < 4\pi \ (SG) \longleftrightarrow d=2 \ (\varphi^4)$$
$$\beta < 6\pi \ (SG) \longleftrightarrow d=3 \ (\varphi^4)$$
$$\beta = 8\pi \ (SG) \longleftrightarrow d=4 \ (\varphi^4)$$

In fact, there is an infinite sequence of thresholds that one encounters in the construction occuring at

$$\beta_n = 8\pi\left(1 - \tfrac{1}{2n}\right).$$

The physical meaning of these thresholds remains debated, but the consensus appears to be that physically nothing happens at these thresholds.

There are many conjectures describing very precise behavior, mostly concerning the $m^2 \downarrow 0$ limit in infinite volume.

## Relation to Yukawa gas (sine-Gordon transformation).

Let $\varphi$ be a Gaussian field with cov. $C = (-\Delta^\varepsilon + m^2)^{-1}$.

Yukawa potential

Coulomb if $m^2 = 0$

$$\Rightarrow \mathbb{E}\left(e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi_{x_i}\sigma_i}\right) = e^{-\frac{\beta}{2}\sum_{i,j} C(x_i,x_j)\sigma_i\sigma_j}$$

$$= e^{-\frac{\beta}{2}\sum_{i,j}\sigma_i(-\Delta^\varepsilon + m^2)^{-1}(x_i,x_j)\sigma_j}$$

$$\Rightarrow \mathbb{E}\left(e^{\frac{\beta}{2}C(0,0)n}\, e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi_{x_i}\sigma_i}\right) = e^{-\frac{\beta}{2}\sum_{i\neq j}\sigma_i(-\Delta^\varepsilon + m^2)^{-1}(x_i,x_j)\sigma_j}$$

For the 2D Yukawa potential:

$$(-\Delta^\varepsilon + m)^{-1}(0,0) = \frac{1}{2\pi}\log\varepsilon^{-1} + c(m) + o(1)$$

$$e^{\frac{\beta}{2}C(0,0)} \approx \text{const. } \varepsilon^{-\beta/4\pi}$$

Partition function of 2D Yukawa gas:

$$Z^{CG} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\varepsilon^{2n}\sum_{\substack{x_1,\dots,x_n\in\Omega_\varepsilon\\ \sigma_1,\dots,\sigma_n\in\{\pm1\}}} e^{-\frac{\beta}{2}\sum_{i\neq j}\sigma_i\sigma_j(-\Delta^\varepsilon+m^2)^{-1}(x_i,x_j)}$$

$\xi_i = (x_i,\sigma_i)$

$$= \mathbb{E}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}e^{\frac{\beta}{2}C(0,0)n}\underbrace{\varepsilon^{2n}\sum_{\xi_1,\dots,\xi_n} e^{i\sqrt{\beta}\sum_{j=1}^{n}\varphi(x_i)\sigma_i}}_{\left(\varepsilon^2\sum_{x,\sigma}e^{i\sqrt{\beta}\varphi(x)\sigma}\right)^n = \left(\varepsilon^2\sum_x 2\cos(\sqrt{\beta}\varphi(x))\right)^n}\right)$$

$$= \mathbb{E}\left(\exp\left(\varepsilon^2\sum_x 2z\underbrace{e^{\frac{\beta}{2}C(0,0)}}_{\sim\,\text{const. }\varepsilon^{-\beta/4\pi}}\cos(\sqrt{\beta}\varphi(x))\right)\right) = Z^{SG}$$

<u>Glauber dynamics</u> (Dynamical sine-Gordon model)

$$d\varphi = \Delta^\varepsilon \varphi - m^2 \varphi - 2z\, \varepsilon^{-\beta/4\pi} \sqrt{\beta}\, \sin(\sqrt{\beta}\varphi) + dW^\varepsilon$$

$W^\varepsilon$ standard Brownian motion with inner product

$$\langle f, g \rangle = \varepsilon^d \sum_{x \in \Omega_\varepsilon} f(x) g(x).$$

(space-time white noise)

i.e. $W^\varepsilon(x)$, $x \in \Omega_\varepsilon$ are independent BM with $\langle W^\varepsilon(x) \rangle_t = \varepsilon^{-d} t$

Chandra-Hairer-Shen: limit well-posed for all $\beta < 8\pi$

as $\varepsilon \to 0$ solutions exist for short time

We'll keep $\varepsilon > 0$ but derive estimates uniform in $\varepsilon$.

Dirichlet form: $\quad D^\varepsilon(F) = \frac{1}{\varepsilon^d} \sum_{x \in \Omega_\varepsilon} \left( \frac{\partial F}{\partial \varphi_x} \right)^2$

Thm. (BB '19). Let $0<\beta<6\pi$, $z\in\mathbb{R}$, $m^2>0$, $L>0$. Then there is $\gamma=\gamma(\beta,z,m,L)>0$ indep. of $\varepsilon$ s.t.

$$\mathrm{Ent}_{\nu^\varepsilon} F \le \frac{2}{\gamma} D^\varepsilon(\sqrt{F})$$

Moreover, if $m^{-2+\beta/4\pi}|z|\le\delta_\beta$ then

$$\gamma \ge m^2 + O_\beta(m^{\beta/4\pi}|z|).$$

independent of L

Rk. Analogous result holds for conservative (Kawasaki) dynamics. Dirichlet form:

$$D_0^\varepsilon(F) = \frac{1}{\varepsilon^{d+2}} \sum_{x\sim y\in\Omega_\varepsilon} \left(\frac{\partial F}{\partial\varphi_x}-\frac{\partial F}{\partial\varphi_y}\right)^2$$

Measure: sine-Gordon measure cond. on $\sum\varphi_x=0$.

Again there is LS ineq. in fin. vol. and $m^{-2+\beta/4\pi}|z|\le\delta_\beta$ implies

$$\gamma \ge \frac{c}{L^2}(m^2+\cdots).$$

# Normalisation

$$V_0(\varphi) = \varepsilon^2 \sum_{x\in\Omega_\varepsilon} 2\underbrace{z\,\varepsilon^{-\beta/4\pi}}_{\text{macroscopic normal.}} \cos(\sqrt{\beta}\varphi_x)$$

$$= \sum_{x\in\Omega_\varepsilon} 2\underbrace{z\,\varepsilon^{2-\beta/4\pi}}_{z_0 \text{ microscopic normal.}} \cos(\sqrt{\beta}\varphi_x)$$

$A = -\Delta^\varepsilon + m^2$ with respect to $(u,v)_\varepsilon = \varepsilon^2 \sum_{x\in\Omega_\varepsilon} u_x v_x$

macroscopic normalisation

$\equiv -\Delta + \varepsilon^2 m^2$ with respect to $(u,v) = \sum_{x\in\Omega_\varepsilon} u_x v_x$

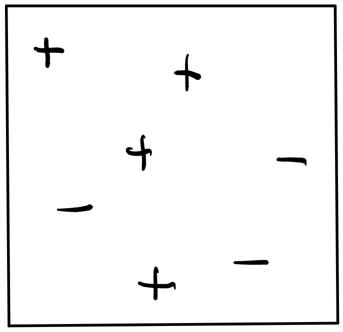
microscopic normalisation

Macroscopic normalisation: SPDE as $\varepsilon \to 0$

Microscopic normalisation: Lattice model with weak interaction

We'll use the lattice model point of view.

# Yukawa gas representation of the renormalised potential (Brydges & Kennedy)



We will write

$\tilde V(\xi_1,\dots,\xi_n) = \tilde V^n(\xi_1,\dots,\xi_n)$

$$V_t(\varphi) = \sum_{n=0}^{\infty} V_t^n(\varphi),$$

charges $\xi_i = (x_i,\sigma_i)$

$$V_t^n(\varphi) = \frac{1}{n!} \sum_{\substack{x_1\cdots x_n\in\Omega_\varepsilon \\ \sigma_1\cdots\sigma_n\in\{\pm1\}}} \tilde V_t(\xi_1,\dots,\xi_n)\, e^{i\sqrt{\beta}\sum \varphi_{x_i}\sigma_i} \qquad (*)$$

$\xi_i = (x_i,\sigma_i)$

factor $\varepsilon^{2n}$ considered part of $\tilde V$

Initial condition: $\tilde V_0(\xi_1) = z_0 = \varepsilon^{2-\beta/4\pi} z$

$$\tilde V_0(\xi_1,\dots,\xi_n) = 0$$

Polchinski equation: $\partial_t V_t = \frac{1}{2}\Delta_{\dot C_t} V_t - \frac{1}{2}(\nabla V_t)^2_{\dot C_t}$

$$\frac{1}{2}\Delta_{\dot C_t} V_t = \frac{1}{n!}\sum \widetilde{\tfrac{1}{2}\Delta_{\dot C_t} V_t}\,(\xi_1,\dots,\xi_n)\, e^{i\sqrt{\beta}\sum \varphi_{x_i}\sigma_i}$$

where $\widetilde{\tfrac{1}{2}\Delta_{\dot C_t} V_t} = \underbrace{-\beta\frac{1}{2}\sum_{j,k=1}^{n} \sigma_j\sigma_k \dot C_t(x_j,x_k)\, \tilde V_t(\xi_1,\dots,\xi_n)}_{=:\dot W_t(\xi_1,\dots,\xi_n)}$

$$\widetilde{(\nabla V_t)^2_{\dot C_t}} = \sum_{\substack{I_1\cup I_2=[n] \\ j\in I_1,\, k\in I_2}} \underbrace{(-\beta \dot C_t(x_j,x_k)\sigma_j\sigma_k)}_{\dot U_t(\xi_j,\xi_k)}\, \tilde V_t(\xi_{I_1})\, \tilde V_t(\xi_{I_2})$$

$(\xi_i)_{i\in I_1}$

Polchinski equation in 'Fourier' space:

$$\partial_t \tilde{V}_t(\xi_1,\dots,\xi_n) = -\dot{W}_t(\xi_1,\dots,\xi_n)\tilde{V}_t(\xi_1,\dots,\xi_n) - \frac{1}{2}\widetilde{(\nabla V_t)^2_{\dot{C}_t}}(\xi_1,\dots,\xi_n)$$

Duhamel formula:

$$\tilde{V}_t(\xi_1,\dots,\xi_n) = e^{-W_t(\xi_1,\dots,\xi_n)}\,\tilde{V}_0(\xi_1,\dots,\xi_n) + \frac{1}{2}\int_0^t e^{-(W_t-W_s)(\xi_1,\dots,\xi_n)} \sum_{\substack{I_1\dot\cup I_2\\ j\in I_1,\,k\in I_2}} \dot{u}_s(\xi_j,\xi_k)\,\tilde{V}_s(\xi_{I_1})\,\tilde{V}_s(\xi_{I_2})\,ds$$

depend on $\leq n-1$ particles since $I_1 \neq \emptyset$, $I_2 \neq \emptyset$

$$n=1:\quad \tilde{V}_t(\xi_1) = e^{-W_t(\xi_1)}\,\tilde{V}_0(\xi_1) = \underbrace{e^{-\frac{\beta}{2}C_t(0,0)}\,z_0}_{=: z_t \approx \varepsilon^2(\varepsilon \ell_t)^{-\beta/4\pi}}$$

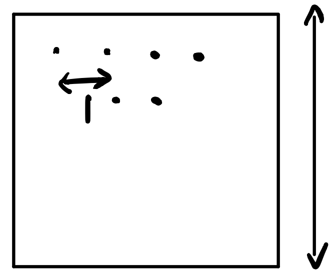
$$n\geq 2:\quad \tilde{V}_t(\xi_1,\dots,\xi_n) = \frac{1}{2}\int_0^t e^{-(W_t-W_s)(\xi_1,\dots,\xi_n)}\sum(\cdots)$$

By induction, $\tilde{V}_t(\xi_1,\dots,\xi_n)$ is defined for all $n$, $t$.

Brydges-Kennedy: $\tilde{V}_t$ are the Ursell functions of a (regularised) Yukawa gas.

Fact. If the series (*) converges, it gives the unique solution to the Polchinski eqn.

We'll show this for $\beta < 4\pi$. (Paper: $\beta < 6\pi$.)




## Preliminaries

Heat kernel $\dot{C}_t(x,y) = e^{t\Delta}(x,y)\, e^{-\varepsilon^2 m^2 t}$

unit lattice Laplacian

$\approx e^{-c|x-y|^2/t}$

Define length scale $\ell_t = (1 \vee \sqrt{t}) \wedge \frac{1}{m^2\varepsilon^2}$

Exercise: $C_t(x,x) = \frac{1}{2\pi}\log \ell_t + O(1)$ ← $\int_1^t \frac{ds}{4\pi s} = \frac{1}{2\pi}\log\sqrt{t}$

$\max_x \sum_y \dot{C}_t(x,y) = O(e^{-m^2 t})$

$$\Rightarrow \tilde{V}_t(\xi_1) = e^{-\frac{\beta}{2}C_t(0,0)} z_0 = \varepsilon^2(\varepsilon \ell_t)^{-\beta/4\pi} z$$

singularity disappears as $\ell_t \to \frac{1}{\varepsilon}$ (macroscopic scale)

Good normalisation:

$$\ell_t^2 \tilde{V}_t(\xi_1) = (\varepsilon \ell_t)^{2-\beta/4\pi} z =: z_t$$

(microscopic) renormalised coupling constant

**Thm.** Let $\beta < 4\pi$. Then for $n \geq 2$,

$$\underbrace{\ell_t^2 \sup_{\xi_1} \sum_{\xi_2 \cdots \xi_n} |\tilde{V}_t(\xi_1, \dots, \xi_n)|}_{\|\tilde{V}_t^n\|} \leq n^{n-2} C_\beta^{n-1} |z_t|^n$$

$$z_t = (\varepsilon \ell_t)^{2-\beta/4\pi} z$$

**Proof.** Assume bound holds for $k < n$. Then

$$|\tilde{V}_t(\xi_1, \dots, \xi_n)| \leq \frac{1}{2} \int_0^t \sum_{\substack{I_1 \dot\cup I_2 \\ j \in I_1, k \in I_2}} |\dot{u}_s(\xi_j, \xi_k)|\, |\tilde{V}_s(\xi_{I_1})|\, |\tilde{V}_s(\xi_{I_2})|\, ds$$

exp. factor dropped
— sufficient for $\beta < 4\pi$

$$\Rightarrow \underbrace{\sup_{\xi_1} \sum_{\xi_2 \cdots \xi_n} |\tilde{V}_t(\xi_1, \dots, \xi_n)|}_{\|\tilde{V}^{(n)}\|} \leq \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} \int_0^t \underbrace{\|\dot{u}_s\|}_{\leq e^{-m^2 s}} \underbrace{\|\tilde{V}_s^{(k)}\|}_{\substack{\leq k^{k-2} C^{k-1} |z_s|^k \ell_s^{-2} \\ \text{induct. assumpt.}}} \|\tilde{V}_s^{(n-k)}\| ds$$

$$\leq \frac{1}{2} \underbrace{\left( \sum_{k=1}^{n-1} \binom{n}{k} k^{k-2} (n-k)^{n-k-2} \right)}_{\substack{\leq \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1} (n-k)^{n-k-1} \\ = 2(n-1) n^{n-2}}} C^{n-2} \underbrace{\int_0^t |z_s|^n \ell_s^{-4} \, ds}_{\substack{\lesssim \frac{1}{n-1} |z_t|^n \ell_t^{-2} \\ \text{if } \beta < 4\pi,\ n \geq 2}}$$

$$\int_0^t \left(\ell_s^{-\beta/4\pi}\right)^n \ell_s^{2n-4} \, ds \underset{\ell_s \approx \sqrt{s}}{\approx} \int_0^t s^{n(1-\beta/8\pi)-2} \, ds \underset{n \geq 2,\ \beta < 4\pi}{\lesssim} \frac{1}{n} t^{n(1-\beta/8\pi)}$$

$$\approx \frac{1}{n} \ell_t^{2-\beta/4\pi}$$

<u>Cor.</u> Let $\beta < 4\pi$ and assume $C_\beta |z_t| < \frac{1}{e}$. Then

$$V_t(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\xi_1 \cdots \xi_n} \tilde{V}_t(\xi_1, \dots, \xi_n) e^{i\sqrt{\beta} \sum_{i=1}^{n} \varphi_{x_i} \sigma_i}$$

converges and thus gives the soln. to Polchinski equation. Moreover,

$$\ell_t^2 |\nabla V_t(\varphi, x)| \lesssim |z_t|$$

$$\ell_t^2 \sum_y |\mathrm{Hess}\, V_t(\varphi, x, y)| \lesssim |z_t|$$

<u>Proof.</u>

$$\frac{n^n}{n!} \leq e^n$$

$$\frac{\ell_t^2 |V_t(\varphi)|}{|\Omega_t|} \leq \sum_{n=0}^{\infty} \frac{1}{n!} n^{n-2} (C_\beta |z_t|)^n$$

$$\leq \sum_{n=0}^{\infty} (e\, C_\beta |z_t|)^n \lesssim |z_t|$$

$$\ell_t^2 |\nabla V_t(\varphi, x)| \leq \sum_{n=0}^{\infty} \frac{1}{n!} n^{n-1} (C_\beta |z_t|)^n \lesssim |z_t|$$

$$\ell_t^2 \sum_y |\mathrm{Hess}(\varphi, x, y)| \leq \cdots$$

Note that $|z_t| = (\varepsilon \ell_t)^{2-\beta/4\pi}|z| \underset{\uparrow \, \ell_t \le \frac{1}{m\varepsilon}}{\le} m^{-2+\beta/4\pi}|z|$

so this provides the required estimates on Hess $V_t$ for all $t$ when $|z|\, m^{-2+\beta/4\pi}$ is small enough.

<u>Cor.</u> For $\beta < 4\pi$, $|z|\, m^{-2+\beta/4\pi}$ small, the Log-Sobolev constant satisfies

$$\gamma \ge m^2 + \underset{\uparrow \text{ indep. of } L}{O}(|z|\, m^{\beta/4\pi}).$$

<u>Proof.</u>

$$Q_t \operatorname{Hess} V_t(\varphi) \underbrace{Q_t}_{\le e^{-m^2 t/2}} \ge \dot\mu_t \,\mathrm{id} \quad \text{with } \dot\mu_t = \frac{|z_t|}{\ell_t^2} e^{-m^2 t}$$

$$\int_0^t \dot\mu_s \, ds = \int_0^t |z_s| e^{-m^2 s} \frac{ds}{\ell_s^2} \lesssim |z_t|$$

Since $A = -\Delta + \varepsilon^2 m^2 \ge \varepsilon^2 m^2$ and taking into account that the Dirichlet form is

$$D^\varepsilon(F) = \frac{1}{\varepsilon^2} \sum_x (\dots),$$

we find

$$\frac{1}{\gamma} = \varepsilon^2 \int_0^\infty e^{-\varepsilon^2 m^2 t + O(|z_t|)} = \frac{1}{m^2}\left(1 + O(|z_\infty|)\right).$$

If $m^{-2+\beta/4\pi}|z|$ is not small, then the 'Fourier' series does not converge absolutely for all $t$, but it still does for $t$ such that $C_\beta |z_t| < \frac{1}{e}$.

Recall: $|z_{t_0}| = (\varepsilon \ell_{t_0})^{2-\beta/4\pi} \overset{!}{\leq} \frac{1}{C_\beta e}$

$\Leftrightarrow \varepsilon \ell_{t_0} \leq \text{const.} \Leftrightarrow \ell_{t_0} \leq \frac{\alpha_0}{\varepsilon} \leftarrow$ macroscop.!

$\Rightarrow$ Fourier expansion converges up to macroscopic length scales, $V_{t_0}$ is macroscopically smooth.

Recall that $e^{-V_t(\varphi)} = \mathbb{E}_{C_t - C_{t_0}}\left(e^{-V_{t_0}(\varphi+\zeta)}\right)$

$$\Rightarrow \nabla V_t(\varphi) = e^{+V_t(\varphi)} \mathbb{E}_{C_t - C_{t_0}}\left(e^{-V_{t_0}(\varphi+\zeta)} \nabla V_{t_0}(\varphi+\zeta)\right)$$

$$= \left(P_{t_0,t_0} \nabla V_{t_0}\right)(\varphi)$$

and similarly

$$\operatorname{Hess} V_t = P_{t_0,t} \operatorname{Hess} V_{t_0} - \Big(P_{t_0,t}\left(\nabla V_{t_0} \otimes \nabla V_{t_0}\right) - \left(P_{t_0,t} \nabla V_{t_0}\right) \otimes \left(P_{t_0,t} \nabla V_{t_0}\right)\Big)$$

For $V_{t_0}$ we already have the following estimates:

$$(Q_t f, \nabla V_{t_0})^2 \leq O_{\beta,L,m}(|z|) \|f\|_2^2 \, e^{-m^2 t} \quad \leftarrow \text{L-dep. const.}$$

$$(Q_t f, \operatorname{Hess} V_{t_0} \, Q_t f) \geq - O_{\beta,L,m}(|z|) \|f\|_2^2 \, e^{-m^2 t}$$

Hence

$$\begin{aligned}(Q_t f, \operatorname{Hess} V_t \; Q_t f) &= P_{t_0,t}\big((Q_t f, \operatorname{Hess} V_{t_0} \, Q_t f)\big) \\ &\quad - \Big(P_{t_0,t}\big((Q_t f, \nabla V_{t_0})^2\big) - \big(P_{t_0,t}(Q_t f, \nabla V_{t_0})\big)^2\Big). \\ &\geq - O_{\beta,L,m,z}(1) \, e^{-m^2 t} \|f\|_2^2\end{aligned}$$

This gives the Log-Sobolev inequality as before, but note that the constant is not uniform in $L$ anymore (still uniform in $\varepsilon$).

What goes wrong if $\beta \geq 4\pi$?

Consider $V_t(\xi_1, \xi_2)$. Explicitly,

$$V_t(\xi_1,\xi_2) = \int_0^t e^{-(W_t - W_s)(\xi_1,\xi_2)} V_s(\xi_1) V_s(\xi_2)\, \dot{u}_s(\xi_1,\xi_2)\, ds$$

$$\ell_t^2 V_t(\xi_1,\xi_2) = |z_t|^2 \Big( \underbrace{1 - e^{-\beta\sigma_1\sigma_2 C_t(x_1,x_2)}}_{\substack{\text{if } \sigma_1\sigma_2 = -1 \text{ than } \approx \left(\frac{|x_1 - x_2|}{\ell_t}\right)^{-\beta/2\pi} \\ \text{(neutral)}}} \Big)$$

spoils convergence

$\Rightarrow \ell_t^2 \sup_x \sum_y |V_t((x,+1),(y,-1))|$ is not bounded uniformly in $\varepsilon$.

n=2 contribut.

and Hess $V_t^2$ is likewise not bounded below.

However, we need $\underset{\overset{\uparrow}{e^{-tA}}}{Q_t}(\text{Hess } V_t) Q_t$ and the smoothing heat kernel $Q_t$ helps:

$Q_t \text{ Hess } V_t^2 Q_t$ is good.

Conj. For Yukawa gas, the Mayer expansion (this more or less what we have done above), converges up to $\beta < 8\pi(1 - \frac{1}{2n})$ if the first n terms are removed. (Converges ≡ type of bound as for $\beta < 4\pi$.)

Further open problems: Lower bound on Hess $V_t$ for

- Continuum $\varphi^4$ $(d=2,3)$
- Lattice sine-Gordon $(\beta > 8\pi)$
- Lattice $\varphi^4$ $(d \geq 4)$