Log-Sobolev inequality & renormalisation

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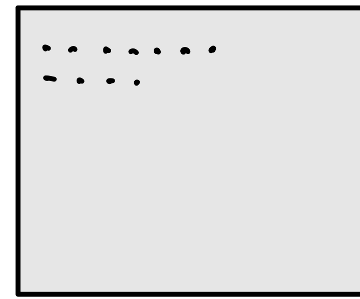
$$d\varphi_t = -\nabla H(\varphi_t)\,dt + \sqrt{2}\,dW_t, \quad \varphi_t \in \mathbb{R}^N$$

invariant measure $\nu_\infty(d\varphi) \propto e^{-H(\varphi)}\,d\varphi$

$$= -A\varphi_t\,dt - \nabla V(\varphi_t)\,dt + \sqrt{2}\,dW_t$$

Laplacian on $\Lambda \cong \mathbb{R}^d$ or $\mathbb{Z}^d$

Local non-linear potential



$\Lambda = \{1, \dots, N\}$

(cont.) Glauber dynamics

Examples: $H(\varphi) = \frac{1}{2}(\varphi, A\varphi) + V(\varphi)$

- Lattice models: $A = -\Delta^\Lambda$, $\Lambda \subset \mathbb{Z}^d$

  $\Delta^\Lambda f(x) = \sum_{y \sim x} (f(y) - f(x))$

  $(f,g)_\Lambda = \sum_{x \in \Lambda} f(x) g(x)$

  $V(\varphi) = \sum_{x \in \Lambda} (g\varphi^4 + \nu\varphi^2)$, $g > 0$, $\nu < 0$

  $V(\varphi) = \sum_{x \in \Lambda} z\cos(\sqrt{\beta}\varphi)$, $z \in \mathbb{R}$, $\beta > 0$

- Continuum models ($\to$ SPDE): $A = -\Delta^\varepsilon + m^2$

  $\Omega_\varepsilon = \varepsilon\mathbb{Z}^d \cap \Omega$

  $\Delta^\varepsilon f(x) = \varepsilon^{-2} \sum_{y \sim x} (f(y) - f(x))$

  $(f,g)_\varepsilon = \varepsilon^d \sum_{x \in \Omega_\varepsilon} f(x) g(x)$

  counterterms (∞)

  $V(\varphi) = \varepsilon^d \sum_{x \in \Omega_\varepsilon} (g\varphi^4 + \nu_\varepsilon \varphi^2)$ $(2 \leq d < 4)$

  $V(\varphi) = \varepsilon^d \sum_{x \in \Omega_\varepsilon} \left(z\,\varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta}\varphi)\right)$ $(d=2)$

$$d\varphi_t = -\nabla H(\varphi_t)\,dt + \sqrt{2}\,dW_t$$

Infinitesimal generator $\Delta_H = \Delta - (\nabla H, \nabla) = e^{+H}(\nabla, e^{-H}\nabla)$

$$\sum_{x\in\Lambda} \frac{\partial^2}{\partial \varphi(x)^2} \qquad \sum_{x\in\Lambda} \frac{\partial H}{\partial \varphi_x}\frac{\partial}{\partial \varphi_x}$$

- ν_∞ is reversible $\mathbb{E}_{\nu_\infty} F(-\Delta_H G) = \mathbb{E}_{\nu_\infty}(\nabla F, \nabla G)$
- Itô $\Rightarrow$ $F_t(\varphi) = \mathbb{E}_{\varphi_0=\varphi} F(\varphi_t)$ solves $\frac{\partial}{\partial t} F_t = \Delta_H F_t$

Law of φ_t satisfies $\mathbb{E}_{\nu_t} F = \mathbb{E}_{\nu_0} F_t$.

- If $d\nu_t = F_t\, d\nu_\infty$ then $\partial_t F_t = \Delta_H^* F_t = \Delta_H F_t$

adj. w.r.t. ν_∞

Ergodicity implies $\nu_t \longrightarrow \nu_\infty$. How fast?

Relative entropy: $H(\nu_t | \nu_\infty) = \underbrace{\mathbb{E}_{\nu_\infty} F \log F}_{\mathrm{Ent}_{\nu_\infty} F}$ where $F = \frac{d\nu_t}{d\nu_\infty}$

Pinsker's inequality: $\|\nu_t - \nu_\infty\|_{TV}^2 \leq 2H(\nu_t | \nu_\infty)$

Fisher information: $I(\nu_t | \nu_\infty) = \mathbb{E}_{\nu_\infty} \frac{(\nabla F)^2}{F} = 4\,\mathbb{E}_{\nu_\infty} (\nabla \sqrt{F})^2 > 0$

de Bruin identity: $\frac{\partial}{\partial t} H(\nu_t | \nu_\infty) = -\, I(\nu_t | \nu_\infty) < 0$

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Proof. $\frac{\partial}{\partial t} \mathbb{E}_{\nu_\infty} \Phi(F_t) = \mathbb{E}_{\nu_\infty} \Phi'(F_t) \dot{F}_t$

$\Phi(x) = x \log x$

$\Phi'(x) = \log x + 1$

$$= \mathbb{E}_{\nu_\infty} \Phi'(F_t) \Delta_H F_t$$

$$= - \mathbb{E}_{\nu_\infty} \left(\underbrace{\nabla \Phi'(F_t)}_{\nabla \log F_t}, \nabla F_t \right)$$

$$= - \mathbb{E}_{\nu_\infty} \frac{(\nabla F_t)^2}{F_t} = - I(\nu_t | \nu_\infty)$$

de Bruin identity: $\frac{\partial}{\partial t} H(\nu_t | \nu_\infty) = - I(\nu_t | \nu_\infty) < 0$

Upshot: $H(\nu_t | \nu_\infty) \leq e^{-2\gamma t} H(\nu_0 | \nu_\infty)$

$$\Leftrightarrow \underbrace{\frac{\partial}{\partial t} H(\nu_t | \nu_\infty)} \leq -2\gamma H(\nu_t | \nu_\infty)$$

$$\Leftrightarrow I(\nu_t | \nu_\infty) \geq 2\gamma H(\nu_t | \nu_\infty) \qquad \text{Log-Sob.}$$

Log-Sobolex inequality: $\underbrace{\mathrm{Ent}_{\nu_\infty} F}_{H(\nu | \nu_\infty) \text{ if } \nu = F \nu_\infty} \leq \frac{2}{\gamma} \underbrace{\mathbb{E}_{\nu_\infty} (\nabla \sqrt{F})^2}_{\frac{1}{4} I(\nu | \nu_\infty)}$

(LSI)

Exercise: (LSI) $\Rightarrow$ Spectral gap: $\mathrm{Var}_{\nu_\infty} F \leq \frac{1}{\gamma} \mathbb{E}_{\nu_\infty} (\nabla F)^2$

Bakry-Émery Theorem. $\nu_\infty(d\varphi) \propto e^{-H(\varphi)} d\varphi = e^{-\frac{1}{2}(\varphi, A\varphi) - V(\varphi)} d\varphi$

$\operatorname{Hess} H(\varphi) \geq \lambda \, \mathrm{id}$ $(\lambda > 0)$ $\Rightarrow$ LS ineq. with $\gamma \geq \lambda$.

unif. in φ $\qquad$ $A \geq \lambda \mathrm{id}$, $\operatorname{Hess} V \geq 0$

Proof.

$$H(\nu_0 | \nu_\infty) = -\int_0^\infty \frac{\partial}{\partial t} H(\nu_t | \nu_\infty) \, dt \quad \text{using } H(\nu_t | \nu_\infty) \longrightarrow 0$$

$$= \int_0^\infty I(\nu_t | \nu_\infty) \, dt$$

$$\leq \frac{1}{2\gamma} I(\nu_0 | \nu_\infty)$$

if $I(\nu_t | \nu_\infty) \leq e^{-2\gamma t} I(\nu_0 | \nu_\infty)$.

To see this, differentiate again:

$$\frac{\partial}{\partial t} I(\nu_t | \nu_\infty) = \frac{\partial}{\partial t} \mathbb{E}_{\nu_\infty} \frac{(\nabla F_t)^2}{F_t}$$

$\dot{F}_t = \Delta_H F$

$$= -2\,\mathbb{E}_{\nu_\infty}\Big(\underbrace{F_t \, |\mathrm{Hess} \log F_t|_2^2}_{\geq 0} + F_t \big(\nabla \log F_t, \underbrace{(\mathrm{Hess}\, H)}_{\geq \lambda\, \mathrm{id}} \nabla (\log F_t)\big)\Big)$$

elementary but tedious exercise

$$\leq -2\lambda\, \mathbb{E}_{\nu_\infty} \frac{(\nabla F_t)^2}{F_t}$$

$$= -2\lambda\, I(\nu_t | \nu_\infty)$$

Summary:

$$\nu_0 \xrightarrow{\quad H(\nu_t|\nu_\infty) \quad} \nu_\infty$$

$$\nu_t \xrightarrow[\quad I(\nu_t|\nu_\infty) \quad]{} \nu_\infty$$

Remarks. • Tensorization: if μ and ν satisfy LS inequalities with constant γ then $\mu \otimes \nu$ satisfies the LS inequality with constant γ on the product space.

• Holley-Stroock perturbation theorem: if ν satisfies the LS inequality with const. γ and $d\mu = F d\nu$ then μ satisfies the LS inequality with const. $\tilde{\gamma} = \frac{\inf F}{\sup F} \gamma$.

- Herbst argument (Gaussian concentration): if ν satisfies the LS inequality with constant γ and F is 1-Lipschitz then

$$\mathbb{E}_\nu e^{\alpha F - \alpha \mathbb{E}_\nu F} \le e^{-\alpha^2/2\gamma}.$$

- Variational formula for the entropy

$$H(\mu|\nu) = \sup\{\mathbb{E}_\mu F - \log \mathbb{E}_\nu e^F\}$$

Entropy inequality

$$\mathbb{E}_\mu F \le \frac{1}{\alpha} H(\mu|\nu) + \frac{1}{\alpha}\log \mathbb{E}_\nu (e^{\alpha F}).$$

Example (Gaussian free field).

$H(\varphi) = \frac{1}{2}(\varphi, A\varphi)$, $\quad A = -\Delta^{\varepsilon} + m^2$ on $\mathbb{T}^d \cap \varepsilon\mathbb{Z}^d$

Invariant measure is GFF on $\mathbb{T}^d$. Dynamics is diagonal in Fourier basis:

$$d\hat{\varphi}_t(k) = -(|k|^2 + m^2)\,\hat{\varphi}(k)\,dt + \sqrt{2}\,d\hat{W}(k)$$

$\Rightarrow$ Different Fourier modes (= length scales) equilibrate at different rates.

Small scales ($|k| \gg 1$) converge quickly.

Macroscopic modes ($|k| \sim 1$) are slowest.

Difficulty: $H(\varphi) = \frac{1}{2}(\varphi, A^\varepsilon \varphi) + V^\varepsilon(\varphi)$

$\sum_x v^\varepsilon(\varphi_x)$ local non-linear

$\rightarrow$ non-local in Fourier space

Gaussian integration.

Let $C_s = \int_0^s \dot{C}_{s'}\, ds'$, $\dot{C}_s$ is pos-def. matrix on $\mathbb{R}^N$.

P_{C_s} be Gaussian measure with cov. C_s.

Example. $\dot{C}_s = e^{-sA} \Rightarrow C_\infty = A^{-1}$

$A = -\Delta^\varepsilon + m^2 \Rightarrow P_{C_\infty}$ is GFF.

Defn. $\Delta_{\dot{C}_s} = \sum_{x,y} \dot{C}_s(x,y) \frac{\partial^2}{\partial\varphi(x)\partial\varphi(y)} = (\nabla, \dot{C}_s \nabla)$

careful: two very different Laplacians!

Rk. Let $g_s^{-1} = \dot{C}_s$. Then $\Delta_{\dot{C}_s} = \Delta_{g_s}$ is the Laplace-Betrami operator on $\mathbb{R}^\Lambda (\cong \mathbb{R}^N)$ associated to the metric $g_s = e^{+sA}$.

$$|f|_{g_s} \le 1 \iff |e^{+sA/2} f|_2 \le 1$$

heat kernel

$$\iff f = e^{-sA/2} f_0, \quad |f_0|_2 \le 1$$

The unit ball consists of functions smooth at scale $\sqrt{s}$.

Prop. $\frac{\partial}{\partial s} P_{C_s} = \frac{1}{2} \Delta_{\dot{C}_s} P_{C_s}$.

Prop. $\frac{\partial}{\partial s} P_{C_s} = \frac{1}{2} \Delta_{\dot{C}_s} P_{C_s}$. $\quad \left(\Delta_{\dot{C}_s} = \sum_{x,y} \dot{C}_s(x,y) \frac{\partial^2}{\partial \varphi(x) \partial \varphi(y)} \right)$

Proof. $$\frac{\partial}{\partial s} \frac{e^{-\frac{1}{2}(\varphi, C_s^{-1}\varphi)}}{Z_{C_s}} = \frac{1}{2} \underbrace{(C_s^{-1}\varphi, C_s^{-1}\varphi)_{\dot{C}_s}}_{(C_s^{-1}\varphi,\, \dot{C}_s C_s^{-1}\varphi) = (C_s^{-1}\varphi)^2_{\dot{C}_s}} P_{C_s}(\varphi) - (\text{const.}) P_{C_s}(\varphi)$$

$$\frac{1}{2} \Delta_{\dot{C}_s} P_{C_s} = \frac{1}{2} (C_s^{-1}\varphi)^2_{\dot{C}_s} P_{C_s}(\varphi) - (\text{const.}) P_{C_s}(\varphi)$$

$$\Rightarrow \left(\partial_s - \frac{1}{2} \Delta_{\dot{C}_s}\right) P_{C_s}(\varphi) = (\text{const.}) P_{C_s}(\varphi).$$

Since the integral of the LHS is 0, the RHS is 0.

Cor. Let $F_s = P_{C_s} * F_0$, i.e., $F_s(\varphi) = E_{C_s} F(\varphi+\zeta)$. Then

$$\partial_s F_s = \frac{1}{2}\Delta_{\dot{C}_s} F_s$$

$$F_s(0) = E_{C_s} F_0 .$$

In particular, if F is a polynomial in φ then (Wick's rule)

$$F_s = e^{\frac{1}{2}\Delta_{C_s}} F_0 = \left(1 + \frac{1}{2}\Delta_{C_s} + \cdots\right) F_0$$

Exercise. Again F is a polyn. Define $:F:_{C_s} = e^{-\frac{1}{2}\Delta_{C_s}} F$ (Wick ordering).

Then $e^{\frac{1}{2}\Delta_{C_s}} :F:_{C_s} = F$ and in particular

$$E_{C_s} :F:_{C_s} = F(0).$$

Renormalised potential

$$e^{-V_s(\varphi)} = (P_{C_s} * e^{-V_0})(\varphi) = \mathbb{E}_{C_s}\left(e^{-V_0(\varphi+\zeta)}\right)$$

$$\Leftrightarrow \frac{\partial}{\partial s} e^{-V_s} = \frac{1}{2}\Delta_{\dot{C}_s} e^{-V_s} \qquad (u,v)_{\dot{C}_s} = \sum_{x,y} \dot{C}_s(x,y)\, u(x)\, v(y)$$

diffusion H.-J.

$$\Leftrightarrow \frac{\partial}{\partial s} V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$$

Polchinski equation (P)

(→ Polchinski, Renormalization and Effective Lagrangians, Nucl. Phys. B 1984)

Prop. Suppose V_s satisfies (P), and

$$\frac{\partial}{\partial s} F_s = \frac{1}{2}\Delta_{\dot{C}_s} F - (\nabla V_s, \nabla F)_{\dot{C}_s} = L_s F.$$

Then the following is indep. of s:

$$\int P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}\, F_s(\varphi)\, d\varphi$$

Prop. Suppose V_s satisfies (P), and

$$\frac{\partial}{\partial s} F_s = \frac{1}{2}\Delta_{\dot{C}_s} F - (\nabla V_s, \nabla F)_{\dot{C}_s} = L_s F.$$

Then the following is indep. of s:

$$\int \underbrace{P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}}_{\nu^s(d\varphi)} \underbrace{F_s(\varphi)}_{P_{0,s}F_0(\varphi)}\, d\varphi$$

Proof. Define $Z_s(\varphi) = e^{-V_s(\varphi)} F_s(\varphi)$. Then $\frac{\partial}{\partial s} Z_s = \frac{1}{2}\Delta_{\dot{C}_s} Z_s$.

$$\Rightarrow \frac{\partial}{\partial s} \int \underbrace{P_{C_\infty - C_s}(\varphi)}\, \underbrace{e^{-V_s(\varphi)} F_s(\varphi)}\, d\varphi = \int\left(\left(-\frac{1}{2}\Delta_{\dot{C}_s} P_{C_\infty - C_s}\right) Z_s + P_{C_\infty - C_s}\left(\frac{1}{2}\Delta_{\dot{C}_s} Z_s\right)\right) d\varphi$$

$$= 0$$

Renormalised measure $\quad \nu^s(d\varphi) = P_{C_\infty - C_s}(\varphi)\, e^{-V_s(\varphi)}\, d\varphi$

Polchinski semigroup $\quad P_{s',s} F(\varphi) = e^{+V_s(\varphi)}\, \mathbb{E}_{C_s - C_{s'}}\left(e^{-V_{s'}(\varphi+\zeta)} F(\varphi+\zeta)\right)$

$$\Rightarrow \mathbb{E}_{\nu_0} F = \mathbb{E}_{\nu_s} P_{0,s} F$$

Exercise: $\quad \frac{\partial}{\partial s} \mathbb{E}_{\nu_s} F = -\mathbb{E}_{\nu_s} L_s F, \qquad L_s = \frac{1}{2}\Delta_{\dot{C}_s} - (\nabla V_s, \nabla)_{\dot{C}_s}$

$$\frac{\partial}{\partial s} P_{s',s} F = L_s P_{s',s} F$$

$$\frac{\partial}{\partial s'} P_{s',s} F = -P_{s',s} L_{s'} F$$

Summary: two ways to evolve measure

$d\nu_t = F_t \, d\nu_\infty$ with $\frac{\partial}{\partial t} F_t = \Delta F_t - (\nabla H, \nabla F_t) = \Delta_H F_t$

Glauber semigroup

Tends to invariant measure ν_∞.

$d\nu_t^s = F_t^s \, d\nu_\infty^s$ with $\frac{\partial}{\partial s} F^s = \frac{1}{2} \Delta_{\dot{C}_s} F^s - (\nabla V_s, \nabla F^s)_{\dot{C}_s} = L_s F^s$

Polchinski semigroup

(time-dependent)

Tends to $\nu^\infty = \delta_0$.

Example. $\mathrm{Hess}\, V_0 \geq 0 \Rightarrow \mathrm{Hess}\, V_s \geq 0 \quad \forall s > 0.$

Proof 1. $$e^{-V_s(\varphi)} \propto \int \underbrace{e^{-(\zeta, C_s^{-1}\zeta) - V_0(\varphi+\zeta)}}_{\text{log-concave in } (\varphi,\zeta)} d\zeta$$

Brascamp-Lieb ineq.: marginals of log-concave meas. are log-concave.

Proof 2. $$\partial_s V_s = \frac{1}{2}\Delta_{\dot{C}_s} V_s - \frac{1}{2}(\nabla V_s)^2_{\dot{C}_s}$$

$$\Rightarrow \partial_s \nabla V_s = \frac{1}{2}\Delta_{\dot{C}_s} \nabla V_s - (\nabla V_s, \mathrm{Hess}\, V_s)_{\dot{C}_s} = L_s \nabla V_s$$

$$\Rightarrow \partial_s \mathrm{Hess}\, V_s = L_s \mathrm{Hess}\, V_s - (\mathrm{Hess}\, V_s)\, \dot{C}_s\, (\mathrm{Hess}\, V_s)$$

Suppose $N=1$. Then this is simply

$$\partial_s f_s = L_s f_s - f_s^2 \qquad f_s = V_s''$$

Maximum principle: $f \geq 0 \Rightarrow L_s f \geq 0$

$\Rightarrow \partial_s f_s \geq -f_s^2, \quad f_0 \geq 0 \quad \Rightarrow \quad f_s \geq 0.$

General dimension: Maximum principle for symmetric tensors.

Thm (BB'19). Assume $A \geq \lambda\, \mathrm{id}$ ($\lambda > 0$) and allowed to be negative!

$$Q_s \operatorname{Hess} V_s(\varphi) Q_s \geq \dot{\mu}_s \,\mathrm{id}. \qquad Q_s = e^{-sA/2}$$

$\Rightarrow$ Log-Sob. with $\frac{1}{\gamma} = \int_0^\infty e^{-\lambda s - 2\mu_s}\, ds, \quad \mu_s = \int_0^s \dot{\mu}_{s'}\, ds'.$

Rk. In terms of the metric $g_s = e^{+sA}$ the condition is

$$\operatorname{Hess}_{g_s} V_s \geq \dot{\mu}_s g_s.$$

Rk. If V_0 is convex this recovers Bakry-Émery:

$$A = \lambda\, \mathrm{id}, \qquad V = H - \tfrac{1}{2}(\varphi, A\varphi)$$

$\Rightarrow \dot{\mu}_s \geq 0 \;\; \forall s \quad \Rightarrow \frac{1}{\gamma} \leq \int_0^\infty e^{-\lambda s} = \frac{1}{\lambda}$

Proof idea. Like Bakry–Émery but use Polchinski semigroup.

$$\frac{\partial}{\partial s}\mathbb{E}_{\nu_s}\Phi(F^s) = \mathbb{E}_{\nu_s}\Big(\underline{-L_s\Phi(F^s)} + \Phi'(F^s)\dot{F}^s\Big)$$

$$= \mathbb{E}_{\nu_s}\Big(\cancel{-\Phi'(F^s)L_sF^s} - \frac{1}{2}\Phi''(F^s)(\nabla F^s)^2_{\dot{C}_s} + \cancel{\Phi'(F^s)L_sF^s}\Big)$$

$$= -\frac{1}{2}\mathbb{E}_{\nu_s}\Big(\frac{(\nabla F^s)^2_{\dot{C}_s}}{F_s}\Big)$$

$$= -2\,\mathbb{E}_{\nu_s}(\nabla\sqrt{F^s})^2_{\dot{C}_s}$$

ν^s renorm meas.

$F^s = P_{0,s}F$

$\Phi(x) = x\log x$

'Fisher information' at scale s

Next step: consider change of $\mathbb{E}_{\nu_s} (\nabla \sqrt{F^s})^2_{\dot{C}_s}$.

$$\left(\frac{\partial}{\partial s} - L_s\right) (\nabla \sqrt{F^s})^2_{\dot{C}_s} = + (\nabla \sqrt{F^s})^2_{\ddot{C}_s}$$

$$\dot{C}_s = e^{-sA}$$
$$\ddot{C}_s = -A \dot{C}_s$$

$$- 2 (\nabla \sqrt{F^s}, \dot{C}_s \operatorname{Hess} V_s \dot{C}_s \nabla \sqrt{F^s})$$

$$\underbrace{- \frac{1}{4} F^s \left| \dot{C}_s^{1/2} (\operatorname{Hess} \log F^s) \dot{C}_s^{1/2} \right|_2^2}_{\geq 0}$$

$$Q_s = \dot{C}_s^{1/2} = e^{-sA/2}$$

$$\leq - (\nabla \sqrt{F^s}, Q_s (\underbrace{A + 2 Q_s \operatorname{Hess} V_s Q_s}_{\geq (\lambda + 2\dot{\mu}_s)\, \mathrm{id}}) Q_s \nabla \sqrt{F^s})$$

assumption

$$\Rightarrow \left(\frac{\partial}{\partial s} - L_s\right)\left(\nabla\sqrt{F^s}\right)^2_{\dot{C}_s} \le -(\lambda + 2\dot{\mu}_s)\left(\nabla\sqrt{F^s}\right)^2_{\dot{C}_s}$$

$\psi(s) = \mathbb{E}_{\nu_s}\left(\nabla\sqrt{F^s}\right)^2_{\dot{C}_s}$ satisfies

$$\dot{C}_s = e^{-sA}$$

$$\dot{\psi}(s) \le -(\lambda + 2\dot{\mu}_s)\,\psi(s)$$

$$\psi(s) \le e^{-\lambda s - 2\mu_s}\,\psi(0) = e^{-\lambda s - 2\mu_s}\,\mathbb{E}\left(\nabla\sqrt{F}\right)^2_{\cancel{\dot{C}_0}}$$

$$\Rightarrow \mathrm{Ent}_\nu F \le 2\underbrace{\left(\int_0^\infty e^{-\lambda s - 2\mu_s}\,ds\right)}_{1/\gamma}\mathbb{E}_\nu\left(\nabla\sqrt{F}\right)^2$$

Rk. • The same proof gives the LS inequality for the renormalised measure ν^s.

• Moreover, it shows that the 'fluctuation measures' defined by

$$\mathbb{E}_{\mu_{0,s}^{\varphi}} F = P_{0,s} F(\varphi)$$

satisfy LS inequalities with constants $\gamma_s \gg \gamma$:

$$\frac{1}{\gamma_s} = \int_0^s (\cdots).$$

Small scales equilibrate quickly.

Summary of argument:

$$\frac{\partial}{\partial t} H(\nu_t | \nu_\infty) = - I_0(\nu_t | \nu_\infty)$$ Glauber

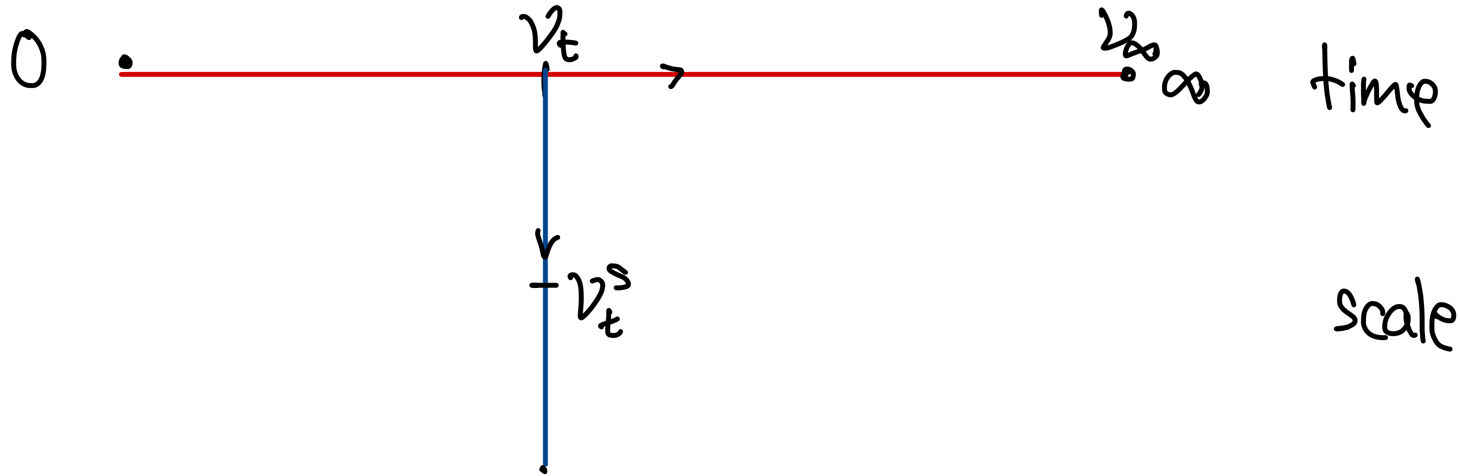
$$\frac{\partial}{\partial s} H(\nu_t^s | \nu_\infty^s) = - I_s(\nu_t^s | \nu_\infty^s)$$ Polchinski

$$\frac{\partial}{\partial s} I_s(\nu_t^s | \nu_\infty^s) \le -(\lambda + 2\dot{\mu}_s) I_s(\nu_t^s | \nu_\infty^s)$$

$$\Rightarrow I_s(\nu_t^s | \nu_\infty^s) \le e^{-\lambda s - 2\mu_s} I_0(\nu_t^0 | \nu_\infty^0)$$

$$\Rightarrow H(\nu_t | \nu_\infty) \le \frac{1}{2}\left(\int_0^\infty (\cdots)\right) I_0(\nu_t | \nu_\infty)$$

$$\Rightarrow \frac{\partial}{\partial t} H(\nu_t | \nu_\infty) \le -2\gamma H(\nu_t | \nu_\infty)$$


0
ν_t
ν_∞
∞
time
ν_t^s
scale

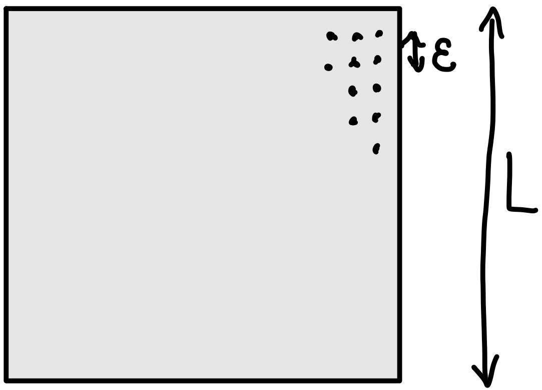
Exercise. Let $A = \lambda\, \mathrm{id}$ and $V = V^c + V^b$ with

$$\operatorname{Hess} V^c \geq 0 \qquad \text{and} \qquad \|V^b\|_{C^2} < \infty.$$

Show that

$$\operatorname{Hess} V_s \geq -C \quad \text{for all } s \geq 0.$$

Continuum sine-Gordon model



$\Omega_{\varepsilon,L} = L\mathbb{T}^2 \cap \varepsilon\mathbb{Z}^2$

$m^2 > 0$

$$\nu(d\varphi) \propto \exp\Big[-\varepsilon^2 \sum_{x\in\Omega_{\varepsilon,L}} \Big(\tfrac{1}{2}\varphi(-\Delta^\varepsilon\varphi) + \tfrac{1}{2}m^2\varphi^2 + 2z\,\varepsilon^{-\frac{\beta}{4\pi}}\cos(\sqrt{\beta}\,\varphi)\Big)\Big]\prod_{x\in\Omega_{\varepsilon,L}} d\varphi_x$$

In various works (under varying assumpt. on β, z, m^2), it is known that:

For $\beta < 8\pi$ the limits $\varepsilon\to 0$ (and $L\to\infty$) exist and define non-Gaussian measure + properties.

[Fröhlich, F-Seiler, F-Park, Benfatto-Gallavotti-Nicolò, N-Renn-Steinmann, Brydges-Kennedy, Dimock-Hurd, Lacoin-Rhodes-Vargas, ...]

For $\beta \geq 8\pi$ all continuum limits are expected to be Gaussian.

Comparing with the φ^4 model, β roughly plays the role of dimension d with the correspondence

$$\begin{aligned}
\beta &< 4\pi \ (\mathrm{SG}) &&\longleftrightarrow && d=2 \ (\varphi^4)\\
\beta &< 6\pi \ (\mathrm{SG}) &&\longleftrightarrow && d=3 \ (\varphi^4)\\
\beta &= 8\pi \ (\mathrm{SG}) &&\longleftrightarrow && d=4 \ (\varphi^4)
\end{aligned}$$

In fact, there is an infinite sequence of thresholds that one encounters in the construction occuring at

$$\beta_n = 8\pi\left(1-\frac{1}{2n}\right).$$

The physical meaning of these thresholds remains debated, but the consensus appears to be that they have no physical meaning.

Relation to Yukawa gas (= sine-Gordon transformation)

Let φ be a Gaussian field with cov. $C = (-\Delta^\varepsilon + m^2)^{-1}$ Yukawa potential

$$\Rightarrow \mathbb{E}\left(e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}\right) = e^{-\frac{\beta}{2}\sum_{i,j}\sigma_i\sigma_j C(x_i,x_j)}$$

$$= e^{-\frac{\beta}{2}\sum_{i,j}\sigma_i\sigma_j(-\Delta^\varepsilon+m^2)^{-1}(x_i,x_j)}$$

$$\Rightarrow \mathbb{E}\left(e^{\frac{\beta}{2}C(0,0)n}e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}\right) = e^{-\frac{\beta}{2}\sum_{i\neq j}\sigma_i\sigma_j(-\Delta^\varepsilon+m^2)^{-1}(x_i,x_j)}$$

For the 2D Yukawa potential: $(-\Delta^\varepsilon+m^2)^{-1}(0,0) = \frac{1}{2\pi}\log\varepsilon^{-1} + c(m) + o(1)$

$$e^{\frac{\beta}{2}C(0,0)} \approx \text{const.}\ \varepsilon^{-\beta/4\pi}$$

Partition function of 2D Yukawa gas

$$Z^{YG} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \varepsilon^{2n} \sum_{\substack{x_1,\dots,x_n \in \Omega_\varepsilon \\ \sigma_1,\dots,\sigma_n \in \{\pm 1\}}} \underbrace{e^{-\frac{\beta}{2} \sum_{i \neq j} \sigma_i \sigma_j (-\Delta^\varepsilon + m^2)^{-1}(x_i, x_j)}}$$

$\xi_i = (x_i, \sigma_i)$

$$= \mathbb{E}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} e^{\frac{\beta}{2} C(0,0) n} \underbrace{\varepsilon^{2n} \sum_{\xi_1,\dots,\xi_n} e^{i\sqrt{\beta} \sum_{i=1}^{n} \varphi(x_i)\sigma_i}} \right)$$

$$= \left(\varepsilon^2 \sum_{x,\sigma} e^{i\sqrt{\beta}\varphi(x)\sigma} \right)^n = \left(\varepsilon^2 \sum_x 2\cos(\sqrt{\beta}\,\varphi(x)) \right)^n$$

$$= \mathbb{E}\left(\exp\left(\varepsilon^2 \sum_{x \in \Omega_\varepsilon} 2z \underbrace{e^{\frac{\beta}{2} C(0,0)}}_{\approx \varepsilon^{-\beta/4\pi}} \cos(\sqrt{\beta}\,\varphi(x)) \right) \right) = Z^{SG}.$$

Glauber dynamics (Dynamical sine-Gordon model)

$$d\varphi = \Delta^\varepsilon \varphi - m^2\varphi + 2z\,\varepsilon^{-\beta/4\pi}\sqrt{\beta}\,\sin(\sqrt{\beta}\,\varphi) + \sqrt{2}\,dW^\varepsilon$$

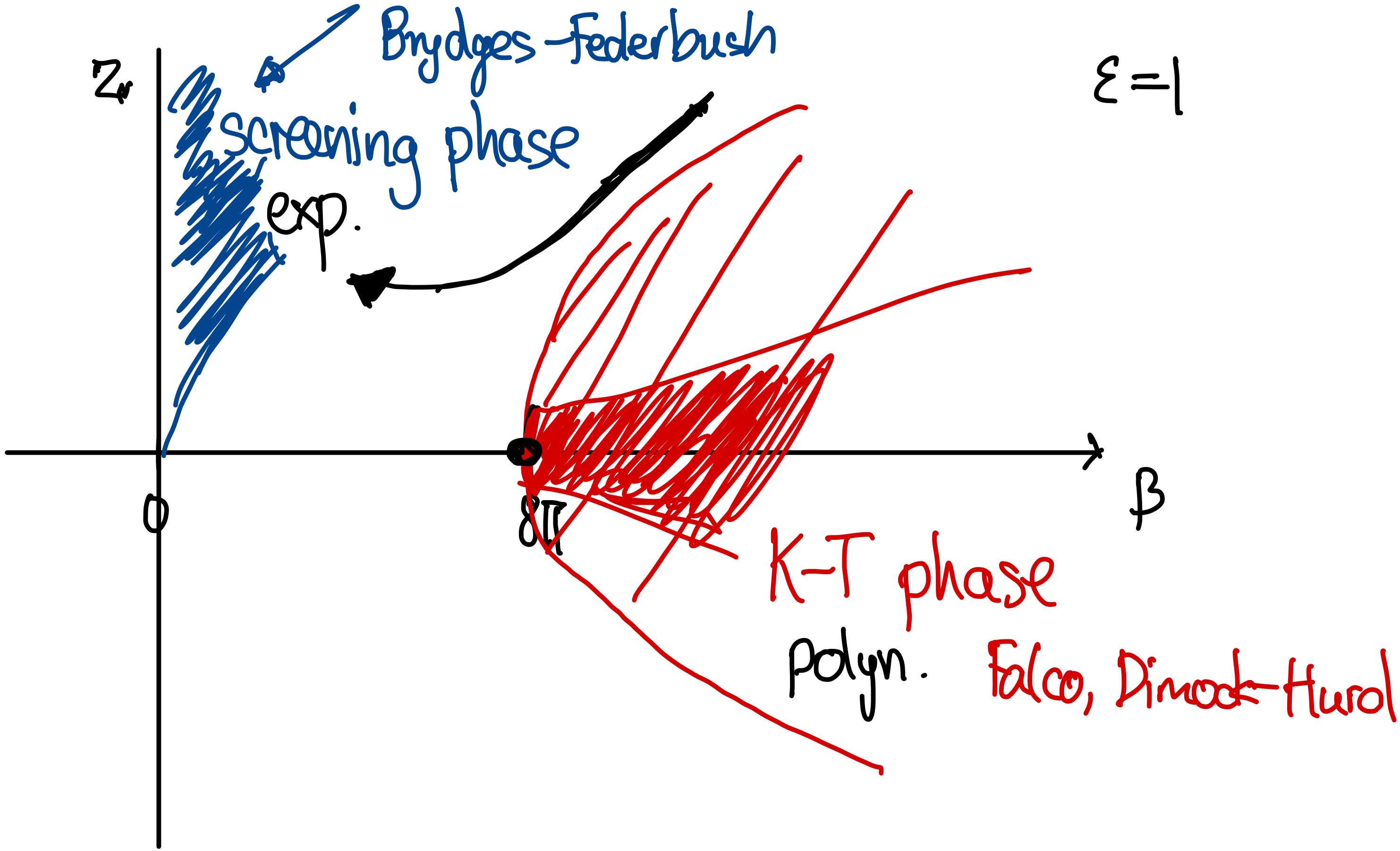
W^ε standard BM with respect to inner product

$$\langle f, g\rangle = \varepsilon^2 \sum_{x\in\Omega_\varepsilon} f(x)g(x).$$

Chandra–Hairer–Shen: limit well-posed for all $\beta < 8\pi$

We'll keep $\varepsilon > 0$ but will derive estimates uniform in ε.

Dirichlet form: $\displaystyle D^\varepsilon(F) = \frac{1}{\varepsilon^2}\sum_{x\in\Omega_\varepsilon} \mathbb{E}_{\nu^\varepsilon}\left(\frac{\partial F}{\partial\varphi(x)}\right)^2.$


Brydges-Federbush
z
screening phase
exp.
$\varepsilon = 1$
0
8π
β
K-T phase
polyn.
Falco, Dimock-Hurd

Thm (BB'19). Let $\beta < 6\pi$, $z \in \mathbb{R}$, $m^2 > 0$, $L > 0$. Then there is $\gamma = \gamma(\beta, z, m, L) > 0$ and independent of ε s.t.

$$\mathrm{Ent}_{\nu^\varepsilon} F \leq \frac{2}{\gamma} D^\varepsilon(\sqrt{F}).$$

Moreover, if $m^{-2+\beta/4\pi} |z| \leq \delta_\beta$ then

$$\gamma \geq m^2 + O_\beta(m^{\beta/4\pi} |z|).$$ ← independent of L

Rk. Analogous result for Kawasaki (conservative) dynamics.

$$D_0^\varepsilon(F) = \frac{1}{\varepsilon^4} \sum_{x \sim y} \mathbb{E}_{\nu^\varepsilon}\left(\frac{\partial F}{\partial \varphi(x)} - \frac{\partial F}{\partial \varphi(y)} \right)^2$$

Measure restricted to $\sum_x \varphi(x) = \text{const}$.

Normalisation

$$V(\varphi) = \varepsilon^2 \sum_{x\in\Omega_\varepsilon} 2z\, \varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta}\varphi(x)) \quad \text{macro. norm.}$$

$$= \sum_{x\in\Omega_\varepsilon} 2z\, \varepsilon^{2-\beta/4\pi} \cos(\sqrt{\beta}\varphi(x)) \quad \text{micro. norm.}$$

$A = -\Delta^\varepsilon + m^2$ with respect to $(u,v)_\varepsilon = \varepsilon^2 \sum_{x\in\Omega_\varepsilon} u(x)v(x)$ macro.

$A = -\Delta + \varepsilon^2 m^2$ with respect to $(u,v) = \sum_{x\in\Omega_\varepsilon} u(x)v(x)$ micro.

unit lattice Laplacian

Macros. norm.: SPDE as $\varepsilon \to 0$

Micros. norm.: Lattice system with weak interaction.

Yukawa gas representation of the renorm. potential (Brydges & Kennedy)

We write

$$V_t(\varphi) = \sum_{n=0}^{\infty} V_t^n(\varphi), \quad V_t^n(\varphi) = \frac{1}{n!} \sum_{\substack{x_1 \dots x_n \in \Omega_\varepsilon \\ \sigma_1, \dots, \sigma_n \in \{\pm 1\}}} \tilde{V}_t^n(\xi_1, \dots, \xi_n)\, e^{i\sqrt{\beta}\sum_{i=1}^{n} \varphi(x_i)\sigma_i}$$

(*)

ε^{2n} considered part of $\tilde{V}$

$\xi_i = (x_i, \sigma_i)$

Initial potential: $\tilde{V}_0(\xi_1) = z_0 = \varepsilon^{2-\beta/4\pi} z$

$\tilde{V}_0(\xi_1, \dots, \xi_n) = 0 \qquad (n \geq 2)$

Polchinski equation: $\partial_t V_t = \frac{1}{2}\Delta_{\dot{C}_t} V_t - \frac{1}{2}(\nabla V_t)^2_{\dot{C}_t}, \quad \dot{C}_t = e^{-tA}$

Polchinski equation: $\partial_t V_t = \frac{1}{2}\Delta_{\dot{C}_t} V_t - \frac{1}{2}(\nabla V_t)^2_{\dot{C}_t}$, $\quad \dot{C}_t = e^{-tA}$

$$\frac{1}{2}\Delta_{\dot{C}_t} V(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\xi_1,\dots,\xi_n} \underbrace{\frac{1}{2}\widetilde{\Delta_{\dot{C}_t} V}(\xi_1,\dots,\xi_n)} \; e^{i\sqrt{\beta}\sum_{i=1}^{n}\varphi(x_i)\sigma_i}$$

where $\frac{1}{2}\widetilde{\Delta_{\dot{C}_t} V}(\xi_1,\dots,\xi_n) = -\underbrace{\left(\frac{1}{2}\sum_{j,k=1}^{n}\beta\sigma_j\sigma_k\dot{C}_t(x_j,x_k)\right)}_{\dot{W}_t(\xi_1,\dots,\xi_n)} \tilde{V}(\xi_1,\dots,\xi_n)$

$$\widetilde{(\nabla V)^2_{\dot{C}_t}}(\xi_1,\dots,\xi_n) = \sum_{\substack{I_1 \dot\cup I_2 = [n] \\ j\in I_1,\, k\in I_2}} \Big(-\underbrace{\beta\dot{C}_t(x_j,x_k)\sigma_j\sigma_k}_{\dot{u}(\xi_j,\xi_k)}\Big)\tilde{V}(\xi_{I_1})\tilde{V}(\xi_{I_2})$$

$\xi_{I_1} = (\xi_i)_{i\in I_1}$

Polchinski equation in 'Fourier space':

$$\partial_t \tilde{V}_t(\xi_1,\dots,\xi_n) = -\dot{W}_t(\xi_1,\dots,\xi_n)\,\tilde{V}_t(\xi_1,\dots,\xi_n) - \frac{1}{2}\widetilde{(\nabla V_t)^2_{\dot{C}_t}}(\xi_1,\dots,\xi_n).$$

Duhamel formula:

$$\tilde{V}_t(\xi_1,\dots,\xi_n) = e^{-W_t(\xi_1,\dots,\xi_n)}\,\tilde{V}_0(\xi_1,\dots,\xi_n) + \frac{1}{2}\int_0^t e^{-(W_t-W_s)(\xi_1,\dots,\xi_n)} \sum_{\substack{I_1 \dot\cup I_2 = [n] \\ j\in I_1,\, k\in I_2}} \dot{u}_s(\xi_j,\xi_k)\,\tilde{V}_s(\xi_{I_1})\,\tilde{V}_s(\xi_{I_2})\,ds$$

depend on $\leq n-1$ particles since $I_1 \neq \emptyset$.

$n=1$: $\hat{V}_t(\xi_1) = e^{-W_t(\xi_1)}\hat{V}_0(\xi_1) = e^{-\frac{\beta}{2}C_t(0,0)} z_0$ $\varepsilon^{2-\beta/4\pi} z$

$n\geq 2$: $\hat{V}_t(\xi_1,\dots,\xi_n) = \frac{1}{2}\int_0^t \underbrace{e^{-(W_t-W_s)(\xi_1,\dots,\xi_n)}}_{\leq 1} \sum_{\substack{I_1\cup I_2=[n] \\ j\in I_1,\, k\in I_2}} \dot{u}_s(\xi_j,\xi_k)\hat{V}_s(\xi_{I_1})\hat{V}_s(\xi_{I_2})\, ds$

By induction, $\hat{V}_t(\xi_1,\dots,\xi_n)$ is well-defined for all t, n.

Fact. If the series (*) converges absolutely, then it gives the unique solution to the Polchinski equation.

(Brydges-Kennedy: $\hat{V}_t$ are the Ursell function of a regularised Yukawa gas)

Heat kernel $\dot{C}_t(x,y) = e^{-tA}(x,y) = e^{\Delta t}(x,y)\, e^{-m^2\varepsilon^2 t}$

$A = -\Delta + m^2\varepsilon^2$

unit lattice Laplacian

$\approx e^{-c|x-y|^2/t}$

Define length scale $\ell_t = (1 \vee \sqrt{t}) \wedge \frac{1}{m\varepsilon}$

Exercise: $C_t(x,x) = \int_0^t \dot{C}_s(x,x)\, ds = \frac{1}{2\pi} \log \ell_t + O(1)$

$\int_1^t \frac{ds}{4\pi s} = \frac{1}{2\pi} \log \sqrt{t}$

$\sup_x \sum_y \dot{C}_t(x,y) \leq e^{-m^2\varepsilon^2 t}$

$$\Rightarrow \underset{n=1}{\ell_t^2 \tilde{V}_t(\xi_1)} = \ell_t^2 e^{-\frac{\beta}{2} C_t(0,0)} z_0 = \ell_t^{2-\frac{\beta}{4\pi}} \varepsilon^{2-\beta/4\pi} z = \underbrace{(\varepsilon \ell_t)^{2-\beta/4\pi} z}_{=: z_t \text{ micro. ren. coupling}}$$

Thm. (Brydges & Kennedy). Let $\beta < 4\pi$. Then for $n \geq 2$,

$$\ell_t^2 \sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} |\tilde{V}_t(\xi_1,\dots,\xi_n)| \leq n^{n-2} C_\beta^{n-1} |z_t|^n.$$

$\xi_i = (x_i, \sigma_i)$

e.g. $\dot{C}(x_j,x_k)\tilde{V}_s(x_j,x_p)\tilde{V}_s(x_k,x_q)$

$j,k,p,q \in \{1,\dots,4\}$

Proof. Assume bound holds for $k<n$. Then

$$|\tilde{V}_t(\xi_1,\dots,\xi_n)| \leq \int_0^t \sum_{\substack{I_1 \dot\cup I_2 \subset [n] \\ j \in I_1,\, k \in I_2}} |\sigma_j \sigma_k \dot{C}_s(x_j,x_k)|\, |\tilde{V}_s(\xi_{I_1})|\, |\tilde{V}_s(\xi_{I_2})|$$

ξ_{I_1}: $(x_i,\sigma_i)_{i \in I_1}$

$$\Rightarrow \underbrace{\sup_{\xi_1} \sum_{\xi_2,\dots\xi_n} |\tilde{V}_t(\xi_1,\dots,\xi_n)|}_{\|\tilde{V}^n\|} \leq \sum_{m=1}^{n-1} \binom{n}{m} \int_0^t \underbrace{\|\dot{C}_s\|}_{\leq e^{-m^2\varepsilon^2 t}} \underbrace{\|\tilde{V}_s^m\|}_{m^{m-2} C_\beta^{m-1} |z_s|^m} \underbrace{\|\tilde{V}_s^{n-m}\|}_{(\cdots)} \, ds$$

$$\sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} |\tilde V_t(\xi_1,\dots,\xi_n)| \le \frac12 \sum_{m=1}^{n-1}\binom{n}{m}\int_0^t \underbrace{\|\dot C_s\|}_{\le e^{-m^2\varepsilon^2 t}}\, \underbrace{\|\tilde V_s^m\|}_{m^{m-2} C_\beta^{m-1}|z_s|^m \ell_s^{-2}}\, \|\tilde V_s^{n-m}\|\, ds$$

$$\le \frac12 \underbrace{\left(\sum_{m=1}^{n-1}\binom{n}{m} m^{m-2}(n-m)^{n-m-2}\right)}_{\le \sum_{m=1}^{n-1}\binom{n}{m} m^{m-1}(n-m)^{n-m-1} = 2(n-1)\,n^{n-2}} C_\beta^{n-2} \underbrace{\int_0^t |z_s|^n \ell_s^{-4}\, ds}_{\lesssim \frac1n |z_t|^n \ell_t^{-2} \text{ if } \beta<4\pi,\ n\ge 2}$$

$$z(\varepsilon \ell_s)^{2-\frac{\beta}{4\pi}}$$

$$\int_0^t \ell_s^{(2-\frac{\beta}{4\pi})n}\, \ell_s^{-4}\, ds \underset{\ell_s \approx \sqrt{s}}{\approx} \int_0^t s^{\overbrace{(1-\frac{\beta}{8\pi})}^{\ge 1 \text{ if } \beta<4\pi,\ n\ge2} n-2}\, ds \underset{n\ge 2,\ \beta<4\pi}{\lesssim} \frac1n t^{(1-\frac{\beta}{8\pi})n-1} = \frac1n \ell_t^{(2-\frac{\beta}{4\pi})n}\, \ell_t^{-2}$$

$$\sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} |\tilde{V}_t(\xi_1,\dots,\xi_n)| \le \frac{1}{2}\sum_{m=1}^{n-1} \binom{n}{m} \int_0^t \underbrace{\|\dot{C}_s\|}_{\le e^{-m^2\varepsilon^2 t}} \underbrace{\|\tilde{V}_s^{m}\| \, \|\tilde{V}_s^{n-m}\|}_{m^{m-2} C_\beta^{m-1} |z_s|^m \ell_s^{-2}} \, ds$$

$$\le \frac{1}{2} \underbrace{\left(\sum_{m=1}^{n-1} \binom{n}{m} m^{m-2} (n-m)^{n-m-2} \right)}_{\le 2(n-1)\, n^{n-2}} C_\beta^{n-2} \underbrace{\int_0^t |z_s|^n \ell_s^{-4} \, ds}_{\lesssim \frac{1}{n} |z_t|^n \ell_t^{-2}} \qquad z(\varepsilon \ell_s)^{2-\beta/4\pi}$$

$$\le n^{n-2} C_\beta^{n-1} |z_t|^n \ell_t^{-2}$$

End of proof.

Cor. Let $\beta < 4\pi$ and assume $C_\beta |z_t| < \frac{1}{e}$. Then

$$V_t(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\xi_1, \dots, \xi_n} \tilde{V}_t(\xi_1, \dots, \xi_n)\, e^{i\sqrt{\beta} \sum_{i=1}^{n} \varphi(x_i)\sigma_i}$$

converges absolutely and gives the soln. to the Polchinski eqn.

Moreover,

$$\ell_t^2 |\nabla V_t(\varphi, x)| \lesssim |z_t|$$

$$\ell_t^2 \sum_y |\mathrm{Hess}\, V_t(\varphi, x, y)| \lesssim |z_t|$$

Proof.

$$\frac{\ell_t^2 |V_t(\varphi)|}{|\Omega_\varepsilon|} \le \sum_{n=0}^{\infty} \frac{1}{n!} n^{n-2} (C_\beta |z_t|)^n \le \sum_{n=0}^{\infty} \underbrace{(e\, C_\beta |z_t|)}_{<1/2}^{n}$$

$$\frac{n^n}{n!} \le e^n$$

pts. in Ω

$$\ell_t^2 |\nabla V_t(\varphi, x)| \le \sum_{n=0}^{\infty} (\cdots) \le |z_t|$$

$$\ell_t^2 \sum_y |\mathrm{Hess}\, V_t(\varphi, x, y)| \le (\cdots)$$

Note that $|z_t| = (\varepsilon \ell_t)^{2-\beta/4\pi} |z| \leq m^{-2+\beta/4\pi} |z|$.

$\ell_t = (1 \vee \sqrt{t}) \wedge \frac{1}{\varepsilon m}$.

So the last corollary provides the required estimates for all t when $m^{-2+\beta/4\pi} |z| \leq \delta_\beta$.

<u>Cor.</u> Let $\beta < 4\pi$, $|z| m^{-2+\beta/4\pi} \leq \delta_\beta$. Then the Log-Sob. const. satisf.

$$\gamma \geq m^2 + O(|z| m^{\beta/4\pi}).$$

independent of L

Proof. $\underbrace{Q_t}_{\leq e^{-m^2\varepsilon^2 t/2}} \operatorname{Hess} V_t(\varphi) Q_t \geq \dot{\mu}_t \, \mathrm{id}$ with $\dot{\mu}_t = \frac{|z_t|}{\ell_t^2} e^{-m^2\varepsilon^2 t}$

$$\int_0^t \dot{\mu}_s \, ds = \int_0^t |z_s| e^{-m^2\varepsilon^2 s} \underbrace{\frac{ds}{\ell_s^2}}_{\approx ds/s} \lesssim |z_t|$$

Since $A = -\Delta + \varepsilon^2 m^2 \geq \varepsilon^2 m^2$ and taking into account that the Dirichlet form is

$$D^{\varepsilon}(F) = \frac{1}{\varepsilon^2} \sum_{x} (\cdots),$$

we find

$$\frac{1}{\gamma} = \varepsilon^2 \int_0^\infty e^{-\varepsilon^2 m^2 t + O(|z_t|)} = \frac{1}{m^2}\left(1 + O(|z_t|)\right)$$

If $m^{-2+\beta/4\pi}|z|$ is not small, then the 'Fourier' series does not converge for all t, but it still does for t s.t. $C_\beta |z_t| < \frac{1}{e}$.

$$|z_{t_0}| = (\varepsilon\, l_{t_0})^{2-\beta/4\pi}|z| \leq \frac{1}{C_\beta e}$$

$\Leftrightarrow$ $\varepsilon\, l_{t_0} \lesssim \alpha_0$ $\leftarrow$ const. that depends on β, $|z|$

$\Leftrightarrow$ $l_{t_0} \leq \frac{\alpha_0}{\varepsilon}$ $\longleftarrow$ macroscoptic!

$\Rightarrow$ 'Fourier' expansion converges up to macroscoptic length scales, γ_{t_0} is macroscoptically smooth.

Recall that $e^{-V_t(\varphi)} = E_{C_t - C_{t_0}}\left(e^{-V_{t_0}(\varphi+\zeta)}\right)$

$$\Rightarrow \quad \nabla V_t(\varphi) = e^{+V_t(\varphi)} E_{C_t - C_0}\left(e^{-V_{t_0}(\varphi+\zeta)} \nabla V_{t_0}(\varphi+\zeta)\right)$$

$$= P_{t_0,t}(\nabla V_{t_0})(\varphi)$$

$$\mathrm{Hess}\, V_t(\varphi) = P_{t_0,t}\left(\mathrm{Hess}\, V_{t_0}\right)$$

$$- \left(P_{t_0,t}(\nabla V_{t_0} \otimes \nabla V_{t_0})\right.$$

$$\left. - (P_{t_0,t} \nabla V_{t_0}) \otimes (P_{t_0,t} \nabla V_{t_0})\right)$$

For V_{t_0} we already have the following estimates:

$$(Q_t f, \nabla V_{t_0})^2 \leq O_{\beta,L,m}(|z|^2) \|f\|_2^2 e^{-m^2 t}$$

$$(Q_t f, \mathrm{Hess}\, V_{t_0} Q_t f) \geq -O_{\beta,z,m}(1) \|f\|_2^2 e^{-m^2 t}$$

$$\Rightarrow \mathrm{Hess}\, V_t(\varphi) \geq \cdots$$

This gives the Log-Sob. ineq. as before, but note that the constant now depends on L (it is still unif. in ε).

What goes wrong if $\beta > 4\pi$?

Consider $\tilde V_t(\xi_1,\xi_2) = \int_0^t e^{-(W_t - W_s)(\xi_1,\xi_2)}\, \tilde V_s(\xi_1)\, \hat V_s(\xi_2)\, \dot u_s(\xi_1,\xi_2)\, ds$

$$\ell_t^2\, \tilde V_t(\xi_1,\xi_2) = |z_t|^2 \underbrace{\left(1 - e^{-\beta\sigma_1\sigma_2 C_t(x_1,x_2)}\right)}_{\text{if } \sigma_1\sigma_2 = -1 \text{ then } \approx \left(\frac{|x_1 - x_2|}{\ell_t}\right)^{-\beta/2\pi}} \qquad \xi_i = (x_i,\sigma_i)$$

$\Rightarrow$ $\ell_t^2 \sup_x \sum_y |N_t((x,+1),(y,-1))|$ is not bd. in ε.

$\Rightarrow$ Hess V_t^2 is not bd. below.

2-part. contr.

But we need $Q_t \,\mathrm{Hess}\, V_t^2\, Q_t$.

Open problems:

- Conj. Mayer exp. for the Yukawa gas converges up to $\beta < 8\pi(1-\frac{1}{2n})$ if the first n coefficients are removed.
- Continuum φ^4? $(d=2,3)$
- Lattice sine-Gordon? $(\beta \geq 8\pi)$
- Lattice critical φ^4? $(d \geq 4)$
- Lattice critical Ising? $(d \geq 4)$

$\vdots$