Applied Probability (Lent 2021)

Primary references:
N. Berestycki and P. Sousi, Applied Probability, Lecture notes from previous years
J. Norris, Markov Chains, Cambridge University Press
G. Grimmett and D. Stirzaker, Probability and Random Processes, Oxford University Press

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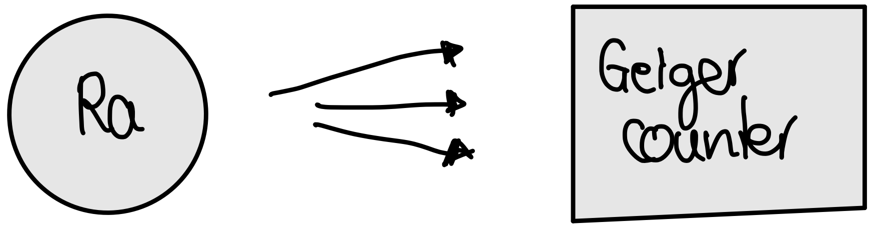
April 25, 2021

1. Poisson and birth processes

1.1. Poisson process



$N(t)$

Defn. A Poisson process with intensity (or rate) λ is a random process $N = (N(t) : t \geq 0)$ taking values in $\{0, 1, 2, \dots\}$ such that

(a) $N(0) = 0$, $\quad N(s) \leq N(t)$ if $s < t$;

(b) $$\mathbb{P}(N(t+h) = n+m \mid N(t) = n) = \begin{cases} \lambda h + o(h) & (m=1) \\ o(h) & (m>1) \\ 1 - \lambda h + o(h) & (m=0) \end{cases}$$

(c) if $s<t$, then $\underbrace{N(t) - N(s)}_{\text{events in } (s,t]}$ is independent of $\underbrace{N(s)}_{\text{events in } [0,s]}$

Existence: later

Thm. $N(t)$ has Poisson(λt) distribution:

$$\underbrace{\mathbb{P}(N(t) = n)}_{p_n(t)} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \qquad (+)$$

Proof.

$$p_n(t+h) = \mathbb{P}(N(t+h)=n) = \sum_m \overbrace{\mathbb{P}(N(t+h)=n \mid N(t)=m)}^{n-m \text{ arrivals in } (t,t+h]} \, \mathbb{P}(N(t)=m)$$

$$= \mathbb{P}(N(t)=n-1)(\lambda h + o(h)) + \mathbb{P}(N(t)=n)(1-\lambda h + o(h)) + o(h)$$

We'll be more careful about the infinite sum in the general setting (Section 2)

$$= p_{n-1}(t)\lambda h + p_n(t)(1-\lambda h) + o(h)$$

$$\Rightarrow \frac{p_n(t+h)-p_n(t)}{h} = p_{n-1}(t)\lambda - p_n(t)\lambda + o(1)$$

$$\left.\begin{aligned} \Rightarrow p_n'(t) &= \lambda(p_{n-1}(t) - p_n(t)) \quad \text{if } n \geq 1. \\ p_0'(t) &= -\lambda p_0(t) \end{aligned}\right\} (*)$$

Together with $p_n(0) = \delta_{n0}$ the system has a unique solution.

Method A: Induction

$p_0' = -\lambda p_0$, $p_0(0) = 1 \Rightarrow p_0(t) = e^{-\lambda t} \Rightarrow (+)$ with $n=0$

Assume $(+)$ holds for some n. Then

$$p_{n+1}' = -\lambda\Big(p_{n+1} - \underbrace{\frac{(\lambda t)^n}{n!}e^{-\lambda t}}_{p_n(t)}\Big), \quad p_{n+1}(0)=0$$

has unique solution $p_{n+1}(t) = \frac{(\lambda t)^{n+1}}{(n+1)!}e^{-\lambda t}$.

Method B: Generating function.

Let $G(s,t) = \sum_{n=0}^{\infty} p_n(t)\, s^n = \mathbb{E}(s^{N(t)})$, $s \in [0,1)$.

$$\Rightarrow \frac{\partial}{\partial t} G(s,t) = \sum_{n=0}^{\infty} p_n'(t)\, s^n = \sum_{n=0}^{\infty} \lambda\big(p_{n-1}(t) - p_n(t)\big)\, s^n$$

$$= \lambda s\, G(s,t) - \lambda G(s,t)$$

$$G(s,0) = 1$$

$$\Rightarrow G(s,t) = e^{-(1-s)\lambda t} = \sum_{n=0}^{\infty} \Big(\underbrace{\frac{(\lambda t)^n}{n!} e^{-\lambda t}}_{p_n(t)}\Big)\, s^n .$$

Let $T_0, T_1, \dots$ be the arrival times

$T_0 = 0$, $\quad T_n = \inf\{t : N(t) = n\}$

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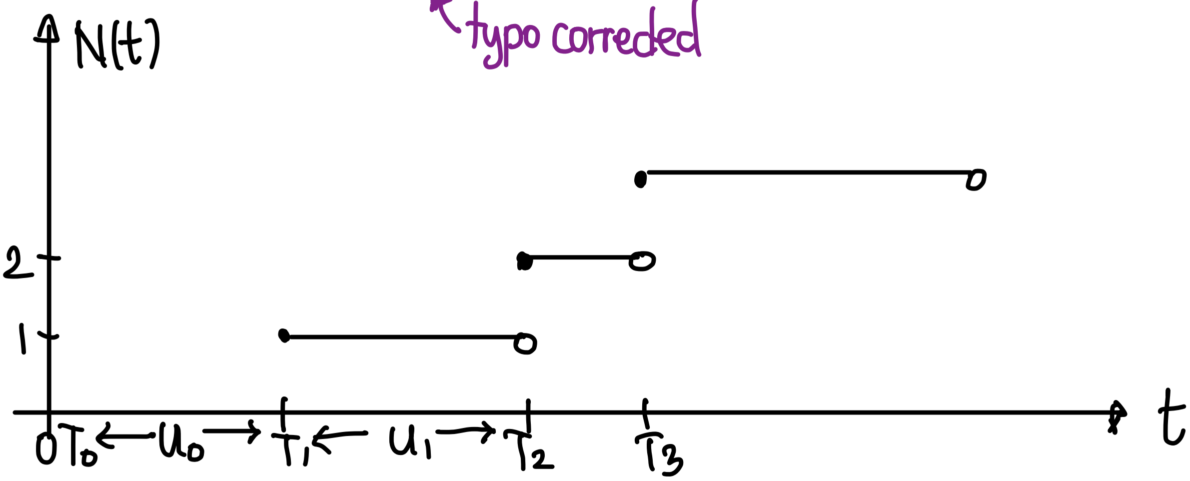


The inter arrival times are defined as

$$U_n = T_{n+1} - T_n$$

$$\Rightarrow T_n = \sum_{i=0}^{n-1} U_i, \qquad N(t) = \max\{n : T_n \leq t\}.$$

<u>Prop.</u> The $U_0, U_1, \dots$ are independent and each have $\mathrm{Exp}(\lambda)$ distribution.

<u>Proof.</u>

$$\mathbb{P}(U_0 > t) = \mathbb{P}(N(t) = 0) = e^{-\lambda t} \Rightarrow U_0 \sim \mathrm{Exp}(\lambda t)$$

$$\Rightarrow \mathbb{P}(U_1 > t \mid U_0 = t_0) = \mathbb{P}(\underbrace{N(t_0+t) - N(t_0)}_{\substack{N(t_0+\cdot)-N(t_0) \text{ again Poisson process}}} = 0)$$

$$= e^{-\lambda t} \Rightarrow \begin{array}{l} U_1 \sim \mathrm{Exp}(\lambda) \\ U_1 \perp U_0 \end{array}$$

The general case follows analogously by induction. ✓

<u>Exercise.</u> Let $U_0, U_1, \dots$ be i.i.d. $\mathrm{Exp}(\lambda)$. Then

$$T_n = \sum_{i=0}^{n-1} U_i, \qquad N(t) = \max\{n : T_n \leq t\}$$

defines a Poisson process of intensity λ.

<u>Exercise.</u> T_n is a $\Gamma(\lambda, n)$ random variable

$$\Rightarrow \mathbb{P}(N(t)=n) = \mathbb{P}(T_n < t < T_{n+1}) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

<u>Prop.</u> The distribution of $(T_1, \dots, T_n)$ conditional on $N(t)=n$ is the same as the order statistics of n uniform $[0,t]$ random variables.

<u>Proof.</u> By rescaling, WLOG, $t=1$.

$U = (U_0, \dots, U_{n-1})$ has density function

$$f(u) = \lambda^n \exp\left(-\lambda \sum_{i=0}^{n-1} u_i\right), \quad u \in [0,\infty)^n.$$

$T = (T_1, \dots, T_n)$ has density function

$$g(t) = \lambda^n \exp(-\lambda t_n)\, \mathbb{1}\{0 < t_1 < \cdots t_n\}$$

For any $A \in \mathbb{R}^n$, thus

$$\mathbb{P}(T \in A \mid N(1)=n) = \frac{\mathbb{P}(N(1)=n, T\in A)}{\mathbb{P}(N(1)=n)}$$

$$\mathbb{P}(N(1)=n, T\in A) = \int_A \underbrace{\mathbb{P}(N(1)=n \mid T=t)}_{\mathbb{P}(N(1)=n \mid T_n = t_n)} g(t)\, dt$$

$$= \mathbb{P}(U_n > 1-t_n)\, \mathbb{1}_{t_n \le 1}$$

$$= e^{-\lambda(1-t_n)}\, \mathbb{1}_{t_n \le 1}$$

$$\Rightarrow \mathbb{P}(T \in A \mid N(1)=n) = \frac{n!}{\lambda^n} e^{\lambda} \int_A e^{-\lambda(1-t_n)} \, \lambda^n e^{-\lambda t_n} \, \mathbb{1}\{0<t_1<\cdots<t_n<1\}$$

$$= n! \int_A \mathbb{1}\{0<t_1<\cdots<t_n\leq 1\}.$$

<u>Exercise.</u> The last RHS is the density function of the order statistics of n i.i.d. uniform $[0,1]$ r.v.

<u>1.2. Birth processes</u>

<u>Defn.</u> A birth process with intensities (or rates) λ_n is a random process $N=(N(t): t\geq 0)$ taking values in $\{0,1,2,\dots\}$ such that

(a) $N(s)\leq N(t)$ if $s<t$;

(b) $$\mathbb{P}(N(t+h)=n+m \mid N(t)=n) = \begin{cases} \lambda_n h + o(h) & (m=1) \\ o(h) & (m>1) \\ 1-\lambda_n h + o(h) & (m=0) \end{cases}$$

(c) if $s<t$, then $N(t)-N(s)$ is independent of $N(s)$

<u>Examples.</u> (i) Poisson process: $\lambda_n = \lambda \ \forall n$.

(ii) Simple birth: $\lambda_n = n\lambda$

Motivation: $N(t)$ individuals each give birth at rate λ.

$$\Rightarrow \mathbb{P}\left(\#\text{births in } (t,t+h) = m \mid N(t)=n\right)$$
$$= \binom{n}{m} (\lambda h + o(h))^m (1-\lambda h + o(h))^{n-m}$$
$$= \begin{cases} 1 - n\lambda h + o(h) & (m=0) \\ n\lambda h + o(h) & (m=1) \\ o(h) & (m>1) \end{cases}$$

(iii) Simple birth with immigration: $\lambda_n = n\lambda + \nu$

Let T_n be the time of the n-th arrival.

$$T_\infty = \lim_{n\to\infty} T_n \in [0,+\infty]$$

<u>Defn</u>. The process N is non-explosive (or honest) if
$$\mathbb{P}(T_\infty = +\infty) = 1.$$

<u>Thm</u>. For a birth process with rates $\lambda_n > 0$,
$$\mathbb{P}(T_\infty = +\infty) = \begin{cases} 1 & \text{if } \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = +\infty \\ 0 & \text{if } \sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty. \end{cases}$$

Thus it is non-explosive iff $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = +\infty$

<u>Proof</u> (i) Assume $\sum \frac{1}{\lambda_n} < \infty$.

$$\Rightarrow \mathbb{E}\, T_\infty = \mathbb{E}\Big(\sum_{i=0}^{\infty} U_i\Big) = \sum_{i=0}^{\infty} \mathbb{E}(U_i) = \sum_{i=0}^{\infty} \frac{1}{\lambda_i} < \infty.$$

$U_i \sim \mathrm{Exp}(\lambda_i)$
(analogous to Poisson process).

$$\Rightarrow \mathbb{P}(T_\infty = +\infty) = 0$$

(ii) Assume $\sum \frac{1}{\lambda_n} = \infty$.

Define $e^{-T_\infty} = \lim_{n\to\infty} e^{-T_n} \in [0,1]$.

$$\Rightarrow \mathbb{E}(e^{-T_\infty}) = \mathbb{E}\Big(\prod_{i=0}^{\infty} e^{-U_i}\Big)$$

monotonicity

$$= \lim_{n\to\infty} \mathbb{E}\Big(\prod_{i=0}^{n} e^{-U_i}\Big)$$

independence

$$= \lim_{n\to\infty} \prod_{i=0}^{n} \underbrace{\mathbb{E}(e^{-U_i})}_{\frac{1}{1+1/\lambda_i}} = \prod_{i=0}^{\infty} \frac{1}{1+1/\lambda_i}$$

Since $\prod_{i=0}^{n} \left(1+\frac{1}{\lambda_i}\right) \geq 1 + \underbrace{\sum_{i=0}^{n} \frac{1}{\lambda_i}}_{\to\infty}$ (using $\lambda_i \geq 0\ \forall i$)

$$\Rightarrow \mathbb{E}(e^{-T_\infty}) = 0$$

$$\Rightarrow \mathbb{P}(T_\infty = +\infty) = \mathbb{P}(e^{-T_\infty} = 0) = 1 \quad (\text{since } e^{-T_\infty} \geq 0)$$

Let $p_{n,m}(t) = \mathbb{P}(X(t)=m \mid X(0)=n)$.

<u>Prop.</u> For $m<n$, $p_{n,m}(t)=0$, and for $m \geq n$, $p_{n,m}(t)$ satisfies the following systems of ODEs:

(forward) $p'_{n,m}(t) = \lambda_{m-1} p_{n,m-1}(t) - \lambda_m p_{n,m}(t)$

(backward) $p'_{n,m}(t) = \lambda_n p_{n+1,m}(t) - \lambda_n p_{n,m}(t)$.

<u>Sketch.</u> For (forward), start from

$$p_{n,m}(t+h) = \sum_k \underbrace{\mathbb{P}(X(t)=k \mid X(0)=n)}_{p_{n,k}(t)} \underbrace{\mathbb{P}(X(t+h)=m \mid X(t)=k)}_{p_{k,m}(h)}$$

$$p_{k,m}(h) = 1_{k=m}(1-\lambda_m h) + 1_{k=m-1}\lambda_{m-1} h + o(h)$$

$\Rightarrow p'_{n,m}(t) = \lambda_m p_{n,m}(t) - \lambda_{m-1} p_{n,m-1}(t)$.

For (backward), start from

$$p_{n,m}(t+h) = \sum_k \underbrace{\mathbb{P}(X(h)=k \mid X(0)=n)}_{p_{n,k}(h)} \underbrace{\mathbb{P}(X(t+h)=m \mid X(h)=k)}_{p_{k,m}(t)}$$

$$p_{n,k}(h) = 1_{k=n}(1-\lambda_n h) + 1_{k=n+1}\lambda_n h + o(h)$$

and proceed analogously.

<u>Thm</u>. (forward) has a unique solution which satisfies (backward).

<u>Proof</u>. Existence and uniqueness to (forward) can be shown by induction:

$m < n$: $\quad p_{n,m}(t) = 0$

$m = n$: $\quad p_{n,n}'(t) = -\lambda_n p_{n,n}(t), \quad p_{n,n}(0) = 1$

$$\Rightarrow p_{n,n}(t) = e^{-\lambda_n t}$$

By induction, if the unique solution for $p_{n,m}(t)$ when $m = n+k$, substitute it into (forward) for $m = n+k+1$ to see that there is also a unique solution for $m = n+k+1$.

Alternatively, we could have studied the Laplace transform

$$\tilde{p}_{n,m}(\theta) = \int_0^\infty e^{-\theta t} p_{n,m}(t)\, dt.$$

(forward) $\quad p_{n,m}'(t) = \lambda_{m-1} p_{n,m-1}(t) - \lambda_m p_{n,m}(t)$

$$\Rightarrow \underbrace{\int_0^\infty e^{-\theta t} p_{n,m}'(t)\, dt}_{\theta \tilde{p}_{n,m}(t) - p_{n,m}(0)} = \lambda_{m-1} \tilde{p}_{n,m-1}(\theta) - \lambda_m \tilde{p}_{n,m}(\theta)$$

$$\Leftrightarrow \quad (\theta+\lambda_m)\tilde{p}_{n,m}(\theta) = \delta_{n,m} + \lambda_{m-1}\tilde{p}_{n,m-1}(\theta)$$

$$\Leftrightarrow \quad \tilde{p}_{n,m}(\theta) = \frac{\lambda_{m-1}}{\theta+\lambda_m}\,\frac{\lambda_{m-2}}{\theta+\lambda_{m-1}}\cdots\cdots\frac{1}{\theta+\lambda_n}\cdot \quad (m>n)$$

In principle, $p_{n,m}(t)$ can now be recovered by inverse Laplace transform.

Let $\pi_{n,m}(t)$ be a solution to (backward)

$$\tilde{\pi}_{n,m}(t) = \int_0^\infty e^{-\theta t}\pi_{n,m}(t)\,dt$$

$$\Rightarrow (\theta+\lambda_n)\tilde{\pi}_{n,m}(\theta) = \delta_{n,m} + \lambda_n\tilde{\pi}_{n+1,m}(\theta).$$

Now note that $\tilde{p}_{n,m}$ satisfies this equation. Inverting Laplace transforms, it follows that $p_{n,m}$ satisfies (backward).

<u>Remark</u> Uniqueness to (backward) may fail when there is explosion.

It is always true (without proof) that

$$p_{n,m}(t) \le \pi_{n,m}(t)$$

for p the unique solution to (forward) and π any solution to (backward).

Now when $\sum_m p_{n,m}(t) = 1$ this implies $\rho = \Pi$.

However, $\sum_m p_{n,m}(t) < 1$ is a possibility that corresponds to explosion ($\mathbb{P}(T_\infty = \infty) < 1$).

See Section 2.5 for more details.

2. Continuous-time Markov processes

From now on:

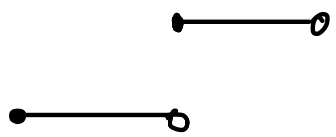
- I denotes a countable (or finite) state space;
- $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which all relevant random variables are defined.

2.1. Right-continuity and Markov property

<u>Defn</u> $X = (X(t) : t \geq 0)$ is a (right-continuous) random process with values in I if

(a) for every $t \geq 0$, $X(t)$ is a random variable $X(t) = X(t, \omega)$, $\omega \in \Omega$, with values in I;

(b) for every $\omega \in \Omega$, $t \mapsto X(t, \omega)$ is right-continuous
$\forall \omega\ \forall t\ \exists \varepsilon :\ X(t, \omega) = X(s, \omega)$ for $t \leq s \leq t + \varepsilon$.



<u>Example</u>. In the construction as
$$N(t) = \max\{n : T_n \leq t\}$$
the Poisson process is right-continuous.

Fact. (without proof). A right-continuous random process is determined by its finite-dimensional distributions:

$$P(X(t_0)=i_0, \dots, X(t_n)=i_n), \quad n \geq 0, t_k \geq 0, i_k \in I.$$

For a right-continuous random process, we can define as before the jump times

$$T_0 = 0, \qquad T_{n+1} = \inf\{t > T_n : X(t) \neq X(T_n)\}$$

and the holding times

$$U_n = \begin{cases} T_{n+1} - T_n & \text{if } T_n < \infty \\ +\infty & \text{if } T_n = +\infty. \end{cases}$$

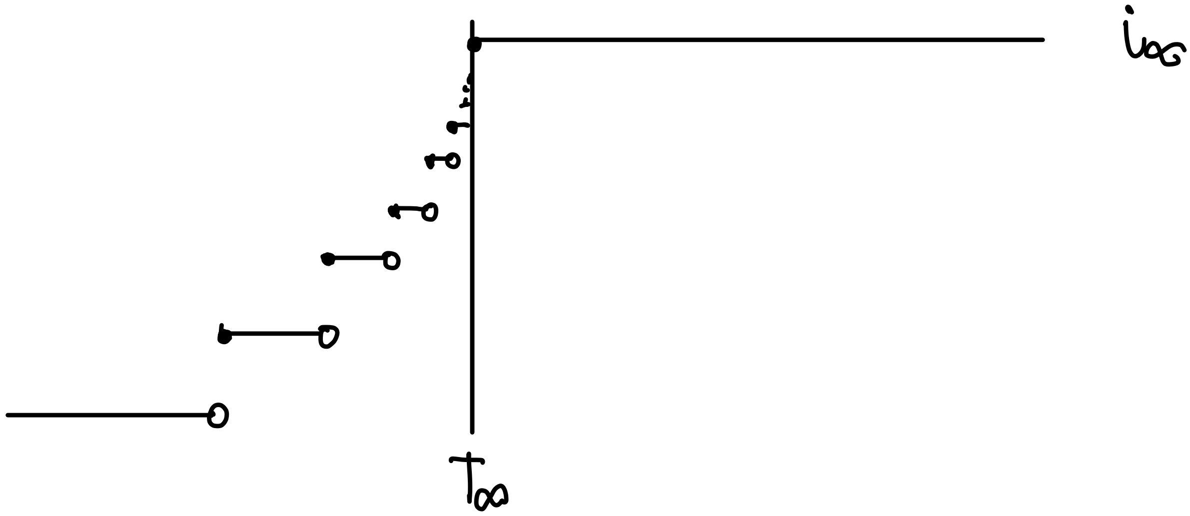
By right-continuity, $U_n > 0$ almost surely, but a process might explode. The explosion time is

$$T_\infty = \sup_n T_n = \sum_n U_n \in (0, +\infty]$$

Defn. A random process is minimal if

$$X(t) = i_\infty \quad \text{for } t > T_\infty$$

for some $i_\infty \in I$ (that we may adjoin to I).



<u>Defn</u>. A random process X has the Markov property (and is then called a Markov process) if

$$\mathbb{P}(X(t_n)=i_n \mid X(t_1)=i_1, \dots, X(t_{n-1})=i_{n-1})$$
$$= \mathbb{P}(X(t_n)=i_n \mid X(t_{n-1})=i_{n-1})$$

for all $i_1, \dots, i_n \in I$ and $t_1 < \cdots < t_n$.

<u>Rk</u>. For any $h>0$, $Y_n = X(hn)$ defines a discrete-time Markov process.

<u>Defn</u>. The transition probabilities are

$$p_{ij}(s,t) = \mathbb{P}(X(t)=j \mid X(s)=i), \quad s \le t, \ i,j \in I.$$

A Markov process is homogeneous if $p_{ij}(s,t) = p_{ij}(0,t-s)$ and we then write $p_{ij}(t-s)$.

Also write: $\mathbb{P}_i = \mathbb{P}(\cdot \mid X(0)=i)$ and $\mathbb{E}_i = \mathbb{E}(\cdot \mid X(0)=i)$.

From now on, all Markov processes will be homogeneous and as in the case of discrete-time Markov chain is then characterised by

- its initial distribution $\lambda_i = \mathbb{P}(X(0)=i)$, $i \in I$.
- its transition matrix $P(t) = (p_{ij}(t))_{i,j \in I}$.

<u>Thm.</u> $(P(t) : t \geq 0)$ is a Markov semigroup:

(a) $P(0) = I$ (identity)

(b) $P(t)$ is a stochastic matrix: $p_{ij}(t) \geq 0$, $\sum_{j \in I} p_{ij}(t) = 1$.

(c) $P(t+s) = P(t)P(s)$, $t, s \geq 0$

Chapman–Kolmogorov equations.

<u>Proof.</u> Identical to the discrete-time setting. E.g. (c)

$$p_{ij}(t+s) = \mathbb{P}(X(t+s)=j \mid X(0)=i)$$

$$= \sum_k \mathbb{P}(X(t+s)=j \mid \cancel{X(0)=i}, X(t)=k) \times \mathbb{P}(X(t)=k \mid X(0)=i)$$

Markov property

$$= \sum_k p_{kj}(s)\, p_{ik}(t)$$

2.2. Construction of Markov processes

Defn. $Q = (q_{ij})_{i,j \in I}$ is called a Q-matrix if

(a) $0 \le -q_{ii} < \infty$ for all i;

(b) $\infty > q_{ij} \ge 0$ for all $i \ne j$;

(c) $\sum_{j \in I} q_{ij} = 0$ for all i.

typo corrected

Write $q_i := -q_{ii} = +\sum_{j \ne i} q_{ij}$.

Example Let $x \in \mathbb{R}^I$. Then

$$(Qx)_i = \sum_{j \in I} q_{ij} (x_j - x_i).$$

Defn. Let Q be a Q-matrix. Then $\Pi = (\pi_{ij})_{i,j \in I}$ where

missing in video

$$\pi_{ij} = \begin{cases} \frac{q_{ij}}{q_i} 1_{i \ne j} & (q_i \ne 0) \\ 1_{i=j} & (q_i = 0) \end{cases}$$

is called the jump matrix associated with Q.

Note that Π is a stochastic matrix.

Defn. Let Q be a Q-matrix. Then a (minimal) random process X is a Markov process with generator Q if

(a) $Y_n := X(T_n)$ is a discrete-time Markov chain with transition matrix Π

(b) conditional on $Y_0, \dots, Y_n$, the holding time $U_n = T_{n+1} - T_n$ is independent with

$$U_n \sim \mathrm{Exp}(q_{Y_n}).$$

We write $X \sim \mathrm{Markov}(\lambda, Q)$ if $X(0) \sim \lambda$.

Example. Birth processes are $\mathrm{Markov}(\lambda, Q)$ with $I = \mathbb{N}$ and

$$q_{ij} = \lambda_i 1_{j=i+1} \qquad (j \neq i)$$

$$\Rightarrow \Pi_{ij} = 1_{j=i+1} \Rightarrow Y_n = Y_0 + n$$

Thm. Let X be $\mathrm{Markov}(\lambda, Q)$. Then X has the Markov property.

The proof requires measure theory ($\rightarrow$ Norris, Section 6.5).

The ingredients are:

- The Markov property (discrete-time) for the jump chain Y
- The memoryless property of the exponential distribution:

$$\mathbb{P}(E > t+s \mid E>s) = \mathbb{P}(E>t) \quad \forall s,t \geq 0$$

iff $E \sim \mathrm{Exp}(\lambda)$ for some $\lambda \geq 0$.

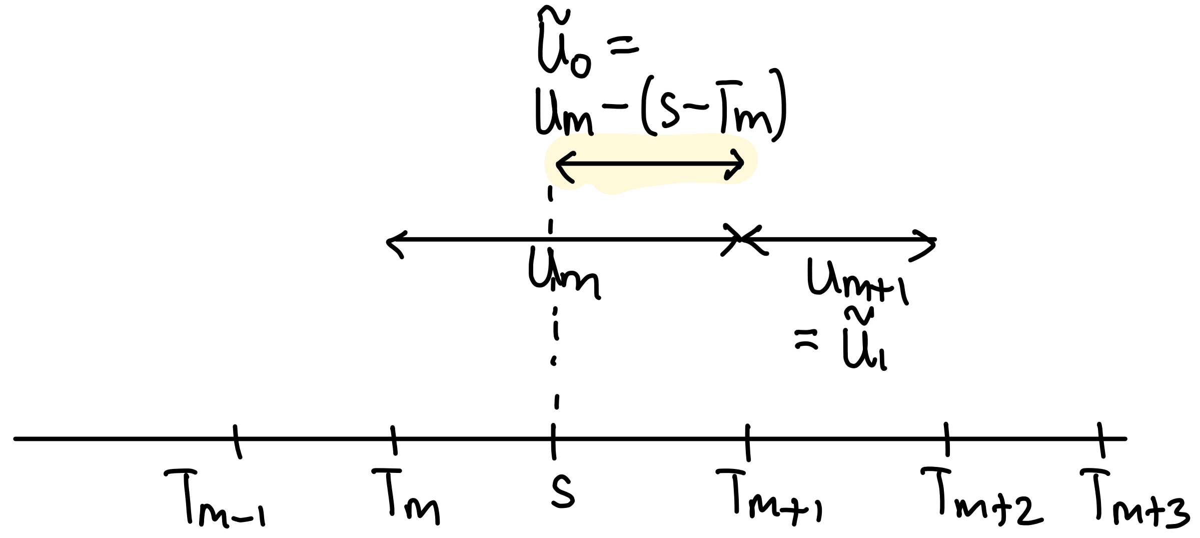
<u>Example</u>. (Poisson process). $I = \mathbb{Z}$, $q_{ij} = \lambda \mathbb{1}_{j=i+1}$.
Condition on $X(s) = m$ and $\tilde{X}(t) = X(s+t) - X(s)$
We will show that $\tilde{X}$ is again a Poisson process independent of $(X(t) : t<s)$ and thus the Markov property holds.

Indeed,

$$\{X_s = m\} = \{T_m \leq s < T_{m+1}\}$$
$$= \{T_m \leq s\} \cap \{U_m > s - T_m\}$$

The interarrival times for $\tilde{X}$ are

$$\tilde{U}_0 = U_m - (s - T_m)$$
$$\tilde{U}_n = U_{n+m}$$



Condition on $T_1, \dots, T_m$ and $\{X(s) = m\}$. Then (U_n) are i.i.d $\mathrm{Exp}(\lambda)$ $\Longrightarrow$ $(\tilde{U}_n)$ are i.i.d $\mathrm{Exp}(\lambda)$ and independent of $U_0, \dots, U_{m-1}$.

memoryless property for $\tilde{U}_0$

Condition only on $\{X(s) = m\}$. Then $(\tilde{U}_n)$ are still $\mathrm{Exp}(\lambda)$ and independent of $U_0, \dots, U_{m-1}$.

Thus conditional on $\{X(s) = m\}$,

$$\tilde{X}(t) = \max\{n : \tilde{T}_n \leq t\}$$

is again a Poisson process independent of $(X(t) : t < s)$.

Defn. A random variable T with values in $[0, +\infty]$ is a stopping time for X if

$\{T \leq t\}$ depends only on $(X(s) : s \leq t)$.

Thm. (Strong Markov property). Let X be Markov(λ, Q) and T a stopping time for X. Then conditional on $T < T_\infty$ and $X(T) = i$ the process $\tilde{X} = (X(T+t) : t \geq 0)$ is Markov(δ_i, Q) and independent of $(X(s) : s \leq T)$.

Constructions of a Markov(λ, Q) process:

Construction 1. Start with

(Y_n) a discrete-time Markov chain with transition propabilities Π, $Y_0 \sim \lambda$

(E_n) i.i.d. Exp(1)

Then set $U_n = E_n / q_{Y_n}$, $T_n = \sum_{i=0}^{n-1} U_i$

$N(t) = \max\{n : T_n \leq t\}$

$X(t) = Y_{N(t)}$

Construction 2. Start with

$(E_{n,i})_{n\geq 0, i\in I}$ i.i.d. $\mathrm{Exp}(1)$

$Y_0 \sim \lambda$, $T_0 = 0$

Inductively, given (Y_n, T_n) define

$$T_{n+1} = T_n + \underbrace{\inf_{j\neq Y_n} \underbrace{\frac{E_{n,j}}{q_{Y_n,j}}}_{\sim \mathrm{Exp}(q_{Y_n,j})}}_{\sim \mathrm{Exp}\left(\sum_{j\neq Y_n} q_{Y_n,j}\right) = \mathrm{Exp}(q_{Y_n})}$$

$$Y_{n+1} = \operatorname{argmin}\left\{ j\neq Y_n : \frac{E_{n,j}}{q_{Y_n,j}} \right\} \text{ if } q_{Y_n} > 0$$

$$Y_{n+1} = Y_n \text{ if } q_{Y_n} = 0.$$

<u>Exercise</u> (Example sheet). Let (U_k) be a sequence of independent $\mathrm{Exp}(q_k)$ random variables, where $0 < \sum q_k < \infty$. Then

(a) $U = \inf_k U_k \sim \mathrm{Exp}(\sum q_k)$

(b) The infimum is attained at a unique K almost surely, $\mathbb{P}(K=k) = q_k / \sum q_k$.

(c) U and K are independent.

Construction 3. Start with

$(N_{ij})_{i \neq j}$ are independent Poisson processes

$N_{ij} = (N_{ij}(t) : t \geq 0)$ with rate q_{ij}

$Y_0 \sim \lambda$

Then define inductively

$$T_{n+1} = \inf\{t > T_n : N_{Y_n j}(t) \neq N_{Y_n j}(T_n) \text{ for some } j \neq Y_n\}.$$

$$Y_{n+1} = j \quad \text{if} \quad T_{n+1} < \infty \text{ and } N_{Y_n j}(T_n) \neq N_{Y_n j}(T_{n+1})$$

$$= Y_n \quad \text{if} \quad T_{n+1} = \infty.$$

2.3. Explosion

For birth chains, we characterised when non-explosions happens. In general, the next theorem gives sufficient conditions.

Thm. Let X be Markov(λ, Q). Then $\mathbb{P}(T_\infty = \infty) = 1$ if any of the following conditions holds:

(a) I is finite;

(b) $\sup_{i \in I} q_i < \infty$;

(c) $X_0 = i$ and i is recurrent for the jump chain Y.

Proof. (b) Assume $\sup q_i = q < \infty$.

$\Rightarrow U_n \geq E_n / q$ where (E_n) are i.i.d. Exp(1)

By the strong law of large numbers (SLLN),

$$T_\infty = \sum_{n=0}^{\infty} U_n \geq \frac{1}{q} \sum_{n=0}^{\infty} E_n = \lim_{N \to \infty} \Big(\frac{N}{q} \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} E_n}_{\to 1 \text{ a.s.}} \Big)$$

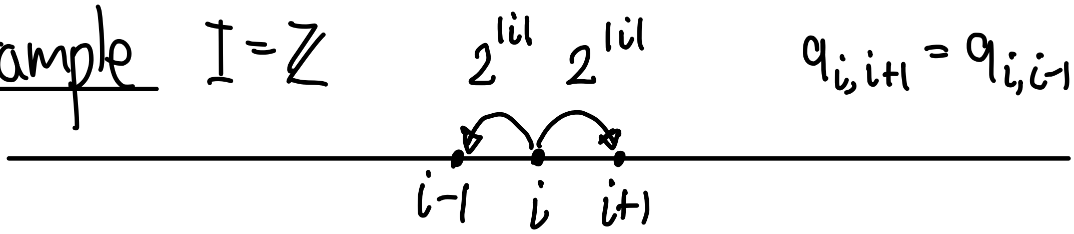
$$= +\infty \text{ a.s.}$$

(c) Assume that $X(0)=i$ and that i is recurrent for Y. Let S_n be the discrete time of the n-th return of Y to i.

By the SLLN,

$$T_\infty = \sum_{n=0}^{\infty} U_{S_n} \geq \frac{1}{q_i} \sum_{n=0}^{\infty} E_{S_n} = +\infty \quad \text{a.s.}$$

<u>Example</u> $I = \mathbb{Z}$



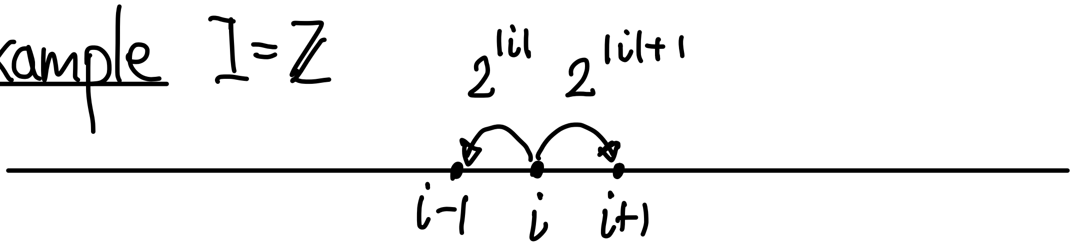
$q_{i,i+1} = q_{i,i-1} = 2^{|i|}$

$\Rightarrow$ Y is a symmetric SRW

$\Rightarrow$ Y is recurrent

$\Rightarrow$ no explosion

<u>Example</u> $I = \mathbb{Z}$



$\Rightarrow$ Y is a biased random walk with

$$\mathbb{P}(Y_{n+1} - Y_n = +1) = \frac{2}{3} = \left(\frac{2^{|i|+1}}{2^{|i|} + 2^{|i|+1}} \right)$$

$$\Rightarrow \mathbb{E} T_\infty = \mathbb{E}\left(\sum_{i=0}^{\infty} U_i \right) = \sum_{j \in \mathbb{Z}} \mathbb{E}\left(\sum_{k=1}^{V_j} U_{S_k^j} \right)$$

k-th visit to j

$$= \sum_{j\in\mathbb{Z}} \frac{1}{2^{|j|}\cdot 3} \underbrace{\mathbb{E} V_j}_{\substack{\leq 1 + \mathbb{E}_j V_j \\ = 1 + \mathbb{E}_0 V_0 < C}} \leq C \sum_{j\in\mathbb{Z}} \frac{1}{2^{|j|}\cdot 3} < \infty.$$

Y is transient

$\Rightarrow \mathbb{P}(T_\infty < +\infty) = 1.$

$\Rightarrow$ explosion.

2.4. Kolmogorov equation — finite state space

Assume I is finite. Let $M = (m_{ij})_{i,j \in I}$ be a matrix. Then the matrix exponential is defined by

$$e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$$

<u>Exercise</u>. If M_1 and M_2 are $I \times I$ matrices that commute. Then

$$e^{M_1 + M_2} = e^{M_1} e^{M_2}$$

<u>Exercise</u>. $P(t) = e^{tM}$ is the unique solution to

$$P'(t) = M P(t), \qquad P(0) = I$$

or

$$P'(t) = P(t) M, \qquad P(0) = I.$$

<u>Prop</u>. M is a Q-matrix iff e^{tM} is a stochastic matrix for all $t \geq 0$.

<u>Proof</u>. Assume e^{tM} is a stochastic matrix for all $t \geq 0$. Then

$$e^{tM} = I + tM + O(t^2)$$

implies:

Using $\sum_j (e^{tM})_{ij} = 1$,

$$1 = \sum_j (e^{tM})_{ij} = \underbrace{\sum_j \delta_{ij}}_{1} + \sum_j m_{ij}\, t + O(t^2), \quad t \to 0$$

$$\Rightarrow \sum_j m_{ij} = O(t) \quad \Rightarrow \quad \sum_j m_{ij} = 0$$

Using $(e^{tM})_{ij} \geq 0$,

$\Rightarrow \; m_{ij} \geq 0$ for $i \neq j$

$$\Rightarrow \; m_{ii} = -\sum_{j \neq i} m_{ij} \leq 0$$

$\Rightarrow$ M is a Q-matrix.

Conversely, assume M is a Q-matrix. Then

$$\sum_j m_{ij} = 0$$

$$\Rightarrow \sum_j (M^n)_{ij} = \sum_{j,k} (M^{n-1})_{ik}\, m_{kj} = 0$$

$$\Rightarrow \sum_j (e^{tM})_{ij} = \sum_j \left(\delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{n!} (M^n)_{ij} \right) = 1.$$

Finally, since $e^{tM} = (e^{(t/k)M})^k$, it suffices to show that $(e^{tM})_{ij} \geq 0$ for t small enough.

If $m_{ij} > 0$ for all $i \neq j$, this follows from

$$e^{tM} = I + tM + O(t^2).$$

In general, let $J_{ij} = -\delta_{ij} + \frac{1}{N}$. Then the above applies to $M + \delta J$, $\delta > 0$, and thus also

$$(e^{tM})_{ij} = \lim_{\delta \downarrow 0} (e^{t(M+\delta J)})_{ij} \geq 0.$$

<u>Thm</u>. Assume I is finite. Let X be a random process with values in I, and let Q be a Q-matrix. Then equivalently:

(a) X is Markov(λ, Q).

(b) for all $t, h \geq 0$, conditional on $X(t) = i$, $X(t+h)$ is independent of $(X(s) : s \leq t)$ and

$$\mathbb{P}(X(t+h) = j \mid X(t) = i) = \delta_{ij} + q_{ij} h + o(h)$$

(c) X has the Markov property with transition semigroup $P(t)$ given by $P(t) = e^{tQ}$:

$$\mathbb{P}(X(t_n) = i_n \mid X(t_{n-1}) = i_{n-1}, \dots, X(t_1) = i_1) = p_{i_{n-1} i_n}(t_n - t_{n-1})$$

for all $n \geq 0$, $t_n \geq \dots \geq t_1$, $i_1, \dots, i_n \in I$.

<u>Proof</u> (a) $\Rightarrow$ (b). The conditional independence follows from the Markov property (→ Section 2.2).

It suffices to consider $t=0$.

$$\mathbb{P}_i(X(h)=i) \geq \mathbb{P}_i(T_1>h) = e^{-q_i h} = 1+q_{ii}h+o(h)$$

$$\begin{aligned}
\mathbb{P}_i(X(h)=j) &\geq \mathbb{P}_i(T_1<h,\ T_2>h,\ Y_1=j) \\
(i\neq j) \quad &\geq \mathbb{P}_i(\underset{\uparrow\, U_0}{T_1}<h,\ \underset{\uparrow\, T_2-T_1}{U_1}>h,\ Y_1=j) \\
&= (1-e^{-q_i h})e^{-q_j h}\pi_{ij} = \underbrace{q_i\pi_{ij}}_{q_{ij}}h+o(h)
\end{aligned}$$

$$\Rightarrow \mathbb{P}_i(X(h)=j) \geq \delta_{ij}+q_{ij}h+o(h)$$

In fact, using $\sum_j \mathbb{P}_i(X(h)=j)=1$ and $\sum_j q_{ij}=0$, and that I is finite, the '$\geq$' are actually '$=$':

$$\sum_{j\neq i}\mathbb{P}_i(X(h)=j) \geq q_i h+o(h)$$

$$\Rightarrow \mathbb{P}_i(X(h)=i) = 1-\underbrace{\sum_{j\neq i}\mathbb{P}_i(X(h)=j)}_{\geq q_i h+o(h)} \leq 1+q_{ii}h+o(h)$$

Analogously,

$$\begin{aligned}
\mathbb{P}_i(X(h)=j) &= 1-\sum_{k\neq j}\mathbb{P}_i(X(h)=k) \\
&\leq 1-\sum_{k\neq j}(\delta_{ik}+q_{ik}h+o(h)) \\
&= \sum_{k\neq j}(-q_{ik}h)+o(h)
\end{aligned}$$

$$\underbrace{\qquad\qquad\qquad}_{+q_{ij}h}$$

$\Rightarrow$ (b) holds.

(b) $\Rightarrow$ (c).

$$p_{ij}(t+h) = \sum_k p_{ik}(t)\, p_{kj}(h)$$
$$= \sum_k p_{ik}(t)\left(\delta_{kj} + q_{kj}h + o(h)\right)$$

$$\Rightarrow \frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_k p_{ik}(t) q_{kj} + o(1)$$

uniform in t

Taking $h \downarrow 0$, we see that p_{ij} is right-differentiable.

Replacing t by $t-h$,

$$\frac{p_{ij}(t) - p_{ij}(t-h)}{h} = \sum_k p_{ik}(t-h) q_{kj} + o(1)$$

$\Rightarrow$ p_{ij} is left-continuous (since the RHS is bounded)

$\Rightarrow$ p_{ij} is left-differentiable

Together, p_{ij} is differentiable and

$$p_{ij}'(t) = \sum_k p_{ik}(t) q_{kj}, \quad p_{ij}(0) = \delta_{ij}.$$

$\Rightarrow \quad P'(t) = P(t)Q, \qquad P(0) = I$

$\Rightarrow \quad P(t) = e^{tQ} \qquad$ (since I is finite).

Also, the conditional independence given by (b) implies

$$\mathbb{P}(X(t_n)=i_n \mid X(t_{n-1})=i_{n-1}, \dots, X(t_1)=i_1)$$
$$= \mathbb{P}(X(t_n)=i_n \mid X(t_{n-1})=i_{n-1})$$
$$= P_{i_{n-1}i_n}(t_n - t_{n-1}).$$

(c) $\Rightarrow$ (a): later (for countable I).

<u>Example</u>.

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix} = U \underbrace{\begin{pmatrix} 0 & & \\ & -2 & \\ & & -4 \end{pmatrix}}_{D} U^{-1}$$

$$\Rightarrow e^{tQ} = \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!} = U \underbrace{\left(\sum_{n=0}^{\infty} \frac{(tD)^n}{n!} \right)}_{e^{tD}} U^{-1}$$

$$= U \begin{pmatrix} 1 & & \\ & e^{-2t} & \\ & & e^{-4t} \end{pmatrix} U^{-1}$$

$$\Rightarrow p_{11}(t) = a + be^{-2t} + ce^{-4t}$$

$$p_{11}(0) = 1, \quad p_{11}'(0) = q_{11} = -2, \quad p_{11}''(0) = (q^2)_{11} = 8$$

fix a, b, c.

2.5. Kolmogorov equations — countable state space

Thm. Let I now be countable. Assume that X is Markov(λ, Q) with associated transition semigroup $(P(t) : t \geq 0)$. Then

(a) $(P(t))$ is the minimal non-negative solution to

$$P'(t) = QP(t), \quad P(0) = I \qquad \text{(backward)}$$

(b) $(P(t))$ is the minimal non-negative solution to

$$P'(t) = P(t)Q, \quad P(0) = I \qquad \text{(forward)}$$

In particular, solutions to both equations exist.

Rk. If X explodes, then $X(t) = i_\infty$ for all $t \geq T_\infty$ and then

$$\underset{\displaystyle \sum_j p_{ij}(t)}{\overset{\displaystyle \sum_{j \neq i_\infty} p_{ij}(t) < 1}{\shortparallel}}$$

On the other hand, if X is non-explosive, then

$$\sum_j p_{ij}(t) = 1.$$

Since P is also the minimal solution to (forward) and (backward), it is the actually the unique solution to these equation.

Rk. If X is a minimal random process with values in I (countable) that X satisfies the Markov property and the associated transition semigroup is the minimal solution to (backward) or (forward) then X is Markov(λ, Q).

This follows from the theorem and the fact that the transition semigroup characterises the finite-dimensional distributions (exercise).

Rk. If $X(0) \sim \lambda$ then $X(t) \sim \lambda(t)$ where $\lambda(t) = \lambda P(t)$ satisfies

$$\lambda'(t) = \lambda(t) Q.$$

Proof. (a) We first show that $P'(t) = QP(t)$.

$$p_{ij}(t) = \mathbb{P}_i(X(t)=j) = \mathbb{P}_i(X(t)=j,\ T_1 > t)$$

$$+\ \mathbb{P}_i(X(t)=j,\ T_1 \leq t)$$

$\delta_{ij} e^{-q_i t}$

$$\sum_{k\neq i} P_i(X(t)=j,\ T_1\le t,\ X(T_1)=k)$$

$$= \sum_{k\neq i} \frac{q_{ik}}{\cancel{q_i}} \int_0^t \cancel{q_i}\, e^{-q_i s}\, P_{kj}(t-s)\, ds$$

$$\Rightarrow p_{ij}(t) = \delta_{ij}\, e^{-q_i t} + \int_0^t \sum_{k\neq i} q_{ik}\, P_{kj}(\underbrace{t-s}_{u})\, e^{-q_i s}\, ds$$

$$\Leftrightarrow e^{q_i t} p_{ij}(t) = \delta_{ij} + \int_0^t \sum_{k\neq i} q_{ik} P_{kj}(u)\, e^{+q_i u}\, du$$

This implies:

- $p_{ij}(t)$ is continuous in t
- $\sum_{k:k\neq i} q_{ik} \underbrace{P_{kj}(u)}_{\le 1}$ is a uniformly convergent sum of continuous functions, so also continuous.

$\Rightarrow p_{ij}$ is differentiable.

Thus the integral equation may be differentiated and

$$\cancel{e^{q_i t}}\left(q_i\, p_{ij}(t) + p_{ij}'(t)\right) = \cancel{e^{q_i t}} \sum_{k\neq i} q_{ik} P_{kj}(t)$$

$$\Leftrightarrow p_{ij}'(t) = \sum_{k\neq i} q_{ik} P_{kj}(t) - q_i \sum p_{ij}(t) = \sum_k q_{ik}\, p_{kj}(t)$$

$$\Leftrightarrow P'(t) = QP(t)$$

To see that P is the minimal solution to (backward), assume $\tilde{P}$ is another (non-negative) solution to (backward)

$$\Rightarrow \tilde{p}_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i s} \sum_{k \neq i} q_{ik} \tilde{p}_{kj}(t-s)\, ds$$

We will show by induction that

$$\mathbb{P}_i(X(t)=j,\ t < T_n) \leq \tilde{p}_{ij}(t) \quad \forall n.$$

Indeed:

$n=1$: $\mathbb{P}_i(X(t)=j,\ t<T_1) = \delta_{ij} e^{-q_i t}$ ✓

$n \to n+1$: follows from

$$\mathbb{P}_i(X(t)=j,\ t<T_{n+1}) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i s} \sum_{k \neq i} q_{ik} \mathbb{P}_k(X(t-s)=j,\ t-s<T_n)\, ds$$

$$\Rightarrow p_{ij}(t) = \mathbb{P}_i(X(t)=j,\ t<T_\infty)$$

X is minimal

$$= \lim_{n\to\infty} \mathbb{P}_i(X(t)=j,\ t<T_n) \leq \tilde{p}_{ij}(t)$$

$\Rightarrow P$ is minimal.

(b) To see that P also satisfies (forward), we will proceed similarly.

$$p_{ij}(t) = \mathbb{P}_i(X(t)=j) \overset{\uparrow}{=} \sum_{n=0}^{\infty} \underbrace{\mathbb{P}_i(X(t)=j,\ T_n \le t < T_{n+1})}_{(*)}$$

minimal proces

$n\geq 2$ (assume $q_j>0$ — the case $q_j=0$ is similar)

$$(*) \overset{\downarrow}{=} \sum_{i_1\neq i} \cdots \sum_{i_{n-1}\neq i_{n-2}} \mathbb{P}_i\left(T_n \le t < T_{n+1},\ Y_1=i_1,\dots,Y_{n-1}=i_{n-1}, Y_n=j\right)$$

$$= \sum_{i_1,\dots,i_{n-1}} \mathbb{P}_i\left(T_n\le t<T_{n+1} \mid Y_1=i_1,\dots,Y_{n-1}=i_{n-1}, Y_n=j\right) \mathbb{P}(Y_1=i_1,\dots,Y_{n-1}=i_{n-1}) \frac{q_{i_{n-1}j}}{q_{i_{n-1}}}$$

Lemma: (time reversal)
$$q_{i_n} \mathbb{P}(T_n \le t < T_{n+1} \mid Y_0=i_0,\dots,Y_n=i_n) = q_{i_0} \mathbb{P}(T_n\le t<T_{n+1} \mid Y_0=i_n,\dots,Y_0=i_0)$$

$$(*) = \sum_{i_1,\dots,i_{n-1}} \frac{q_i}{q_j} \mathbb{P}_j\left(T_n\le t<T_{n+1} \mid Y_1=i_{n-1},\dots,Y_{n-1}=i_1, Y_n=i\right) \mathbb{P}(Y_1=i_1,\dots,Y_{n-1}=i_{n-1}) \frac{q_{i_{n-1}j}}{q_{i_{n-1}}}$$

Markov prop.

$$\overset{\downarrow}{=} \sum_{i_1,\dots,i_{n-1}} \frac{q_i}{\cancel{q_j}} \int_0^t ds\, \cancel{q_j} e^{-q_j s} \mathbb{P}_{i_{n-1}}\left(T_{n-1}\le t-s<T_n \mid Y_1=i_{n-2},\dots,Y_{n-1}=i\right) \mathbb{P}(Y_1=i_1,\dots,Y_{n-1}=i_{n-1}) \frac{q_{i_{n-1}j}}{q_{i_{n-1}}}$$

time reversal

$$\overset{\downarrow}{=} \sum_{i_1,\dots,i_{n-1}} \cancel{q_{i_{n-1}}} \int_0^t ds\, e^{-q_j s} \mathbb{P}_i\left(T_{n-1}\le t-s<T_n \mid Y_1=i_1,\dots,Y_{n-1}=i_{n-1}\right) \mathbb{P}(Y_1=i_1,\dots,Y_{n-1}=i_{n-1}) \frac{q_{i_{n-1}j}}{\cancel{q_{i_{n-1}}}}$$

Undoing the conditioning of $Y_1, \dots, Y_{n-2}$, and renaming i_{n-1} to k,

$$(*) = \sum_{k \neq j} q_{kj} \int_0^t ds\, e^{-q_j s}\, \mathbb{P}_i(X(t)=k, T_{n-1} \leq t-s < T_n).$$

$$\Rightarrow p_{ij}(t) = \delta_{ij} e^{-q_i t} + \sum_{n=1}^{\infty} \int_0^t ds \sum_{k \neq j} \mathbb{P}_i(X(t)=k, T_{n-1} \leq t-s < T_n)\, q_{kj} e^{-q_j s}$$

$$= \delta_{ij} e^{-q_i t} + \int_0^t ds \sum_{k \neq j} p_{ik}(\underbrace{t-s}_{u})\, q_{kj} e^{-q_j s}$$

$$= \delta_{ij} e^{-q_j t} + \int_0^t du \sum_{k \neq j} p_{ik}(u)\, q_{kj} e^{-q_j (t-u)}$$

$$\Rightarrow p_{ij}(t) e^{q_j t} = \delta_{ij} + \int_0^t du \sum_{k \neq j} p_{ik}(u)\, q_{kj} e^{q_j u}$$

Since $e^{q_i t} p_{ij}(t)$ is increasing in t,

$\sum_{k \neq j} p_{ik}(u) q_{kj}$ converges uniformly on $[0, t]$

Since we already know from the study of the backward equation that $p_{ij}(t)$ is continuous, it again follows that the integrand is continuous.

$$\Rightarrow (p_{ij}'(t) + q_j p_{ij}(t))\, \cancel{e^{q_j t}} = \sum_{k \neq j} p_{ik}(t)\, q_{kj}\, \cancel{e^{q_j t}}$$

$$\Leftrightarrow p'_{ij}(t) = \sum_{k \neq j} p_{ik}(t) q_{kj} - q_j p_{ij}(t)$$

$$= \sum_k p_{ik}(t) q_{kj}$$

$$\Leftrightarrow P'(t) = P(t)\,Q.$$

The proof that P is minimal among solutions to (forward) is omitted; see Norris.

<u>Lemma</u>. $q_{i_n} \mathbb{P}(T_n \leq t < T_{n+1} \mid Y_0 = i_0, \ldots, Y_n = i_n)$

$$= q_{i_0} \mathbb{P}(T_n \leq t < T_{n+1} \mid Y_0 = i_n, \ldots, Y_n = i_0)$$

<u>Proof</u>. Conditional on $Y_0, \ldots, Y_n$, the holding time $U_0, \ldots, U_n$ are independent with

$$U_k \sim \mathrm{Exp}(q_{Y_k}) = \mathrm{Exp}(q_{i_k}).$$

$$\Rightarrow \text{LHS} = q_{i_n} \mathbb{P}(U_n > t - U_0 - \cdots - U_{n-1} \geq 0)$$

$$= q_{i_n} \int_{\Delta(t)} e^{-q_{i_n}(t - u_0 - \cdots - u_{n-1})} \prod_{k=0}^{n-1} q_{i_k} e^{-q_{i_k} u_k} du_k$$

$$\Delta(t) = \{ u_0, \ldots, u_{n-1} \geq 0 : u_0 + \cdots + u_{n-1} \leq t \}$$

Now change variables: $\hat{\tilde{u}}_0 = t - u_0 - \cdots - u_{n-1}$

$$\hat{\tilde{u}}_k = u_{n-k} \quad (k = 1, \dots, n-1)$$

$$\hat{\tilde{\Delta}}(t) = \{ \hat{\tilde{u}}_0 + \cdots + \hat{\tilde{u}}_{n-1} \le t \}$$

$$\Rightarrow \text{LHS} = \int \underbrace{q_{i_n} e^{-q_{i_n} \hat{\tilde{u}}_0} \prod_{k=1}^{n} q_{i_{n-k}} e^{-q_{i_{n-k}} \hat{\tilde{u}}_k}}_{\prod_{k=0}^{n-1} q_{i_{n-k}} e^{-q_{i_{n-k}} \hat{\tilde{u}}_k} \quad q_{i_0} e^{-q_{i_0}(t - \hat{\tilde{u}}_0 - \cdots - \hat{\tilde{u}}_{n-1})}} d\hat{\tilde{u}}_0 \cdots d\hat{\tilde{u}}_{n-1}$$

$$= \text{RHS}.$$

<u>Rk.</u> We could assume WLOG that $q_0, \dots, q_{i_n} > 0$ — otherwise both sides are 0.

However if $q_{i_n} = 0$, then one can find a similar statement that replaces the application of the lemma. E.g.

$$\mathbb{P}_{i_0}(T_n \le t < T_{n+1} \mid Y_1 = i_1, \dots, Y_{n-1} = i_{n-1}, Y_n = i_n)$$

$$= q_{i_0} \int_0^t ds\, \mathbb{P}_{i_{n-1}}(T_{n-1} \le t - s < T_n \mid Y_1 = i_{n-2}, \dots, Y_{n-1} = i_0).$$

3. General properties of Markov processes

3.1. Communicating classes

fixed terminology to match disc. time one

Defn. For states $i, j \in I$, write $i \to j$ (i leads to j) if $\mathbb{P}_i(X(t) = j$ for some $t > 0) > 0$ and $i \leftrightarrow j$ (i communicates with j) if $i \to j$ and $j \to i$.

Also define communicating classes, irreducibility, closed classes, and absorbing states exactly as in the discrete-time setting.

Prop. Let X be Markov(Q). Then equivalently:

(a) $i \to j$;

(b) $i \to j$ for the jump chain;

(c) $q_{i_0 i_1} \cdots q_{i_{n-1} i_n} > 0$ for some $i_0 = i, \dots, i_n = j$;

(d) $p_{ij}(t) > 0$ for all $t > 0$;

(e) $p_{ij}(t) > 0$ for some $t > 0$.

Proof. (d) $\Rightarrow$ (e) $\Rightarrow$ (a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (c): $i \to j$ for the jump chain

$\Rightarrow \exists\, i_0 = i, \dots, i_n = j : \pi_{i_0 i_1} \cdots \pi_{i_{n-1} i_n} > 0$

$$\frac{q_{i_0 i_1}}{q_{i_0}} \ (q_{i_0} \neq 0) \quad \text{or} \quad \cancel{1_{i_0 i_1} \ (q_{i_0} = 0)}$$

since product > 0

$\Rightarrow q_{i_0 i_1} \cdots q_{i_{n-1} i_n} > 0 \Rightarrow$ (c)

(c) $\Rightarrow$ (d): for any $k \neq \ell \in I$, $t > 0$,

$$q_{k\ell} > 0 \Rightarrow p_{k\ell}(t) \geq \mathbb{P}_k(T_1 < t,\ Y_1 = \ell,\ U_1 > t)$$
$$= (1 - e^{-q_k t}) \frac{q_{k\ell}}{q_k} e^{-q_\ell t} > 0$$

Since $P(t) = P(t/n)^n$, (c) implies

$$p_{ij}(t) \geq p_{i_0 i_1}(t/n) \cdots p_{i_{n-1} i_n}(t/n) > 0.$$

3.2. Recurrence and transience

<u>Defn</u>. The state i is

recurrent for X if $\mathbb{P}_i(\{t \geq 0 : X(t) = i\} \text{ is unbounded}) = 1$

transient for X if $\mathbb{P}_i(\{t \geq 0 : X(t) = i\} \text{ is unbounded}) = 0$.

<u>Thm</u>. Let X be Markov(Q) with jump chain Y.

(a) If $q_i = 0$ then i is recurrent for X.

(b) If $q_i > 0$ then i is recurrent for X iff it is for Y.

and the same holds for transience.

Proof. (a) If $q_i = 0$, then $X(t) = i$ for all $t \geq 0$ if $X(0) = i$. So recurrence is trivial.

(b) Assume i is transient for Y. Then almost surely there is a last visit to i:

$$N = \sup\{n : Y_n = i\} < \infty$$

and $T_{N+1} < \infty$. Since $t \in \{s : X(s) = i\}$ implies $t \leq T_{N+1}$, and the set $\{s : X(s) = i\}$ must be bounded.

Assume i is recurrent for Y. Then X cannot explode (as seen before): $\mathbb{P}(\underbrace{T_\infty = \infty}_{T_n \to \infty}) = 1$.

Since $X(T_n) = Y_n$ and Y_n visits i infinitely many times, the set $\{t : X(t) = i\}$ must be unbounded.

Cor. Assume $q_i > 0$. Then

i is recurrent $\Leftrightarrow \int_0^\infty p_{ii}(t)\,dt = +\infty$

i is transient $\Leftrightarrow \int_0^\infty p_{ii}(t)\,dt < \infty$

Proof. $$\int_0^\infty p_{ii}(t)\,dt = \int_0^\infty \mathbb{E}_i(1_{X(t)=i})\,dt$$
$$= \mathbb{E}_i\left(\int_0^\infty 1_{X(t)=i}\,dt\right)$$
$$= \mathbb{E}_i\left(\sum_{n=0}^\infty U_n 1_{Y_n=i}\right)$$
$$= \sum_{n=0}^\infty \frac{1}{q_i} \underbrace{\pi_{ii}(n)}_{(\Pi^n)_{ii}} = \frac{1}{q_i} \sum_{n=0}^\infty \pi_{ii}(n)$$

By the corresponding result for discrete-time Markov chains, the RHS is finite iff Y is transient.

3.3. Hitting times

For $A \subset I$, set $H_A = \inf\{t \geq 0 : X(t) \in A\}$
$$h_i^A = \mathbb{P}_i(H_A < \infty)$$
$$k_i^A = \mathbb{E}_i H_A$$

Thm $(h_i^A)_{i\in I}$ and $(k_i^A)_{i\in I}$ are the minimal solutions to

$$(h)\begin{cases} h_i^A = 1 & (i \in A) \\ (Qh^A)_i = 0 & (i \notin A) \end{cases}$$

respectively

$$(k)\begin{cases} k_i^A = 0 & (i\in A) \\ (Qk^A)_i = -1 & (i\notin A) \end{cases}$$

<u>Proof</u>. (h) The hitting probabilities are the same as those for the jump chain. For $i\notin A$, these satisfy

$$h_i^A = \sum_{j\neq i} \pi_{ij} h_j^A \iff \sum_j q_{ij} h_j^A = 0$$

(k) Clearly, $k_i^A = 0$ if $i\in A$. Let $i\notin A$. Then $H_A \geq T_1$ and

$$k_i^A = \mathbb{E}_i H_A = \mathbb{E}_i T_1 + \sum_{j\neq i} \underbrace{\mathbb{E}_i(H_A - T_1 \mid Y_1 = j)}_{\mathbb{E}_j H_A} \pi_{ij}$$

$$= \frac{1}{q_i} + \sum_{j\neq i} \frac{q_{ij}}{q_i} k_j^A$$

$$\iff \sum_j q_{ij} k_j^A + 1 = 0$$

<u>Rk</u>. The eqns (h) are the same as for the jump chain. The eqns (k) are similar to those for the jump chain but in general not the same.

3.4 Invariant distributions

If X is Markov(Q) with $X(0) \sim \lambda$ then $X(t) \sim \lambda P(t)$.

Defn. Suppose X is irreducible and non-explosive with transition semigroup $(P(t))$ and generator Q. Then a measure $\lambda = (\lambda_i)_{i \in I}$ is

- invariant if $\lambda P(t) = \lambda$ for all $t \geq 0$;
- infinitesimally invariant if $\lambda Q = 0$.

It is called an (infinitesimally) invariant distribution if in addition $\sum_i \lambda_i = 1$.

Exercise: If I is finite, show that λ is invariant iff it is infinitesimally invariant.

Lemma. Let Q be a Q-matrix with jump matrix Π. For any measure λ,

$$\lambda Q = 0 \quad \Leftrightarrow \quad \mu \Pi = \mu \text{ where } \mu_i = \lambda_i q_i .$$

Proof. By definition, $(\pi_{ij} - \delta_{ij}) q_i = q_{ij}$, so

$$(\mu \Pi - \mu)_j = \sum_i \underbrace{\mu_i}_{\lambda_i q_i} (\pi_{ij} - \delta_{ij}) = \sum_i \lambda_i q_{ij} = (\lambda Q)_j .$$

Lemma. Assume X is irreducible and recurrent.

(a) There is a unique (up to multiplication by scalars) measure λ s.t. $\lambda Q = 0$.

(b) There is at most one (up to multiplication by scalars) invariant measure.

Proof. (a) Assume $|I|>1$. Then $q_i > 0$ for all $i \in I$ by irreducibility. Thus also Π is irreducible and recurrent. By a result from Markov Chains, there exists a unique μ (up to multiplication) s.t.

$$\mu \Pi = \mu \iff \lambda Q = 0 \quad \text{for } \lambda_i = \frac{\mu_i}{q_i}$$

(b) The discrete time chain $Z_n^h = X(hn)$ for any fixed $h>0$ is recurrent. Indeed,

$$\infty = \int_0^\infty p_{ii}(t)\,dt = \sum_{n=0}^{\infty} \int_{hn}^{h(n+1)} \underbrace{p_{ii}(t)}_{\substack{\text{Markov property} \\ \text{(exercise)}}\ \le e^{hq_i} p_{ii}(h(n+1))} dt \le h e^{hq_i} \underbrace{\sum_{n=0}^{\infty} p_{ii}(h(n+1))}_{\infty}$$

Since (Z_n^h) is recurrent for any $h>0$, there is a unique measure λ s.t.

$$\lambda P(2^{-m} n) = \lambda \quad \text{for all } m,n \ge 0$$

Uniqueness follows immediately. [Existence can be obtained from

$$\lambda P(t) = \lambda \quad \text{for all } t \ge 0,\ t \text{ dyadic.}]$$

Lemma. Let Q be irreducible. Assume λ is a measure with $\sum \lambda_i \neq 0$ s.t. $\lambda Q = 0$. Then

$\lambda_j > 0$ for all j.

Proof. If $\lambda_j = 0$ for some j then

$$\underbrace{q_j}_{-q_{jj}} \lambda_j = \sum_{i: i\neq j} \lambda_i q_{ij} \Rightarrow \lambda_i = 0 \text{ for all } i \text{ s.t. } q_{ij} > 0$$

argument corrected

$\xRightarrow{\text{Induction}}$ $\lambda_k = 0$ for all k s.t. there are i_ℓ with $q_{ki_1} q_{i_1 i_2} \cdots q_{i_n j} > 0$

$\xRightarrow{\text{Irreducibility}}$ $\lambda_k = 0$ for all k

Lemma. Assume X is Markov(Q) and irreducible and recurrent. Let $i \in I$ and set

$$\mu_j^i = \mathbb{E}_i\left(\int_0^{R_i} 1_{X(s)=j} \, ds \right) = \frac{\gamma_j^i}{q_j}$$

where $R_i = \inf\{t > T_1 : X(t) = i\}$. Then

$$\mu^i Q = 0 \quad \text{and} \quad \mu^i P(t) = \mu^i \text{ for all } t.$$

Proof. (a) Since X is irreducible and recurrent, it is non-explosive. Moreover,

$$\sum_{j \in I} \mu_j^i = \mathbb{E}_i(R_i)$$

$$\mu^i_j = \mathbb{E}_i\left(\sum_{n=0}^{\infty} U_n \mathbb{1}_{Y_n=j}\mathbb{1}_{n<N_i}\right)$$

N_i = # of steps to return to i

corrected

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{E}_i(U_n \mid Y_n = j)}_{\frac{1}{q_j}} \underbrace{\mathbb{P}(Y_n = j, n < N_i)}_{\mathbb{E}_i(\mathbb{1}_{Y_n=j}\mathbb{1}_{n\le N_i-1})}$$

$$= \frac{1}{q_j}\underbrace{\mathbb{E}_i\left(\sum_{n=0}^{N_i-1} \mathbb{1}_{Y_n=j}\right)}_{\gamma^i_j} = \frac{\gamma^i_j}{q_j}$$

γ^i_j = exp. # of steps spent at j between visits to i

Since $\gamma^i \Pi = \gamma^i$ by a Markov Chains result, we see $\mu^i Q = 0$.

To see that also $\mu^i P(t) = \mu^i$ for all $t > 0$:

$$\mu^i_j = \mathbb{E}_i\left(\int_0^{R_i} \mathbb{1}_{X(s)=j}\, ds\right)$$

$$= \underbrace{\mathbb{E}_i\left(\int_0^{t} \mathbb{1}_{X(s)=j}\, ds\right)}_{\mathbb{E}_i\left(\int_{R_i}^{R_i+t} \mathbb{1}_{X(s)=j}\, ds\right) \text{ by strong Markov property}} + \mathbb{E}_i\left(\int_t^{R_i} \mathbb{1}_{X(s)=j}\, ds\right)$$

$$= \mathbb{E}_i\left(\int_t^{R_i+t} \mathbb{1}_{X(s)=j}\, ds\right)$$

$$= \mathbb{E}_i \left(\int_t^\infty 1_{X(s)=j} \, 1_{\underbrace{s-t}_{u} \le R_i} \, ds \right)$$

$$= \mathbb{E}_i \left(\int_0^\infty 1_{X(u+t)=j} \, 1_{u \le R_i} \, du \right)$$

$$= \int_0^\infty \sum_{k \in I} \mathbb{P}_i \left(X(u) = k, \, u \le R_i \right) p_{kj}(t)$$

Markov at time u

$$= \sum_{k \in I} \mu_k^i \, p_{kj}(t)$$

$\Rightarrow \mu^i = \mu^i P(t)$ for any $t > 0$.

3.5. Positive recurrence and convergence to equilibrium

Defn. The state $i \in I$ is positive recurrent if

$$m_i = \mathbb{E}_i R_i < \infty \quad \text{or} \quad q_i = 0$$

where $R_i = \inf\{ t > T_1 : X(t) = i \}$.

Thm. Let be Markov(Q), irreducible, and non-explosive. Then:

(a) If some state is positive recurrent then all states are positive recurrent and

$$\lambda_i = \frac{1}{m_i q_i} \quad \text{for } i \in I$$

is the (unique) invariant distribution and also the (unique) distribution with $\lambda Q = 0$.

(b) If there is a distribution λ with $\lambda Q = 0$ then all states are positive recurrent.

(c) If there is an invariant distribution λ, then again all states are positive recurrent.

Proof. By irreducibility, $q_i > 0$ for all $i \in I$.

(a) Assume i is positive recurrent.

$$\Rightarrow \sum_j \mu_j^i = \mathbb{E}_i R_i = m_i < \infty$$

Let $\lambda_j = \frac{\mu_j^i}{m_i}$. Thus λ is a distribution.

By one of the lemmas, $\lambda Q = 0$ and λ is inv.

By another lemma, using recurrence, λ is in fact the unique measure s.t. $\lambda Q = 0$ and also the unique invariant measure.

Also

$$\mu_i^i = \mathbb{E}_i\Big(\underbrace{\int_0^{R_i} 1_{X(s)=i}\, ds}_{U_0}\Big) = \mathbb{E}_i U_0 = \frac{1}{q_i}$$

$$\Rightarrow \lambda_i = \frac{\mu_i^i}{m_i} = \frac{1}{m_i q_i}.$$

By uniqueness of invariant measure, for any $k \in I$,

$$\mu_j^k = C_k \mu_j^i \quad \text{where} \quad C_k \in (0, \infty)$$

$$\Rightarrow m_k = \sum_j \mu_j^k = C_k \sum_j \mu_j^i = C_k m_i < \infty$$

$\Rightarrow$ k is positive recurrent, for every $k \in I$.

(b) Assume $\lambda Q = 0$ and $\sum \lambda_i = 1$. Let $i \in I$ and set

$$\nu_j = \frac{\lambda_j q_j}{\lambda_i q_i}.$$

$\Rightarrow$ $\nu_i = 1$ and $\nu \Pi = \nu$ where Π is the jump matrix, by the first of the lemmas.

By a discrete-time Markov chains result,

$$\nu_j \geq \gamma_j^i \left(= \sum_n \mathbb{E}_i \left(1_{Y_n = j} 1_{n < N_i}\right)\right)$$

$$\Rightarrow m_i = \sum_j \mu_j^i = \sum_j \frac{\gamma_j^i}{q_j} \leq \sum_j \frac{\nu_j}{q_j} = \frac{1}{q_i \lambda_i} \underbrace{\sum_j \lambda_j}_{=1}$$

$\Rightarrow m_i < \infty$ since $q_i > 0$ and $\lambda_i > 0$

$\Rightarrow$ i is positive recurrent.

(c) Assume λ is an invariant distribution.

$\Rightarrow Z_n = X(n)$ is a recurrent discrete-time Markov chain

$\Rightarrow X$ is recurrent

Thus μ_j^i is an invariant measure for X

By uniqueness of invariant measures (for X irred, rec)

$$\lambda_j = C_i \mu_j^i \quad \text{for some } C_i \in (0, \infty)$$

Since λ is distribution,

$$1 = \sum_j \lambda_j = C_i \sum_j \mu_j^i = C_i m_i$$

$\Rightarrow m_i < \infty \Rightarrow i$ is positive recurrent.

Thm. Let X be Markov(Q), irreducible, and non-explosive.

(a) If there is an (inf.) invariant distribution λ, then
$$p_{ij}(t) \xrightarrow{(t\to\infty)} \lambda_j \quad \text{for all } i,j\in I.$$

(b) If there is no (inf.) invariant distribution, then
$$p_{ij}(t) \xrightarrow{(t\to\infty)} 0 \quad \text{for all } i,j\in I.$$

Lemma. Let X be Markov(Q). Then
$$|p_{ij}(t+h)-p_{ij}(t)| \le q_i h$$

Proof.
$$\begin{aligned}
|p_{ij}(t+h)-p_{ij}(t)| &= \Big|\sum_k p_{ik}(h)p_{kj}(t) - p_{ij}(t)\Big| \\
&= \Big|\underbrace{\sum_{k\neq i} p_{ik}(h)p_{kj}(t)}_{\ge 0} - \underbrace{(1-p_{ii}(h))p_{ij}(t)}_{\ge 0}\Big| \\
&\le 1-p_{ii}(h) \le P_i(T_1 \le h) \\
&= 1-e^{-q_i h} \\
&\le q_i h.
\end{aligned}$$

<u>Proof of theorem.</u> By irred., $q_i > 0$ for all i.

(a) If λ is an invariant distribution, it is also one for the discrete-time Markov chain $Z_n^h = X(hn)$ for an arbitrary $h>0$.

$\Rightarrow$ $p_{ij}(hn) \xrightarrow{n\to\infty} \lambda_j$ (by discrete-time M.C. result)

$\Rightarrow$ $|p_{ij}(hn) - \lambda_j| \leq \varepsilon$ for $n \geq n_0(h,\varepsilon)$

$$\Rightarrow |p_{ij}(t) - \lambda_j| \leq \underbrace{|p_{ij}(hn) - p_{ij}(t)|}_{\substack{\leq q_i |t-nh| \\ \leq q_i h \\ \text{for } n \text{ such that} \\ |t-nh| \leq h}} + \underbrace{|p_{ij}(hn) - \lambda_j|}_{\substack{\leq \varepsilon \\ \text{for } n \geq n_0}}$$

$$\Rightarrow \limsup_{t\to\infty} |p_{ij}(t) - \lambda_j| \leq q_i h + \varepsilon$$

Since $h>0$ and $\varepsilon>0$ are arbitrary, thus

$$\lim_{t\to\infty} p_{ij}(t) = \lambda_j.$$

(b) Essentially the same argument with $\lambda = 0$.

3.6. Reversibility

Thm. Let X be Markov(λ, Q), irreducible, and non-explosive, with invariant distribution λ. For any fixed $T>0$, set $\hat{X}(t) = X(T-t)$ for $t \le T$. Then $\hat{X}$ is Markov$(\lambda, \hat{Q})$ where

$$\hat{q}_{ij} = q_{ji} \frac{\lambda_j}{\lambda_i}$$

positive (strictly) because $\lambda Q = 0$

and $\hat{X}$ is also irreducible and non-explosive.

Proof. $\hat{Q}$ is a Q-matrix: clearly $\hat{q}_{ij} \ge 0$ for $i \ne j$,

$$\sum_j \hat{q}_{ij} = \sum_j \frac{\lambda_j}{\lambda_i} q_{ji} = \frac{1}{\lambda_i} (\lambda Q)_i = 0$$

corrected

Irreducibility of $\hat{Q}$ is also clear (noting that $\lambda_i > 0$ for all i by irreducibility of Q) and $\lambda \hat{Q} = 0$.

Define $\hat{p}_{ij}(t) = \frac{\lambda_j}{\lambda_i} p_{ji}(t)$. Then

$$\hat{p}_{ij}'(t) = \frac{\lambda_j}{\lambda_i} p_{ji}'(t) = \frac{\lambda_j}{\lambda_i} \sum_k p_{jk}(t) q_{ki}$$

forward eqn. for $P(t)$

$$= \sum_k \underbrace{\frac{\lambda_j}{\lambda_k} p_{jk}(t)}_{\hat{p}_{kj}(t)} \underbrace{\frac{\lambda_k}{\lambda_i} q_{ki}}_{\hat{q}_{ik}}$$

$$= \sum_k \hat{q}_{ik} \hat{p}_{kj}(t) = (\hat{Q}\hat{P}(t))_{ij}$$

$\Rightarrow \hat{P}$ satisfies the backward equation for $\hat{Q}$.

Claim: $\hat{P}$ is the minimal solution to $\hat{P}' = \hat{Q}\hat{P}$.

Let $\hat{T}$ be another be another solution: $\hat{T}' = \hat{Q}\hat{T}$.
Then set

$$T_{ij}(t) = \frac{\lambda_j}{\lambda_i} \hat{T}_{ji}(t)$$

$\Rightarrow T$ satisfies the forward equation for Q

$\Rightarrow T_{ij}(t) \geq P_{ij}(t)$ for all i, j, t.

$\Rightarrow \hat{T}_{ij}(t) \geq \hat{P}_{ij}(t)$ for all i, j, t.

Finally, note

$$\mathbb{P}(\hat{X}(t_0) = i_0, \dots, \hat{X}(t_n) = i_n)$$

$$= \mathbb{P}(X(T-t_0) = i_0, \dots, X(T-t_n) = i_n)$$

$$= \lambda_{i_n} \underbrace{p_{i_n i_{n-1}}(t_n - t_{n-1})}_{\frac{\lambda_{i_{n-1}}}{\lambda_{i_n}} \hat{p}_{i_{n-1} i_n}(t_n - t_{n-1})} \cdots p_{i_1 i_0}(t_1 - t_0)$$

$$= \lambda_{i_0} \hat{p}_{i_0 i_1}(t_1 - t_0) \cdots \hat{p}_{i_{n-1} i_n}(t_n - t_{n-1})$$

$\Rightarrow \hat{X}$ has transition semigroup $\hat{P}$.

That $\hat{X}$ does not explode follows from $\sum_j \hat{p}_{ij} = 1$.

<u>Defn</u>. Let Q be a Q-matrix and λ a measure. Then Q and λ are in detailed balance if

$$\lambda_i q_{ij} = \lambda_j q_{ji} \quad \text{for all } i,j.$$

<u>Prop</u>. If λ and Q are in detailed balance, then

$$\lambda Q = 0.$$

<u>Proof</u>. $\sum_i \lambda_i q_{ij} = \sum_i \lambda_j q_{ji} = \lambda_j \underbrace{\sum_i q_{ji}}_{=0} = 0.$

<u>Defn</u>. Let X be Markov(Q). Then X is reversible if for all $T>0$, $(X_t)_{t \leq T}$ and $(X_{T-t})_{t \leq T}$ have the same distribution.

<u>Prop</u>. Let X be Markov(λ, Q), irreducible, and non-explosive. Then X is reversible iff λ and Q are in detailed balance.

<u>Proof.</u> Q and λ are in detailed balance

$$\iff \hat{Q} = Q$$

(note that $\lambda Q = 0$ so $\lambda_i > 0$ by irreducibility)

$$\Rightarrow \hat{X} \overset{(d)}{=} X$$

theorem

For the reverse direction, assume X is reversible. Then λ is invariant, so $\lambda_i > 0$ for all i, and

$$\hat{Q} = Q$$

again by the theorem.

<u>Example.</u> A birth-death process is a Markov process on $I = \{0, 1, 2, \ldots\}$ with

$$q_{ij} = \begin{cases} \lambda_i & (j = i+1) \\ \mu_i & (j = i-1) \\ 0 & (|i-j| > 1) \end{cases}$$

for some $\lambda_i, \mu_i \geq 0$. For such a process, π satisfies $\pi Q = 0$ iff π and Q are in detailed balance, i.e.,

$$\pi_{j+1} \mu_{j+1} = \pi_j \lambda_j \text{ for all } j \geq 0.$$

<u>Proof</u>. Clearly, detailed balance implies $\pi Q = 0$.
Suppose $\sum_i \pi_i q_{ij} = 0$ for all $j \geq 0$.

$$(j \geq 1) \quad \sum_i \pi_i q_{ij} = 0 \Leftrightarrow \pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = \pi_j \overbrace{(\lambda_j + \mu_j)}^{-q_{ii}}$$

$$\Leftrightarrow \pi_{j+1}\mu_{j+1} - \pi_j\lambda_j = \pi_j\mu_j - \pi_{j-1}\lambda_{j-1}$$

$$(j = 0) \quad \sum_i \pi_i q_{ij} = 0 \Leftrightarrow \pi_{j+1}\mu_{j+1} = \pi_j \lambda_j$$

$$\pi_1 \mu_1 = \pi_0 \lambda_0$$

By induction, $\pi_{j+1}\mu_{j+1} = \pi_j \lambda_j$ for all $j \geq 0$.
These are the detailed balance equations.

4. Birth-death processes and Markovian queues

4.1. Birth-death processes

<u>Defn</u>. A birth-death process is a Markov process on $I = \{0, 1, 2, \dots\}$ with

$$q_{ij} = \begin{cases} \lambda_i & (j = i+1) \\ \mu_i & (j = i-1) \\ 0 & (|i-j| > 1) \end{cases}$$

for some $\lambda_i \geq 0$ and $\mu_i \geq 0$.

Last lecture, we showed that $\pi Q = 0$ iff

$$\pi_{j+1} \mu_{j+1} = \pi_j \lambda_j \quad \text{for all } j \geq 0.$$

so
$$\pi_j = \pi_0 \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}$$

<u>Thm</u>. Let X be a birth-death process.

(a) There is a distribution π s.t. $\pi Q = 0$ iff

$$0 < \sum_{j=0}^{\infty} \underbrace{\frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}}_{=1 \text{ if } j=0} < \infty$$

does not mean that π is invariant (finite-time sense)

(provided all $\mu_j > 0$ and $\lambda_j > 0$).

(b) The process X is non-explosive, and hence π from (a) is invariant if in addition

$$0 < \sum_{j=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} (\lambda_j + \mu_j) < \infty.$$

<u>Proof</u>. (a) By the discussion above the statement, we have shown that

$$\pi_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} \pi_0 \;, \quad \pi_0 = \frac{1}{\sum_{j=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}}$$

is the unique distribution with $\pi Q = 0$.

(b) To show that X is non-explosive, it suffices to show that the jump chain is recurrent.

But for this, it suffices to show that $\nu_j = \pi_j q_j$ is normalisable, i.e. $0 < \sum \nu_j < \infty$. Then ν is is an invariant distribution (after normalisation) for the jump chain. But since

$$\nu_j = \underbrace{\frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} \pi_0}_{\pi_j} \cdot \underbrace{(\lambda_j + \mu_j)}_{q_j}$$

this is exactly what we have assumed.

Example. Assume $q_i > 0$ and $\lambda_i = \lambda q_i$ and $\mu_i = \mu q_i$ with $\lambda + \mu = 1$. Then

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \frac{1}{q_j}$$

satisfies $\pi Q = 0$. It is normalisable if

$$\sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{q_j} < \infty.$$

But the theorem only implies that π is actually invariant (i.e., X is non-explosive) if

$$\sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j < \infty.$$

For example, if $q_j = 1$ for all j, both conditions hold iff $\lambda < \mu$.

On the other hand, consider the case $q_j = 2^j$ and assume that $\lambda/\mu \in (1,2)$.

In this case, the first condition holds, but the second does not. And indeed, the jump chain is a biased random walk on $\{0, 1, \dots\}$ and thus it is transient, and thus X is also transient.

Thus X is transient but it does have an infinitesimally invariant distribution, i.e. $\pi Q = 0$. If X was non-explosive, then actually π would be invariant and thus positive recurrent. This a contradiction, so we conclude that X has to explode.

Example (Simple death with immigration).

$$\lambda_n = \lambda, \qquad \mu_n = n\mu.$$

Then with $\rho = \lambda/\mu$, for all $n = 0, 1, \dots,$

$$P(X(t) = n) \longrightarrow \underbrace{\frac{\rho^n}{n!} e^{-\rho}}_{\pi_n}$$

Indeed, X is non-explosive with invariant distr.

$$\pi_n = \frac{\rho^n}{n!} e^{-\rho}.$$

Thus $p_{ij}(t) \longrightarrow \frac{\rho^n}{n!} e^{-\rho}$ by the limit theorem.

Example (Simple birth-death).

$$\lambda_n = n\lambda, \qquad \mu_n = n\mu.$$

Note that 0 is an absorbing state and thus we assume that $X(0)=i>0$.

Also, since $\lambda_0=0$, we cannot use the theorem to see that X is non-explosive.

Let $G(s,t)=\mathbb{E}_i(s^{X(t)})=\mathbb{E}_i(s^{X(t)}\underbrace{1_{X(t)\neq i_\infty}}_{=1_{t<T_\infty}})$.

where $s\in[0,1]$, $t\geq 0$.

Then $\dfrac{\partial G}{\partial t}=(\lambda s-\mu)(s-1)\dfrac{\partial G}{\partial s}$, $G(s,0)=s^i$.

Indeed,

$$G(s,t)=\sum_{j=0}^{\infty} p_{ij}(t)s^j,$$

so by the forward equation,

$$\frac{\partial G}{\partial t}=\sum_{j=0}^{\infty}\Big[p_{i,j-1}(t)\underbrace{\lambda(j-1)}_{q_{j-1,j}}+p_{i,j+1}(t)\underbrace{\mu(j+1)}_{q_{j+1,j}}-p_{i,j}(t)(\mu+\lambda)j\Big]s^j$$

$$=\lambda s^2\frac{\partial G}{\partial s}+\mu\frac{\partial G}{\partial s}-s(\mu+\lambda)\frac{\partial G}{\partial s}.$$

The unique solution (without proof) given by

$$G(s,t)=\begin{cases}\left(\dfrac{\lambda t(1-s)+s}{\lambda t(1-s)+1}\right)^i & (\mu=\lambda)\\[2ex] \left(\dfrac{\mu(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}{\lambda(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}\right)^i & (\mu\neq\lambda).\end{cases}$$

Conclusions:

- $G(1,t) = \lim_{s \uparrow 1} G(s,t) = 1$ for all t (and λ, μ)
  $\Rightarrow X$ is non-explosive (for all λ, μ).

- $\mathbb{E}_i X(t) = i e^{(\lambda-\mu)t}$

  Indeed, $\mathbb{E}_i X(t) = \lim_{s \uparrow 1} \frac{\partial}{\partial s} G(s,t)$

  $$= \lim_{s \uparrow 1} \left[i \left(\cdots \right)^{i-1} \frac{\partial}{\partial s} \left(\cdots \right) \right]$$

  $$= i e^{(\lambda-\mu)t}$$

  In particular, $\mathbb{E}_i X(t) \longrightarrow \begin{cases} 0 & (\lambda/\mu < 1) \\ \infty & (\lambda/\mu > 1) \end{cases}$

- $\mathrm{Var}_i X(t) = \begin{cases} 2i\lambda t & (\mu = \lambda) \\ i \frac{\lambda+\mu}{\lambda-\mu} e^{(\lambda-\mu)t} \left(e^{(\lambda-\mu)t} - 1 \right) & (\mu \neq \lambda) \end{cases}$

- $\underbrace{\mathbb{P}_i (X(t) = 0)}_{\text{extinction probability}} = G(0,t) \longrightarrow \begin{cases} 1 & (\lambda/\mu < 1) \\ \mu/\lambda & (\lambda/\mu > 1) \end{cases}$

4.2. M/M/k queues

M/M/k:

- 'Markovian arrival': customers arrive according to a Poisson process of rate λ
- 'Markovian service': service times are i.i.d. $\text{Exp}(\mu)$
- There are k servers.

Let $X(t)$ denote the queue length, a Markov process on $\{0,1,2,\dots\}$ with Q-matrix:

M/M/1: $\quad q_{i,i+1} = \lambda, \quad q_{i,i-1} = \mu$

M/M/∞: $\quad q_{i,i+1} = \lambda, \quad q_{i,i-1} = i\mu$

<u>Thm</u>. The length of an M/M/1 queue is

transient $\iff \rho > 1$ where $\rho = \lambda/\mu$
recurrent $\iff \rho \leq 1$
positive recurrent $\iff \rho < 1$

In the pos. rec. case, the invariant distribution is $\pi_n = (1-\rho)\rho^n$ and if $X(0) \sim \pi$ then the wait time for a customer is $\text{Exp}(\mu - \lambda)$.

Proof. The jump chain Y is a biased random walk on $\{0,1,2,\dots\}$ with reflection at 0:

$$\mathbb{P}(Y_{n+1} - Y_n = +1 \mid Y_n > 0) = \lambda/(\lambda+\mu)$$

$$\mathbb{P}(Y_{n+1} - Y_n = -1 \mid Y_n > 0) = \mu/(\lambda+\mu)$$

$$\mathbb{P}(Y_{n-1} - Y_n = +1 \mid Y_n = 0) = 1$$

Thus Y (and hence X) is transient iff $\lambda > \mu$.

Since $\sup_i q_i = \lambda + \mu < \infty$ there is no explosion.

Thus positive recurrence is equivalent to the existence of a distribution π with $\pi Q = 0$, i.e.,

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n < 1 \iff \rho = \lambda/\mu < 1.$$

So suppose $\rho < 1$ and $X(0) \sim \pi$. Then $X(t) \sim \pi$ and the wait time is

$$W = \sum_{i=1}^{X(t)+1} T_i$$

with $T_i \sim \mathrm{Exp}(\mu)$ and i.i.d. and independent of $X(t)$ by the Markov property. Also

$$X(t) + 1 \sim \mathrm{Geom}(\rho).$$

Exercise (Example sheet): $W \sim \mathrm{Exp}(\mu(1-\rho))$.

Rk. Similarly, $\mathbb{E}X(t) = \frac{1}{1-\rho} - 1 = \frac{\lambda}{\mu-\lambda}$

since $X(t)+1 \sim \mathrm{Geom}(\rho)$.

Thm. The length of an $M/M/\infty$ queue is positive recurrent for all $\mu>0$ and $\lambda>0$, with invariant distribution Poisson(ρ), $\rho=\lambda/\mu$.

Proof. There is a distribution π with $\pi Q=0$ iff

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right) \frac{1}{n!} < \infty$$

and

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} (\mu_n + \lambda_n) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right) \frac{1}{n!} (\lambda + n\mu) < \infty$$

so π is an invariant distribution and X non-explosive (by the theorem from last lecture). Also:

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \Rightarrow \pi \sim \mathrm{Poisson}(\rho).$$

Let A and D denote the arrival and departure processes associated with a queue, i.e. A increases by +1 if X does, but does not decrease, and D increases by +1 if X decreases, so

$$X(t) = X(0) + A(t) - D(t).$$

Rk. A is a Poisson process of rate λ.
(The easiest way to see this is by using 'Construction 3' when we constructed Markov processes.)

But in general D is not a Poisson process.

Rk. A Poisson process does not have an invariant distribution. Still it has the following time reversal property:

If N is a Poisson process, then for any $T>0$,

$$\hat{N}(t) = N(T) - N(T-t)$$

is again a Poisson process on $[0,T]$.

Indeed, conditioned on $N(T)=n$, the distribution ordered jump times is

$$\frac{n!}{T^n} \mathbb{1}\{0 \le t_1 < \cdots < t_n \le T\}$$

Thm (Burke's theorem). Consider an M/M/k queue with invariant distribution (so e.g. the M/M/1 queue with $\mu > \lambda$). At equilibrium, i.e. with $X(0) \sim \pi$, D is a Poisson process of rate λ and $X(t)$ is independent of $(D(s) : s \leq t)$.

Proof. The invariant distribution π satisfies detailed balance because X is a birth-death process. Thus X is reversible:

if $\hat{X}(t) = X(T-t)$

then $(\hat{X}(t) : t \leq T) \overset{D}{=} (X(t) : t \leq T)$.

$\Rightarrow$ Arrival process $\hat{A}$ of $\hat{X}$ is Poisson(λ).

But $\hat{A}(t) = D(T) - D(T-t)$.

Since the time reversal of a Poisson process on $[0,T]$ is a Poisson process on $[0,T]$, this shows that $(D(t) : t \leq T)$ is a Poisson process.

Since $T > 0$ is arbitrary, this determines the finite-dimensional distributions of D (and thus D). So D is a Poisson process.

4.3 Queues in tandem and Jackson networks

Queues in tandem: Suppose there is an $M/M/1$ queue with parameters λ and μ_1. After a customer is served, they immediately join a second queue with service rate μ_2.

Let X_1 and X_2 denote the lengths of the two queues. Thus $I = \{0, 1, \dots\}^2$ and

$$q_{(m,n),(m+1,n)} = \lambda$$

$$q_{(m,n),(m-1,n+1)} = \mu_1 \qquad (m \geq 1)$$

$$q_{(m,n),(m,n-1)} = \mu_2 \qquad (n \geq 1)$$

Thm. (X_1, X_2) is positive recurrent iff $\lambda < \mu_1$ and $\lambda < \mu_2$ and the invariant distribution is then

$$\pi_{m,n} = (1-\rho_1)\rho_1^m (1-\rho_2)\rho_2^n \qquad (\rho_i = \lambda/\mu_i).$$

Thus $X_1(t)$ and $X_2(t)$ are independent in equilibrium.

Proof 1. The rates are bounded, so (X_1, X_2) is non-explosive. Thus it suffices to check

$$\pi Q = 0$$

which is the case.

<u>Proof 2.</u> Note the marginal X_1 is an $M/M/1$ queue. Thus X_1 is positive recurrent iff $\lambda < \mu$, with invariant distribution

$$\pi^1(m) = (1-\rho_1)\rho_1^m.$$

By Burke's theorem, the departure process of X_1 is Poisson of rate λ. But the departure process of X_1 is the arrival process of X_2. So the marginal X_2 is also an $M/M/1$ queue with parameters λ and μ_2 and invariant distribution

$$\pi^2(n) = (1-\rho_2)\rho_2^n.$$

Independence: if $X_1(0) \sim \pi^1$ and $X_2(0) \sim \pi^2$ are independent then $X_1(t)$ and $X_2(t)$ are independent by Burke's theorem.

Careful: $X_1(t)$ and $X_2(t)$ at a fixed time t are independent at equilibrium, but the processes $(X_1(t) : t \geq 0)$ and $(X_2(t) : t \geq 0)$ are not.

Jackson networks: Consider N single server queues with rates λ_k and μ_k where $k=1,\dots,N$. After service, each customer in queue k moves to queue j with probability p_{kj} and exits with probability $p_{k0} = 1-\sum_{j\neq k} p_{kj}$.

We assume $p_{kk}=0$ and $p_{kj}>0$ for all $k\neq j$ (including $j=0$).

Thus $I = \{0,1,2,\dots\}^N$ and

$e_k = (0,\dots,0,1,0,\dots,0)$ (1 in position k)

$$q_{n,n+e_k} = \lambda_k$$

$$q_{n,n-e_k+e_j} = \mu_k p_{kj} \qquad (n_k \geq 1)$$

$$q_{n,n-e_k} = \mu_k p_{k0} \qquad (n_k \geq 1)$$

Traffic equations: $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_N) \in [0,\infty)^N$ satisfies

$$\bar{\lambda}_k = \lambda_k + \sum_{j\neq k} \bar{\lambda}_j p_{jk} \qquad (T)$$

Lemma. There is a (unique) solution to (T).

Proof. Let $p_{00}=1$. Then $P=(p_{kj})_{k,j=0}^N$ is a stochastic matrix. The corresponding Markov Chain $Z=(Z_n)$ is absorbing at 0. Thus the communicating class of $\{1,\dots,N\}$ is transient.

Thus $V_k = \sum_n 1_{Z_n = k}$ has $\mathbb{E} V_k < \infty$ for all $k \in \{1, \dots, N\}$.

Assume $\mathbb{P}(Z_0 = k) = \frac{\lambda_k}{\lambda}$, $\lambda = \sum_{i=1}^N \lambda_i$. Then

$$\mathbb{E} V_k = \underbrace{\mathbb{P}(Z_0 = k)}_{\frac{\lambda_k}{\lambda}} + \sum_{n=0}^{\infty} \underbrace{\mathbb{P}(Z_{n+1} = k)}_{\sum_{j=1}^N \mathbb{P}(Z_n = j, Z_{n+1} = k)}$$

$$= \sum_{j=1}^N \mathbb{P}(Z_n = j) p_{jk}$$

$$= \frac{\lambda_k}{\lambda} + \sum_{j=1}^N (\mathbb{E} V_j) p_{jk}$$

Thus if $\bar{\lambda}_k = \lambda \mathbb{E} V_k$ then $\bar{\lambda}$ satisfies (T).

Uniqueness: Example sheet.

<u>Thm</u>. (Jackson) Assume (T) has a solution with $\bar{\lambda}_k < \mu_k$ for all $k = 1, \dots, N$. Then the Jackson network is positive recurrent with invariant distribution

$$\pi(n) = \prod_{k=1}^N (1 - \bar{\rho}_k) \bar{\rho}_k^{\,n_k}, \quad \bar{\rho}_k = \frac{\bar{\lambda}_k}{\mu_k}.$$

At equilibrium, the departure processes (to outside) from each queue are independent Poisson processes of rates $\lambda_i p_{i0}$.

<u>Lemma</u>. (Partial detailed balance). Let X be a Markov process on I and let π be a measure on I. Assume that for each $i \in I$ there is a partition

$$I \setminus \{i\} = I_1^i \cup I_2^i \cup \cdots$$

such that for all k

$$\sum_{j \in I_k^i} \pi_i q_{ij} = \sum_{j \in I_k^i} \pi_j q_{ji} \qquad \text{(PDB)}.$$

Then π satisfies $\pi Q = 0$.

<u>Proof</u>. $\sum_i \pi_i q_{ij} = \sum_{i \in I \setminus \{j\}} \pi_i q_{ij} + \pi_j q_{jj}$

$$= \sum_k \underbrace{\sum_{i \in I_k^j} \pi_i q_{ij}}_{\sum_{i \in I_k^j} \pi_j q_{ji}} + \pi_j q_{jj}$$

$$= \pi_j \sum_{i \neq j} q_{ji} + \pi_j q_{jj} = 0$$

Proof of theorem. Let

$$\pi_n = \prod_{i=1}^{N} \bar{g}_i^{\,n_i}.$$

We'll check (PDB). Let

$$I_A = \{e_i : i = 1, \dots, N\}$$

$$I_{D_j} = \{e_i - e_j : i \neq j\} \cup \{-e_j\}.$$

Thus: • when a customer arrives: $n \to n+m$, $m \in I_A$

• when a customer departs from j: $n \to n+m$, $m \in I_{D_j}$.

It suffices to show:

$$\sum_{m \in I_A} \cancel{\pi_n}\, q_{n,n+m} = \sum_{m \in I_A} \frac{\pi_{n+m}}{\pi_n} q_{n+m,n} \qquad (A)$$

$$\sum_{m \in I_{D_j}} \cancel{\pi_n}\, q_{n,n+m} = \sum_{m \in I_{D_j}} \frac{\pi_{n+m}}{\pi_n} q_{n+m,n} \qquad (D)$$

(D) $m \in I_{D_j} \Rightarrow q_{n,n+m} = \mu_j P_{j0}$ if $m = -e_j$

$q_{n,n+m} = \mu_j P_{ji}$ if $m = e_i - e_j$

$$\Rightarrow \sum_{m\in I_{D_j}} q_{n,n+m} = \mu_j P_{j0} + \sum_{i\neq j} \mu_j P_{ji} = \mu_j$$

$$\sum_{m\in I_{D_j}} \frac{\pi_{n+m}}{\pi_n} q_{n,n+m} = \sum_{i\neq j} \underbrace{\frac{\pi_{n+e_i-e_j}}{\pi_n}}_{\frac{\bar\rho_i}{\bar\rho_j} = \frac{\bar\lambda_i/\cancel{\mu_i}}{\bar\rho_j}} \underbrace{q_{n+e_i-e_j,n}}_{\cancel{\mu_i} P_{ij}} + \underbrace{\frac{\pi_{n-e_j}}{\pi_n}}_{\frac{1}{\bar\rho_j}} \underbrace{q_{n-e_j,n}}_{\lambda_j}$$

$$= \sum_{i\neq j} \frac{\bar\lambda_i}{\bar\rho_j} P_{ij} + \frac{1}{\bar\rho_j}\lambda_j \overset{(T)}{=} \frac{\bar\lambda_j}{\bar\rho_j} = \mu_j$$

$$(A) \quad \sum_{m\in I_A} q_{n,n+m} = \sum_{i=1}^{N} \lambda_i$$

$$\sum_{m\in I_A} \frac{\pi_{n+m}}{\pi_n} q_{n+m,n} = \sum_{i=1}^{N} \underbrace{\frac{\pi_{n+e_i}}{\pi_n}}_{\bar\rho_i = \frac{\bar\lambda_i}{\cancel{\mu_i}}} \underbrace{q_{n+e_i,n}}_{\cancel{\mu_i} P_{i0}}$$

$$= \sum_i \bar\lambda_i P_{i0}$$

$$= \sum_i \bar\lambda_i \left(1 - \sum_j P_{ij}\right)$$

$$= \sum_i \bar\lambda_i - \sum_j \underbrace{\sum_i P_{ij} \bar\lambda_i}_{\overset{(T)}{=} \bar\lambda_j - \lambda_j} = \sum_j \lambda_j$$

Thus (PDB) holds and $\pi Q = 0$.

The rates are bounded, so there is no explosion. Hence if $\tilde{\rho}_i < 1$ for all i we can normalise π, and it then is an invariant distribution.

Claim about departure processes: Example sheet.

4.4. M/G/1 queue

- Customers arrive according to a Poisson process with rate λ.
- Service times of the n-th customer is ξ_n where the ξ_n are i.i.d. with $\mathbb{E}\xi_n = 1/\mu$.
- There is one server.

Note $(X(t) : t \geq 0)$, the process of the number of customers in the queue, is in general not a Markov process!

Let D_n be the departure time of the n-th customer.

Prop. $Z_n = X(D_n)$, $n = 0, 1, 2, \ldots$, is a discrete-time Markov chain on $\{0, 1, 2, \ldots\}$ with transition probabilities

$$\begin{pmatrix} p_0 & p_1 & p_2 & \cdots & & \\ p_0 & p_1 & p_2 & \cdots & & \\ 0 & p_0 & p_1 & p_2 & \cdots & \\ 0 & 0 & p_0 & p_1 & p_2 & \cdots \\ & \vdots & & & & \end{pmatrix}$$

where $p_k = \mathbb{E}\left(e^{-\lambda\xi} \frac{(\lambda\xi)^k}{k!}\right)$.

<u>Proof.</u> Let A_{n+1} be the number of customers arriving after D_n during the service time ξ_{n+1}. The A_n are i.i.d. and given ξ_n,

$$A_n \sim \text{Poisson}(\lambda \xi_n).$$

$$\Rightarrow \mathbb{P}(A_n = k) = \mathbb{E}\Big(\underbrace{\mathbb{P}(A_n = k \mid \xi_n)}_{e^{-\lambda \xi_n} \frac{(\lambda \xi_n)^k}{k!}}\Big) = p_k$$

Now,

$$X(D_{n+1}) = A_{n+1} \qquad (X(D_n) = 0)$$

$$X(D_{n+1}) = A_{n+1} + X(D_n) - 1 \qquad (X(D_n) > 0)$$

This gives the claim.

<u>Thm.</u> Let $\rho = \lambda/\mu$. If $\rho \leq 1$ the queue is recurrent in the sense will empty out almost surely. If $\rho > 1$, the queue is transient in the sense that it will not empty out, with positive probability.

Lemma. Let (Y_i) be i.i.d. $\mathbb{Z}$-valued random variables. Let $S_n = Y_1 + \cdots + Y_n$. If $\mathbb{E}|Y_i| < \infty$ then S is recurrent iff $\mathbb{E}Y_i = 0$. We will assume for convenience that $\mathbb{E}|Y_i|^3 < \infty$.

Proof. $\mathbb{E}Y_i > 0 \overset{SLLN}{\Rightarrow} S_n \to +\infty$ a.s.

$\phantom{\mathbb{E}Y_i} < 0 \qquad S_n \to -\infty$ a.s.

So we will consider the case $\mathbb{E}Y_i = 0$ now.
It suffices to prove that

$$G_\lambda(0) = \sum_{n=0}^{\infty} \lambda^n \mathbb{P}_0(S_n = 0)$$

$$\lim_{\lambda \uparrow 1} G_\lambda(0) = +\infty.$$

Now: $\mathbb{P}_0[S_n = 0] = \frac{1}{2\pi} \int_0^{2\pi} f(k)^n e^{ik\cdot 0}\, dk$

where $f(k) = \mathbb{E}(e^{iYk}) = \sum_{x \in \mathbb{Z}} \mathbb{P}[Y = x] \underbrace{e^{ikx}}_{1 + ikx - \frac{1}{2}k^2x^2 + O(|kx|^3)}$

$$|f(k)| \leq 1$$

$$f(k) = 1 - \underbrace{\tfrac{1}{2}(\mathbb{E}Y^2)}_{c} k^2 + O(|k|^3)$$

$$\Rightarrow G_\lambda(0) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\sum_{n=0}^{n} (\lambda f(k))^n}_{\frac{1}{1-\lambda f(k)} = \frac{1}{1-\lambda(1-ck^2+O(k^3))}}\, dk = \frac{1}{2\pi} \int_0^{2\pi} \frac{dk}{1 - \lambda + \lambda c k^2 + O(k^3)} \to \infty \text{ as } \lambda \uparrow 1.$$

Proof 1 of thm.

X transient/recurrent $\Leftrightarrow$ $X(D_n)$ transient/recurrent

While $X(D_n) > 0$, $X(D_n)$ has the steps of a random walk on $\mathbb{Z}$ with step distribution $Y = A - 1$.

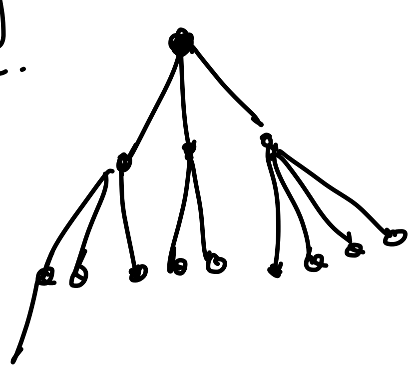
$\sim (p_k)$

$$\begin{aligned} \mathbb{E}Y &= \mathbb{E}A - 1 \\ &= \sum_m \mathbb{E}(A \mid \xi = m)\, \mathbb{P}(\xi = m) - 1 \\ &= \sum_m \lambda m\, \mathbb{P}(\xi = m) - 1 \\ &= \lambda \mathbb{E}\xi - 1 = \rho - 1 \end{aligned}$$

If $\rho = 1$, then X is recurrent.

If $\rho < 1$, then X is drifted to the left. X is then in fact positive recurrent. If $\rho > 1$, X is trans.

The second proof uses a hidden branching structure. A customer C_2 is an offspring of C_1 if C_2 arrives during the service time of C_1. This defines a tree.



The offspring distribution is i.i.d. A.
So this is a branching process.

<u>Proof 2 of thm.</u> Recurrence $\Leftrightarrow$ tree finite.
(emptying out)

By branching process results, this happens iff $\mathbb{E}A \le 1$, i.e., $\rho \le 1$.

<u>Busy period</u>: time between a customer joins an empty queue and another customer leaving behind an empty queue.

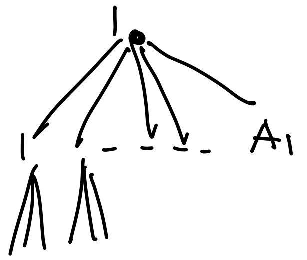
<u>Prop.</u> For the M/G/1 queue with $\lambda < \mu$, the busy period B satisfies

$$\mathbb{E}B = \frac{1}{\mu - \lambda}.$$

<u>Proof.</u> Assume for now that $\mathbb{E}B < \infty$. Since

$$B = \xi_1 + \sum_{i=1}^{A_1} B_i$$

B_i: busy period of i-th subtree



A_1 depends on ξ_1, but the B_i are conditionally independent and have the same distribution as B.

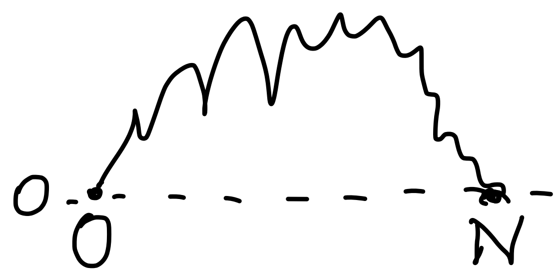
$$\Rightarrow \mathbb{E}B = \mathbb{E}\xi + \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{A_i} B_i \,\middle|\, A_1, \xi_1\right)\right)$$
$$= \mathbb{E}\xi + \mathbb{E}A\,\mathbb{E}B$$

If $\mathbb{E}B < \infty$ then this implies that

$$\mathbb{E}B = \frac{\mathbb{E}\xi}{1-\mathbb{E}A} = \frac{\mathbb{E}\xi}{1-\varrho} = \frac{1}{\mu-\lambda}.$$

Since $\varrho < 1$, we have seen that $X(D_n)$ is in fact positive recurrent. If N is the length of an excursion of $X(D_n)$.

$$\Rightarrow \mathbb{E}B = \mathbb{E}\left(D_N - D_0\right)$$
$$= \mathbb{E}\left(\sum_{n=1}^{N} D_n - D_{n-1}\right)$$



By independence,

$$\mathbb{E}B \leq \mathbb{E}N \underbrace{\mathbb{E}D_n - D_{n-1}}_{\mathbb{E}\xi} < \infty.$$

5. Renewal processes and non-Markovian queues

5.1. Renewal processes and size biased picking

Suppose busses arrive "every 10 minutes" according to the following two models:

(a) Busses always arrive exactly 10 minutes after the previous one.

(b) According to a Poisson process of rate 10, i.e. the next bus arrives after an independ. mean 10 exponential time.

How long do you have to wait on average?

(a) 5 minutes

(b) 10 minutes

Defn Let (ξ_i) be i.i.d. nonnegative random variables with $\mathbb{P}(\xi>0)>0$. Set

$$T_n = \sum_{i=1}^{n} \xi_i, \qquad N(t) = \max\{n \geq 0 : T_n \leq t\}$$

Prop If $\lambda = 1/\mathbb{E}\xi$ then

$$\frac{N(t)}{t} \longrightarrow \lambda \text{ a.s.}, \quad \mathbb{E}\frac{N(t)}{t} \longrightarrow \lambda.$$

We will only prove the first claim.

Proof. First note that $N(t) < \infty$ a.s. and that $N(t) \longrightarrow \infty$ a.s. Then

$$T_{N(t)} \leq t \leq T_{N(t)+1}$$

$$\Rightarrow \frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{N(t)+1}}{N(t)}$$

By the LLN, $\frac{T_n}{n} \longrightarrow \mathbb{E}\xi = \frac{1}{\lambda}$ a.s. Since $N(t) \to \infty$, therefore

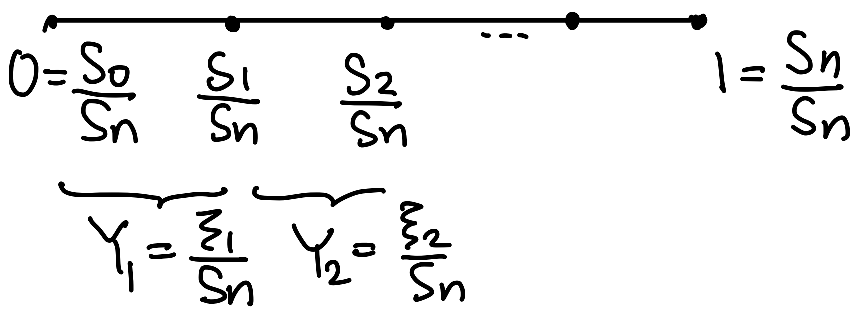
$$\frac{T_{N(t)}}{N(t)} \longrightarrow \frac{1}{\lambda}, \quad \frac{T_{N(t)+1}}{N(t)} \longrightarrow \frac{1}{\lambda}$$

$$\Rightarrow \frac{N(t)}{t} \longrightarrow \lambda.$$

Now suppose $\mathbb{P}(\xi_i > 0) = 1$. Let

$$S_i = \xi_1 + \cdots + \xi_n$$

$\Rightarrow \frac{S_i}{S_n} \in [0,1]$ partition $[0,1]$:



Let $U \sim \text{Unif}[0,1]$ and $\hat{Y}$ be the length of the interval containing U.
What is the distribution of $\hat{Y}$? One might it is the same as that of Y_1, but this is not so because U tends to fall in bigger intervals: size biased picking.

<u>Prop.</u> $\mathbb{P}(\hat{Y} \in dy) = ny\, \mathbb{P}(Y_1 \in dy)$

<u>Proof.</u> $$\mathbb{P}(\hat{Y} \in dy) = \sum_{i=1}^{n} \mathbb{P}\Big(\underbrace{\hat{Y}}_{= \frac{\xi_i}{S_n}} \in dy, \frac{S_{i-1}}{S_n} \leq U \leq \frac{S_i}{S_n}\Big)$$
$$= \sum_{i=1}^{n} \mathbb{E}\left(\mathbb{1}\left\{\frac{\xi_i}{S_n} \in dy\right\} \frac{\xi_i}{S_n}\right)$$
$$= \sum_{i=1}^{n} y\, \mathbb{P}\left(\frac{\xi_i}{S_n} \in dy\right)$$

$$= ny \, P(Y_1 \in dy).$$

<u>Defn</u>. Let X be a nonnegative random variable with distribution μ and $\mathbb{E}X = m$. Then the size biased distribution is

$$\hat{\mu}(dy) = \frac{y}{m}\mu(dy).$$

We will write $\hat{X}$ for a random variable with distribution $\hat{\mu}$.

<u>Rk</u>. $\int \hat{\mu}(dy) = \frac{1}{m}\underbrace{\int y\mu(dy)}_{m} = 1.$

<u>Example</u>. If $X \sim \mathrm{Unif}[0,1]$. Then $\hat{X}$ has distribution

$$\hat{\mu}(dx) = 2x \, dx$$

<u>Example</u> If $X \sim \mathrm{Exp}(\lambda)$. Then $\hat{X}$ has distr.

$$\hat{\mu}(dx) = \frac{x}{1/\lambda}\lambda e^{-\lambda x} dx$$

$$= \lambda^2 x e^{-\lambda x} dx$$

So $\hat{X} \sim \text{Gamma}(2, \lambda)$, i.e., $\hat{X}$ has the same distribution as $X_1 + X_2$ where X_1 and X_2 are independent $\text{Exp}(\lambda)$ random variables.

5.2 Renewal processes: equilibrium

Given a renewal process, set

$A(t) = t - T_{N(t)}$ — age (time since last renewal)

$E(t) = T_{N(t)+1} - t$ — excess (time until next renewal)

$L(t) = A(t) + E(t) = T_{N(t)+1} - T_{N(t)}$ — length of the current interval

For simplicity, assume ξ is $\mathbb{Z}$-valued in the following.

<u>Thm</u>. Assume ξ is $\mathbb{Z}$-valued and non-arithmetic:

$$\forall k > 1 : \ \mathbb{P}(\xi \in k\mathbb{Z}) < 1.$$

Then $\mathbb{P}(L(t) \le x, E(t) \le y) \longrightarrow \mathbb{P}(\hat{\xi} \le x, U\hat{\xi} \le y)$

$(x, y \in \mathbb{N})$ $\mathbb{P}(L(t) \le x, A(t) \le y) \longrightarrow \mathbb{P}(\hat{\xi} \le x, U\hat{\xi} \le y)$

where $U \sim \text{Unif}[0,1]$ and independent of $\hat{\xi}$.

Rk. $\mathbb{P}(U\hat{\xi} \le x) = \lambda \int_0^x \mathbb{P}(\xi > y)\,dy$

$\frac{1}{\lambda} = \mathbb{E}\xi$

Indeed, $\mathbb{P}(U\hat{\xi} \le x) = \int_0^1 \mathbb{P}(\hat{\xi} \le x/u)\,du$

$$= \int_0^1 \left(\int_0^{x/u} \lambda y \,\mathbb{P}(\xi \in dy) \right) du$$

$$= \int_0^\infty \lambda y \,\mathbb{P}(\xi \in dy) \underbrace{\int_0^{x/y \wedge 1} du}_{1 \wedge \frac{x}{y}}$$

$$= \lambda \int_0^\infty (x \wedge y)\,\mathbb{P}(\xi \in dy)$$

$$\lambda \int_0^x \mathbb{P}(\xi > y)\,dy = \lambda \int_0^x \left(\int_y^\infty \mathbb{P}(\xi \in dz) \right) dy$$

$$= \lambda \int_0^\infty \mathbb{P}(\xi \in dz) \underbrace{\int_0^{x \wedge z} dy}_{x \wedge z}$$

$$= \lambda \int_0^\infty (x \wedge y)\,\mathbb{P}(\xi \in dz)$$

Example. If $\xi \sim \mathrm{Unif}[0,1]$ and $y \in [0,1]$,

$$\mathbb{P}(E_\infty \le y) = \lambda \int_0^y \mathbb{P}(\xi > u)\,du \qquad \lambda = 1/\text{mean of } \xi$$

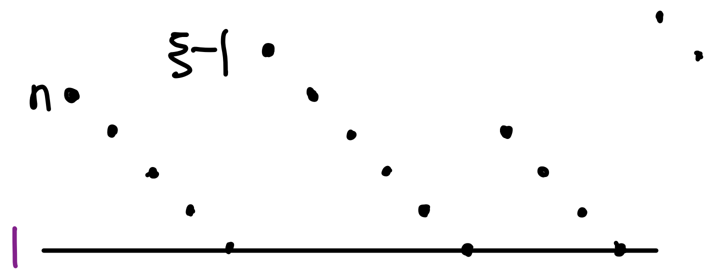
E_∞ ↑ $U\hat{\xi}$

$$= \lambda \int_0^y (1-u)\,du = 2\left(y - \frac{y^2}{2}\right)$$

Proof (of the theorem). Since ξ is $\mathbb{Z}$-valued, $\underline{E}(t)$, $t=0,1,2,\ldots$ is a discrete-time Markov chain with

$$P_{i,i-1} = 1 \quad (i \geq 2)$$

$$P_{1,n} = \mathbb{P}(\xi = n+1)$$



Since $\mathbb{P}[\xi \in k\mathbb{Z}] < 1 \; \forall k > 1$, this Markov Chain is aperiodic.

It is also irreducible (on $\{1,\ldots,N\}$ for suitable N) and recurrent, and $\pi = \pi P$ is

$$\pi_n = \pi_{n+1} + \pi_0 \mathbb{P}(\xi = n+1)$$

$$\Rightarrow \quad \pi_n = \underbrace{\sum_{m \geq n+1} \mathbb{P}(\xi = m)}_{\mathbb{P}(\xi > n)}, \quad \pi_0 = 1$$

is an invariant measure

Since $\mathbb{E}\xi = \sum_n \mathbb{P}(\xi > n)$, this measure can be normalised if $\mathbb{E}\xi = 1/\lambda < \infty$ and thus

$$\pi_n = \lambda \mathbb{P}(\xi > n)$$

is an invariant distribution.

Since we have already noticed that the chain is aperiodic, by convergence to the invariant distribution, for y integer,

$$\mathbb{P}(E(t) \leq y) \longrightarrow \sum_{n \leq y} \pi_n = \lambda \sum_{n=0}^{\lfloor y \rfloor} \mathbb{P}(\xi > n)$$

$$= \lambda \int_0^{\lfloor y \rfloor} \mathbb{P}(\xi > \lfloor x \rfloor)\, dx$$

$$= \lambda \int_0^{y} \mathbb{P}(\xi > x)\, dx$$

Now $(L(t), E(t))$, $t = 0, 1, 2, \ldots$ is also a Markov chain with state space

$$I = \{(n,k) : 1 \leq k \leq n\} \subset \mathbb{N} \times \mathbb{N}$$

and transition probabilities

$$p_{(n,k) \to (n,k-1)} = 1 \qquad (k \geq 2)$$

$$p_{(n,1) \to (k,k-1)} = \mathbb{P}(\xi = k) \quad \longleftarrow \text{indep. of } n$$

This is again an aperiodic irreducible Markov chain and the invariant measure equation is

$$\pi_{(n,k-1)} = \pi_{(n,k)} \qquad (1 \leq k \leq n)$$

$$\pi_{(k,k-1)} = \underbrace{\sum_{m=1}^{\infty} \pi_{(m,1)}}_{\text{indep. of } k} \mathbb{P}(\xi = k).$$

Take $$\pi_{(n,k)} = \frac{P(\xi=n)}{\mathbb{E}\xi} = \underbrace{\frac{n\,P(\xi=n)}{\mathbb{E}\xi}}_{P(\hat{\xi}=n)} \times \underbrace{\frac{1}{n}\mathbb{1}\{1\le k\le n\}}_{\substack{\text{Given } L=n, \\ E \text{ is uniform} \\ \text{on } \{1,\dots,n\}.}}$$

Again by the Markov chain limit theorem,

$$\mathbb{P}(L(t)\le x,\ E(t)\le y) \longrightarrow \sum_{n\le x}\sum_{k\le y}\pi_{n,k}$$
$$\sim (\hat{\xi},\ U\hat{\xi})$$

This completes the proof of convergence for $(L(t), E(t))$. The one for $(L(t), A(t))$ is analogous.

5.3. Renewal-reward processes

Let (ξ_i, R_i) be i.i.d. pairs of random variables. (Here ξ_i and R_i need not be independent.) Assume ξ_i is nonnegative and $\mathbb{E}\xi_i = 1/\lambda < \infty$.

Let $(N(t) : t \geq 0)$ be the renewal process associated with the (ξ_i) and

$$R(t) = \sum_{i=1}^{N(t)+1} R_i \qquad \text{(total reward up to time } t\text{)}$$

<u>Prop.</u> If $\mathbb{E}|R_i| < \infty$ then

$$\frac{R(t)}{t} \longrightarrow \lambda \mathbb{E} R_i \quad \text{a.s.}$$

$$\mathbb{E}\, \frac{R(t)}{t} \longrightarrow \lambda \mathbb{E} R_i .$$

<u>Thm.</u> The expected current reward $r(t) = \mathbb{E} R_{N(t)+1}$ satisfies

$$r(t) \longrightarrow \lambda \mathbb{E}(R\xi)$$

<u>Example:</u> Alternating renewal process

A machine breaks after time X_i then it takes time Y_i for it to get fixed. Thus $\xi_i = X_i + Y_i$ is the length of a cycle and the ξ_i define a renewal process (if we assume that the X_i and Y_i are i.i.d).

What is the fraction of time that the machine runs in the long run?

Let $R_i = X_i$ be the amount of time the machine was on during cycle i. Then (ξ_i, R_i) is a renewal-reward process.

Thus the last proposition suggests that

$$\frac{R(t)}{t} \longrightarrow \frac{\mathbb{E}X_1}{\mathbb{E}X_1 + \mathbb{E}Y_1}$$

$$\mathbb{E}\left(\text{—''—}\right) \longrightarrow \text{——''——}.$$

Now $R(t)/t$ is not precisely the fraction of time that the machine is on because the rewards are always awarded at the end of the cycles.

What is the probability $p(t)$ that the machine is on at time t?

$$\mathbb{E}R(t) = \int_0^t p(s)\,ds$$

so we expect (and it is true under suit. ass.)

$$p(t) \longrightarrow \frac{\mathbb{E}X_1}{\mathbb{E}X_1 + \mathbb{E}Y_1}$$

<u>Example</u> Busy periods of M/G/1 queue

Assume $\rho < 1$. Let I_n and B_n denote the lengths of the n-th idle and busy periods.

Then (B_n, I_n) is an alternating renewal process.

$$\Rightarrow p(t) \longrightarrow \frac{\mathbb{E}I_n}{\mathbb{E}B_n + \mathbb{E}I_n}.$$

By the Markov property, $I_n \sim \mathrm{Exp}(\lambda)$

$$\mathbb{E}I_n = 1/\lambda$$

Earlier we saw that $\mathbb{E}B_n = \frac{1}{\mu - \lambda}$.

Thus

$$p(t) \longrightarrow \frac{1/\lambda}{\frac{1}{\mu-\lambda} + \frac{1}{\lambda}} = \frac{\mu - \lambda}{\mu} = 1 - \frac{\lambda}{\mu}.$$

5.4. Little's formula

Defn. A process $(X(t): t \geq 0)$ is regenerative if there exist random times τ_n such that the law of $(X(t+\tau_n): t \geq 0)$ is the same as that of $(X(t): t \geq 0)$ and independent of $(X(t): t \leq \tau_n)$. Also assume that $\tau_0 = 0$, that $\tau_{n+1} > \tau_n$ and that τ_{n+1} depends only on $(X(t+\tau_n): t \geq 0)$ (so that $(\tau_{n+1} - \tau_n)_{n \geq 0}$ is i.i.d.).

Rk. An M/G/1 queue is regenerative with τ_n the end time of the n-th busy period.

Thm (Little's formula). Let X be a queue that is regenerative with regeneration times τ_n. Let N be the arrival process of X and let W_i the waiting time of the i-th customer (including the service time).

Assume $\mathbb{E}\tau_1 < \infty$ and $\mathbb{E}N(\tau_1) < \infty$. Then the following limits exist almost surely and are deterministic:

(a) Long-run mean queue size:

$$L = \lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)\, ds$$

(b) Long-run average waiting time:

$$W = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} W_i$$

(c) Long-run average arrival rate

$$\lambda = \lim_{t\to\infty} \frac{N(t)}{t}.$$

Moreover, $L = \lambda W$. In fact, $L = \lambda W$ holds only assuming that the limits in (b) & (c) exist and $X(t)/t \longrightarrow 0$.

<u>Proof</u>. Set $Y_n = \sum_{i=1}^{N(\tau_n)} W_i$. Since $X(\tau_n) = 0$, for any $\tau_n \le t < \tau_{n+1}$,

$$\frac{Y_n}{\tau_{n+1}} \le \frac{1}{t}\int_0^t X(s)\,ds \le \frac{Y_{n+1}}{\tau_n}.$$

By the regeneration property, $Y_i - Y_{i-1}$ are i.i.d. Also $Y_0 = 0$. By the SLLN,

$$\frac{Y_{n+1}}{\tau_{n+1}} \longrightarrow \frac{\mathbb{E}Y_1}{\mathbb{E}\tau_1} \quad \text{a.s.}$$

$$\Rightarrow \frac{1}{t}\int_0^t X(s)\,ds \longrightarrow \frac{\mathbb{E}Y_1}{\mathbb{E}\tau_1}.$$

Similarly, since $EN(\tau_1) < \infty$,

$$\frac{N(t)}{t} \longrightarrow \lambda.$$

Also, for $N(\tau_n) \le k < N(\tau_{n+1})$,

$$\frac{Y_n}{N(\tau_{n+1})} \le \frac{1}{k} \sum_{i=1}^{k} W_i \le \frac{Y_{n+1}}{N(\tau_n)}$$

and $$\frac{Y_{n+1}}{N(\tau_{n+1})} \longrightarrow \frac{L}{\lambda}$$

This concludes the main statement.

That $L = \lambda W$ still holds under the stated assumption is also similar:

$$\sum_{k=1}^{N(t)-X(t)} W_k \le \int_0^t X(s)\, ds \le \sum_{k=1}^{N(t)} W_k$$

Since $N(t)/t \longrightarrow \lambda > 0$ and $X(t)/t \longrightarrow 0$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t X(s)\, ds = \lambda W.$$

6. Spatial Poisson processes

6.1. Definition and superposition

The standard Poisson process can be encoded by the set of arrival times

$$\Pi = \{T_1, T_2, T_3, \ldots\} \subset [0,\infty)$$

Π is a countable random subset of $[0,\infty)$.

A spatial Poisson process is a random countable subset $\Pi \subset \mathbb{R}^d$ (with certain properties)

Let $\tilde{\mathcal{B}}(\mathbb{R}^d) = \{ \prod_{i=1}^{d} (a_i, b_i] : a_i < b_i \}$ be the set of boxes in $\mathbb{R}^d$. For $A \in \tilde{\mathcal{B}}(\mathbb{R}^d)$, the volume is

$$|A| = \prod_{i=1}^{d} |b_i - a_i|$$

The Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ is obtained by countable unions and intersections of elements of $\tilde{\mathcal{B}}(\mathbb{R}^d)$. For $A \in \mathcal{B}(\mathbb{R}^d)$ the volume (Lebesgue measure) $|A|$ is still defined.

<u>Defn</u>. A random countable subset $\Pi \subset \mathbb{R}^d$ is a Poisson process with constant intensity $\lambda > 0$ if for all $A \in \mathcal{B}(\mathbb{R}^d)$,

(a) $N(A) := \#(A \cap \Pi) \sim \text{Poisson}(\lambda |A|)$

(b) For any $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^d)$ disjoint, $N(A_1), \dots, N(A_n)$ are independent.

If $|A| = \infty$ we interpret (a) as $\mathbb{P}(N(A) = \infty) = 1$.

<u>Example</u> If Π is a spatial Poisson process on $\mathbb{R}$ then $N(t) := N([0,t])$ is a standard Poisson process.

<u>Defn</u>. Let $\lambda : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a non-negative (measurable) function such that

$$\Lambda(A) := \int_A \lambda(x)\, dx < \infty$$

for every bounded $A \in \mathcal{B}(\mathbb{R}^d)$. Then Π is a non-homogeneous Poisson process with intensity λ if

(a) $N(A) := \#(A \cap \Pi) \sim \text{Poisson}(\Lambda(A))$

(b) For any $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^d)$ disjoint, $N(A_1), \dots, N(A_n)$ are independent.

Λ is called the mean measure of the Poisson process.

Thm. (Superposition theorem). Let Π_1 and Π_2 be independent Poisson processes with intensity functions λ_1 and λ_2. Then $\Pi = \Pi_1 \cup \Pi_2$ is a Poisson process with intensity $\lambda = \lambda_1 + \lambda_2$.

Proof. Let $N_i(A) = \#\{\Pi_i \cap A\}$.

Since $N_i(A) \sim \text{Poisson}(\Lambda_i(A))$ it follows that

$$S(A) = N_1(A) + N_2(A)$$
$$\sim \text{Poisson}(\underbrace{\Lambda_1(A) + \Lambda_2(A)}_{\Lambda(A)})$$

Also if $A_1, \dots, A_n$ are disjoint, $S(A_1), \dots, S(A_n)$ are independent.

To show that $S(A) = \#\{\Pi \cap A\}$ we need to show that $\Pi_1 \cap \Pi_2 \cap A = \emptyset$ almost surely. We will assume A is bounded. Let

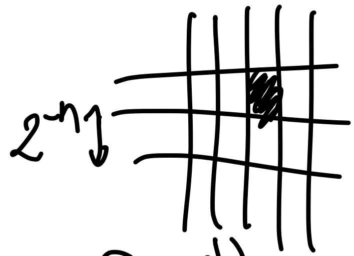
$$Q_{k,n} = \prod_{i=1}^{d} (k_i 2^{-n}, (k_i+1)2^{-n}] \text{ for } k \in \mathbb{Z}^d, n \in \mathbb{N}.$$

$$\Rightarrow \mathbb{P}[\Pi_1 \cap \Pi_2 \cap A \neq \emptyset]$$
$$\leq \sum_{k \in \mathbb{Z}^d} \mathbb{P}[N_1(Q_{k,n} \cap A) \geq 1,\ N_2(Q_{k,n} \cap A) \geq 1]$$

$$= \sum_{k\in\mathbb{Z}^d} \underbrace{\left(1-e^{-\Lambda_1(Q_{k,n}\cap A)}\right)}_{\le \Lambda_1(Q_{k,n}\cap A)} \underbrace{\left(1-e^{-\Lambda_2(Q_{k,n}\cap A)}\right)}_{\le \Lambda_2(Q_{k,n}\cap A)}$$

$$\le \underbrace{\max_{k\in\mathbb{Z}^d} \Lambda_1(Q_{k,n}\cap A)}_{M_n(A)} \times \underbrace{\sum_{k\in\mathbb{Z}^d} \Lambda_2(Q_{k,n}\cap A)}_{\Lambda_2(A)<\infty}$$

Clearly, when λ is constant (or bounded) then
$M_n(A) \le \lambda 2^{-nd} \longrightarrow 0$.



Lemma. $M_n(A) \longrightarrow 0$ for any non-negative measurable λ and $A\in\mathcal{B}(\mathbb{R}^d)$ bounded.

Proof. WLOG, A is a finite union of $Q_{k,0}$.
Clearly, $M_{n+1}(A) \le M_n(A)$ and thus $M_n(A)\to\delta$ for some $\delta\ge 0$.
If $\delta>0$, then for every n there is $k_n\in\mathbb{Z}^d$ s.t. $\Lambda(Q_{k_n,n})\ge\delta$.
Colour a cube $Q_{k,n}$ black if for any $m\ge n$ there is a cube $Q_{k_m,m}\subset Q_{k,n}$ s.t. $\Lambda(Q_{k_m,m})\ge\delta$.

Since A is a finite union of $Q_{k,0}$ there is one of them such that $Q_{k,0}$ contains infinitely many of the $Q_{k_m,m}$. By monoton. $Q_{k,0}$ is then black and we thus have a nested sequence of cubes

$$Q_0 \supset Q_1 \supset Q_2 \supset \cdots$$

s.t. $\Lambda(Q_n) \geq \delta$. But

$$\lim_{n\to\infty} \Lambda(Q_n) \underset{\substack{\uparrow \\ \text{monotone} \\ \text{convergence}}}{=} \Lambda\Big(\underbrace{\bigcap_n Q_n}_{\substack{\text{contains at most} \\ \text{one point}}}\Big) = 0.$$

Rk. The same proof applies to any measure Λ satisfying $\Lambda(\{k\}) = 0$ for any k.

6.2. Mapping and conditioning

Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^s$ be measurable.

For Π a Poisson process on $\mathbb{R}^d$, when is

$$f(\Pi) = \{ f(x) \in \mathbb{R}^s : x \in \Pi \}$$

again a Poisson process (on $\mathbb{R}^s$)?

Thm. (Mapping theorem) Let Π be a non-homogeneous Poisson process with intensity function λ. Assume $f : \mathbb{R}^d \longrightarrow \mathbb{R}^s$ satisfies

$$\Lambda(f^{-1}(\{y\})) = 0 \quad \text{for all } y \in \mathbb{R}^s \qquad (*)$$

and

$$\mu(B) = \Lambda(f^{-1}(B)) = \int_{f^{-1}(B)} \lambda(x)\, dx < \infty$$

for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$. Then $f(\Pi)$ is a Poisson process on $\mathbb{R}^s$ with mean measure μ.

Proof. Assume that the points in $f(\Pi)$ are a.s. distinct, i.e., f is injective on Π.

Then $M(B) = \#\{ f(\Pi) \cap B \}$

$$= \#\{ \Pi \cap f^{-1}(B) \} \sim \text{Poisson}(\underbrace{\Lambda(f^{-1}(B))}_{\mu(B)})$$

If $B_1, \ldots, B_n$ are disjoint, so are the $f^{-1}(B_i)$.

$\Rightarrow M(B_1), \ldots, M(B_n)$ are independent

Thus $f(\Pi)$ is a Poisson process with mean measure μ.

Thus it suffices to show that the points in $f(\Pi) \cap [0,1)^s$ are distinct a.s.

$\nwarrow$ WLOG

Let $Q_{k,n} = \prod_{i=1}^{s} (k_i 2^{-n}, (k_i+1)2^{-n}] \subset \mathbb{R}^s$

Then $\underbrace{\#\{\Pi \cap f^{-1}(Q_{k,n})\}}_{N_k} \sim \text{Poisson}(\underbrace{\mu(Q_{k,n})}_{\mu_k})$

$$\Rightarrow \mathbb{P}(N_k \geq 2) = 1 - e^{-\mu_k} - \mu_k e^{-\mu_k}$$

$$= 1 - \underbrace{e^{-\mu_k}}_{\geq 1-\mu_k}(1+\mu_k) \leq \mu_k^2$$

$$\Rightarrow \sum_k \mu_k^2 \leq \underbrace{\max_k \mu_k}_{M_n} \underbrace{\sum_k \mu_k}_{=\mu([0,1)^s) < \infty}$$

and $M_n \longrightarrow 0$ by the lemma from last time using the assumption $(*)$.

Example. Let Π be a Poisson process on $\mathbb{R}^2$ with constant rate λ. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ denote Polar coordinates:

$$r = \sqrt{x^2+y^2}, \quad \theta = \arctan\left(\frac{x}{y}\right)$$

and $(r,\theta)=0$ if $(x,y)=(0,0)$.

Then $f(\Pi)$ is a Poisson process on $\mathbb{R}^2$ with measure

$$\mu(B) = \int_{f^{-1}(B)} \lambda \, dx\, dy = \int_{B\cap S} \lambda r \, dr\, d\theta$$

where $S = \{(r,\theta): r \geq 0, \ 0 < \theta \leq 2\pi\}$

Thus $f(\Pi)$ is a Poisson process on S with intensity λr.

Thm (Conditioning property). Let Π be a Poisson process on $\mathbb{R}^d$ with intensity function λ and $A \in \mathcal{B}(\mathbb{R}^d)$ such that

$$0 < \Lambda(A) < \infty.$$

Conditional on $\#\{\Pi \cap A\} = n$, the n points in $\Pi \cap A$ have the same distribution as n points chosen independently, all from the probability distribution

$$\nu(B) = \frac{\Lambda(B)}{\Lambda(A)}, \quad B \subseteq A$$

$$= \int_B \frac{\lambda(x)}{\Lambda(A)} \, dx.$$

In particular, if λ is constant, then the n points are uniform in A.

<u>Proof.</u> Write $N(B) = \#\{B \cap \Pi\}$.

Let $A_1, \ldots, A_k$ be a partition of A.

$$\Rightarrow \mathbb{P}(N(A_1) = n_1, \ldots, N(A_k) = n_k \mid N(A) = n)$$

$$= \frac{\prod_i \mathbb{P}(N(A_i) = n_i)}{\mathbb{P}(N(A) = n)} \quad \text{(independence property)}$$

$$= \frac{\prod_i \frac{1}{n_i!} \Lambda(A_i)^{n_i} e^{-\Lambda(A_i)}}{\frac{1}{n!} \Lambda(A)^n e^{-\Lambda(A)}}$$

$$= \frac{n!}{n_1! \cdots n_k!} \nu(A_1)^{n_1} \cdots \nu(A_k)^{n_k}$$

This multinomial distribution is the same as for n independent points chosen from ν.

This holds for any $A_1, \dots, A_k$, and thus characterises the distribution of $\Pi \cap A$.

<u>Rk</u>. One can simulate a Poisson process by using this property. Partion $\mathbb{R}^d$ into say unit cubes $A_1, A_2, \dots$. For each i, simulate a random variable $N_i \sim \text{Poisson}(\lambda \underbrace{|A_i|}_{=1})$.

Then choose N_i points uniformly at random from A_i. The result is a Poisson process with constant intensity λ.

6.3. Colouring

Let Π be a (non-homogeneous) Poisson process on $\mathbb{R}^d$ with intensity function $\lambda : \mathbb{R}^d \to \mathbb{R}$. Colour the points $x \in \Pi$ independently as follows:

- A point $x \in \Pi$ is red with probability $\gamma(x)$;
- a point $x \in \Pi$ is blue otherwise.

Let $\Gamma \subset \Pi$ be the set of red points and let $\Sigma \subset \Pi$ be the of blue points.

Thm. Γ and Σ are independent Poisson processes on $\mathbb{R}^d$ with intensity function $\gamma(x)\lambda(x)$ resp. $(1-\gamma(x))\lambda(x)$.

Proof. Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $\Lambda(A) < \infty$. Then condition on $\#\{\Pi \cap A\} = n$. Then $\Pi \cap A$ consists of n points chosen independently from distribution ν (as last time).

The probability that a point is red is

$$\bar{\gamma} = \frac{1}{\Lambda(A)} \int_A \gamma(x)\lambda(x)\,dx.$$

Let N_r and N_b be the number of red and blue points in A.

$$\Rightarrow \mathbb{P}\left(N_r = n_r,\ N_b = n_b \,\middle|\, N(A) = n\right) = \frac{n!}{n_r!\,n_b!}\,\bar{\gamma}^{n_r}(1-\bar{\gamma})^{n_b}.$$

$$\Rightarrow \mathbb{P}\left(N_r = n_r,\ N_b = n_b\right) = \frac{\cancel{(n_r+n_b)!}}{n_r!\,n_b!}\,\bar{\gamma}^{n_r}(1-\bar{\gamma})^{n_b} \times \frac{\Lambda(A)^{n_r+n_b}}{\cancel{(n_r+n_b)!}}\,e^{-\Lambda(A)}$$

$$= \underbrace{\frac{(\bar{\gamma}\Lambda(A))^{n_r}\,e^{-\bar{\gamma}\Lambda(A)}}{n_r!}}_{\mathbb{P}(\mathrm{Poisson}(\bar{\gamma}\Lambda(A)) = n_r)}\ \underbrace{\frac{((1-\bar{\gamma})\Lambda(A))^{n_b}\,e^{-(1-\bar{\gamma})\Lambda(A)}}{n_b!}}_{\mathbb{P}(\mathrm{Poisson}((1-\bar{\gamma})\Lambda(A)) = n_b)}$$

Thus the number of red and blue points in A are independent and they are distributed as $\mathrm{Poisson}(\bar{\gamma}\Lambda(A))$ respectively $\mathrm{Poisson}((1-\bar{\gamma})\Lambda(A))$.

The independence of the number of red / blue points in disjoint sets $A_1, \dots, A_n$ follows from the independence property of Π.

<u>Example</u>. A museum contains n different rooms that the visitors have to visit in sequence. Assume visitors arrive according to a Poisson

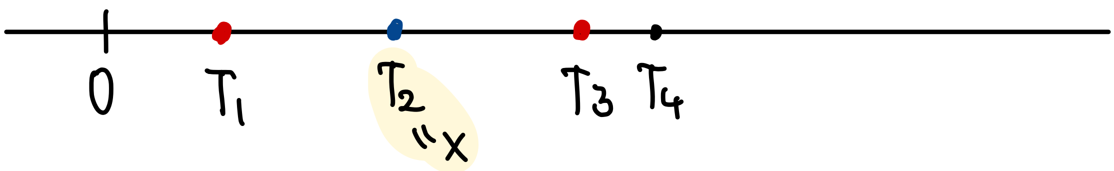
process (on $\mathbb{R}_+$) of (constant) rate λ.

The r-th visitor spends time $X_{s,r}$ in room s where the $X_{r,s}$ are independent random variables and given s the distribution of $X_{r,s}$ does not depend on r.

Let $V_s(t)$ be the number of visitors in room s at time t.

<u>Claim</u>: For any fixed t, the $V_s(t)$, $s=1,\dots,n$, are independent and have Poisson distributions.

<u>Proof</u>. Let $T_1 < T_2 < \cdots$ be the arrival times.



Colour the visitors according to which room they are in at time t. A point x in the Poisson process is coloured c_s if

$$x + \sum_{v=1}^{s-1} X_v \leq t < x + \sum_{v=1}^{s} X_v \qquad (*)$$

where the X_v are the times spent in room v by the visitor that arrived at time x.

If $(*)$ does not hold for any $s=1,\dots,n$, we colour the point x by δ = gray. These are the visitors that have not arrived yet or that have already left the museum.

The colours of different points are indep. Thus we have a (t-dependent) coloured Poisson process.

The (t-dependent) intensity measure for each colour that is not δ is a finite measure because if $x > t$ then x is δ.

So $V_s(t) = N_s([0,t])$.

6.4. Rényi's and Campbell-Hardy theorem

Thm. Let Π be a countable random subset of $\mathbb{R}^d$ and let $\lambda : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative measurable function with

$$\Lambda(A) = \int_A \lambda(x)\, dx < \infty$$

for all bounded $A \in \mathcal{B}(\mathbb{R}^d)$. If

$$\mathbb{P}(\Pi \cap A = \emptyset) = e^{-\Lambda(A)}$$

for any A that is a finite union of dyadic boxes $Q_{k,n} = \prod_{i=1}^{d} (k_i 2^{-n}, (k_i+1)2^{-n}]$ then Π is a Poisson process with intensity function λ.

Proof. Let $A \subset \mathbb{R}^d$ be bounded and open. Let

$$I_{k,n} = \mathbb{1}_{Q_{k,n} \cap A \neq \emptyset}.$$

Then

$$N(A) := \#\{\Pi \cap A\} = \lim_{n \to \infty} N_n(A)$$

$$\text{where } N_n(A) = \sum_{k : Q_{k,n} \subset A} I_{k,n}$$

independent for fixed n

The limit is monotone.

Since the $I_{k,n}$ are independent (for fixed n),

$$\mathbb{E}\left(s^{N_n(A)}\right) = \prod_{k:Q_{k,n}\subset A} \underbrace{\mathbb{E}\left(s^{I_{k,n}}\right)}$$

assumption $\longrightarrow = e^{-\Lambda(Q_{k,n})} + s\left(1-e^{-\Lambda(Q_{k,n})}\right)$

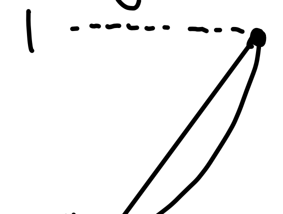
$$= \prod_{k:Q_{k,n}\subset A} \left(s+(1-s)e^{-\Lambda(Q_{k,n})}\right)$$

By monotone convergence,

$$\mathbb{E}\left(s^{N(A)}\right) = \lim_{n\to\infty} \prod_{k:Q_{k,n}\subset A} \underbrace{\left(s+(1-s)e^{-\Lambda(Q_{k,n})}\right)}$$

Also: $e^{-(1-s)\alpha} \leq s+(1-s)e^{-\alpha} \leq e^{-(1-s)\alpha+O(\alpha^2)}$

$e^{-(1-s)\alpha}$ is convex



$$\log\left(s+(1-s)e^{-\alpha}\right)$$
$$= \log\left(e^{-\alpha}\left((e^{\alpha}-1)s+1\right)\right)$$
$$= -\alpha + \log\left((e^{\alpha}-1)s+1\right)$$
$$\leq -\alpha + \underbrace{(e^{\alpha}-1)}_{\alpha+O(\alpha^2)}s$$
$$= -(1-s)\alpha + O(\alpha^2)$$

$$\Rightarrow \lim_{n\to\infty} e^{-(1-s)\sum_{k:Q_{k,n}\subset A} \Lambda(Q_{k,n})} \leq \mathbb{E}(s^{N(A)})$$

$$\leq \lim_{n\to\infty} e^{-(1-s)\sum_{k:Q_{k,n}\subset A} \Lambda(Q_{k,n}) + O(\max_k \Lambda(Q_{k,n}))}$$

Since Λ has a density λ,

$$\Lambda(A) = \lim_{n\to\infty} \sum_{k:Q_{k,n}\subset A} \Lambda(Q_{k,n}).$$

By the lemma from earlier: $\max_{k:Q_{k,n}\subset A} \Lambda(Q_{k,n}) \longrightarrow 0$.

$$\Rightarrow \mathbb{E}(s^{N(A)}) = e^{-(1-s)\Lambda(A)}$$

$$\Rightarrow N(A) \sim \text{Poisson}(\Lambda(A)).$$

Also, $N(A_1),\dots,N(A_k)$ are independent for disjoint open sets $A_1,\dots,A_k$ because the $N_n(A_i)$ are and $N_n(A_i) \longrightarrow N_n(A)$.

Without proof: the statement can be extended from A_k bounded open to general A_k.

Example: Lottery.

- A player wins at the events of a standard Poisson process Π on $\mathbb{R}$ with rate λ.
- The amounts won are i.i.d.
- The player spends gains at exponential rate α.

The gain at time t is

$$G(t) = \sum_{x\in\Pi\cap[0,t]} e^{-\alpha(t-x)} W_x$$

W_x: amount won at time x

$$= \sum_{x\in\Pi} r(t-x)\, W_x$$

where $r(u) = \begin{cases} 0 & \text{if } u<0 \\ e^{-\alpha u} & \text{if } u\geq 0. \end{cases}$

<u>Thm</u>. Let Π be a Poisson process on $\mathbb{R}$ with intensity λ, and let $r: \mathbb{R}\to\mathbb{R}$ be integrable over bounded interval with $r(u)=0$ if $u<0$, and let $(W_x)_{x\in\Pi}$ be i.i.d. and independent of Π. Then

$$\mathbb{E}\left(e^{i\theta G(t)}\right) = \exp\left(\lambda\int_0^t \left(\mathbb{E}\left(e^{i\theta r(s) W}\right)-1\right)ds\right) \quad (*)$$

$\theta\in\mathbb{R}$

Also assume r is smooth.

In particular, if $\mathbb{E}|W| < \infty$,

$$\mathbb{E}G(t) = \lambda\, \mathbb{E}W \int_0^t r(s)\, ds.$$

<u>Proof</u>. By definition,

$$G(t) = \sum_{n=1}^{N(t)} r(t - T_n) W_n$$

Poisson process

events of the Poisson process.

$$\Rightarrow F(t) = \mathbb{E}\left(e^{i\theta G(t)}\right)$$

$$= \mathbb{E}\left(e^{i\theta r(t-T_1) W_1}\, \mathbb{E}\left(e^{i\theta \sum_{n=2}^{N(t)} r(t-T_n) W_n} \,\middle|\, W_1, T_1\right)\right)$$

cond. expectation

$$t - T_n = t - T_1 - (T_n - T_1)$$

$$= \mathbb{E}\left(e^{i\theta r(t-T_1) W_1}\, F(t - T_1)\right)$$

Markov

$$= \lambda \int_0^t \mathbb{E}\left(e^{i\theta r(t-u) W_1}\right) F(t-u)\, e^{-\lambda u}\, du$$

$T_1 \sim \mathrm{Exp}(\lambda)$

$$= \lambda \int_0^t \mathbb{E}\left(e^{i\theta r(s) W_1}\right) F(s)\, e^{-\lambda(t-s)}\, ds$$

$s = t - u$

The solution to this integral equation is (*).

Indeed,

$$\frac{\partial}{\partial t} F(t) = \lambda \mathbb{E}\left(e^{i\theta r(t) W}\right) F(t) \underbrace{- \lambda^2 \int_0^t \mathbb{E}\left(e^{i\theta r(s) W}\right) F(s)\, e^{-\lambda(t-s)}\, ds}_{-\lambda F(t)}$$

$$\Leftrightarrow F(t) = \exp\left(\lambda \int_0^t \left(\mathbb{E}\left(e^{i\theta r(s) W}\right) - 1\right) ds\right).$$

6.5. Applications

Olber's paradox. Suppose a star occurs at the points of a Poisson process Π on $\mathbb{R}^3$ with constant intensity λ.

For $x \in \Pi$, let B_x be the brightness of the star at x, and assume that the B_x are i.i.d. with mean β.

The intensity of light striking an observer at the origin O of all stars within distance a is

$$I_a = \sum_{\substack{x \in \Pi \\ |x| \le a}} \frac{B_x}{|x|^2}.$$

Let $N_a = \#\{\Pi \cap B_a(0)\}$. Then

$$\mathbb{E}(I_a \mid N_a) = N_a \beta \frac{1}{|B_a(0)|} \int_{B_a(0)} \frac{1}{|x|^2}\, dx$$

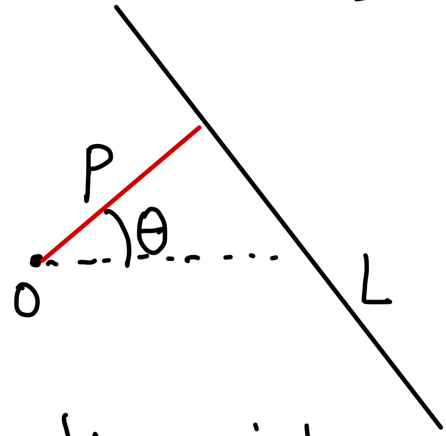
Since $\mathbb{E} N_a = \lambda\, |B_a(0)|$,

$$\mathbb{E}(I_a) = \lambda \beta \int_{B_a(0)} \frac{1}{|x|^2}\, dx = \lambda \beta\, 4\pi a.$$

In particular, the expected intensity is unbounded as $a \to \infty$: Olber's paradox.

Poisson line process Let $S = \{\text{lines in } \mathbb{R}^2\}$

For $L \in S$, define coordinates $(\theta, p) \in [0,\pi) \times \mathbb{R}$ by letting L^T be the line through O perpendicular to L and θ be its angle and p the signed distance of the intersection point.



Note $f: S \longrightarrow [0,\pi) \times \mathbb{R}$ is a bijection.

The Poisson line process on S is now defined via this identification in terms of a Poisson process on $[0,\pi) \times \mathbb{R} \subset \mathbb{R}^2$. We say that the Poisson line process has constant intensity λ if the Poisson process on $\mathbb{R}^2$ has intensity

$$\lambda(\theta, p) = \lambda \quad \text{for } \theta \in [0,\pi),\ p \in \mathbb{R}$$
$$= 0 \quad \text{otherwise}$$

This process on lines is translation and rotation invariant.

For given $(\theta, p) \in [0,\pi) \times \mathbb{R}$ the corresponding line is

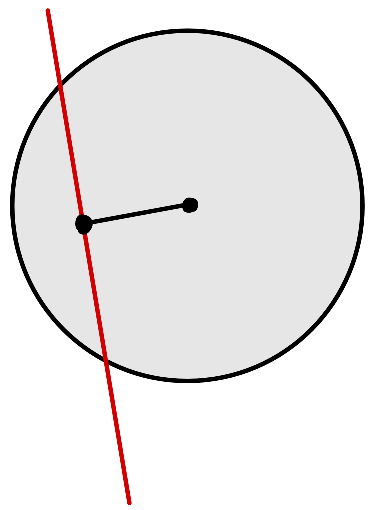
$$L_{\theta,p} = p\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + \mathbb{R}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}.$$

How many lines hit a disk $D_a(0)$?

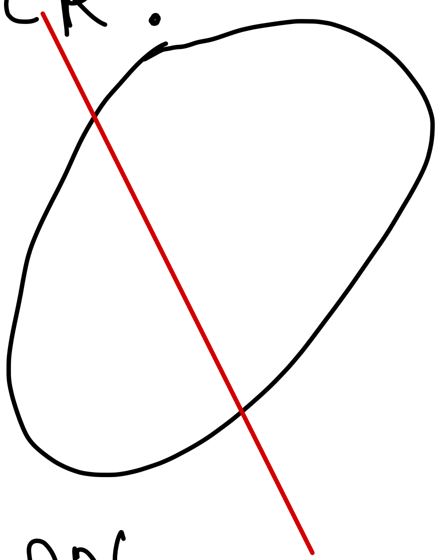
Poisson(μ)

with $\mu = \int_0^\pi \int_{-a}^{a} \lambda \, dr \, d\theta = \lambda 2\pi a$

$= \lambda \times \text{perimeter}(D_a(0))$



How many lines hit a convex $D \subset \mathbb{R}^2$?



Lines hitting D correspond to (θ, p) such that

$$L_{\theta,p} \cap \partial D \neq \emptyset$$

By convexity, for a.a. (p,θ),

$$\mathbb{1}\{L_{\theta,p} \cap \partial D\} = \frac{1}{2} \#\{L_{\theta,p} \cap \partial D\}.$$

$\Rightarrow$ The number of lines hitting D is Poisson with mean given by

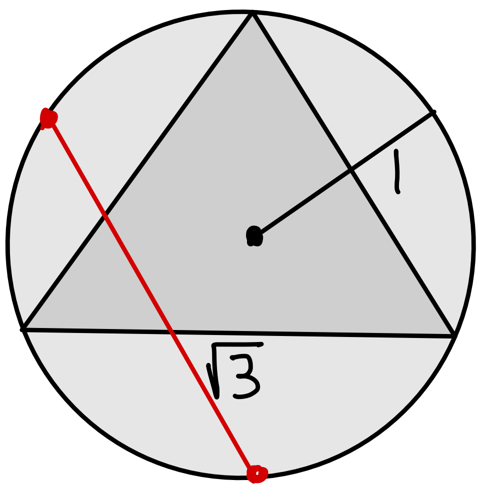
$$\lambda \int_0^\pi \int_{-\infty}^{\infty} \frac{1}{2} \#\{L_{\theta,p} \cap \partial D\} \, d\theta \, dp.$$

The function $C \longmapsto \int_0^\pi \int_{-\infty}^{\infty} \frac{1}{2} \#\{L_{\theta,p} \cap C\} \, d\theta \, dp$ is where C is a curve is additive under concatination, so proportional to arclength.

And from this one can deduce that the number of lines hitting D is Poisson with mean

$$\lambda \times \text{perimeter}(D).$$

<u>Bertrand's paradox</u> What is the probability that a random chord is longer than $\sqrt{3}$?



① Use a Poisson line process (cond. to intersect).

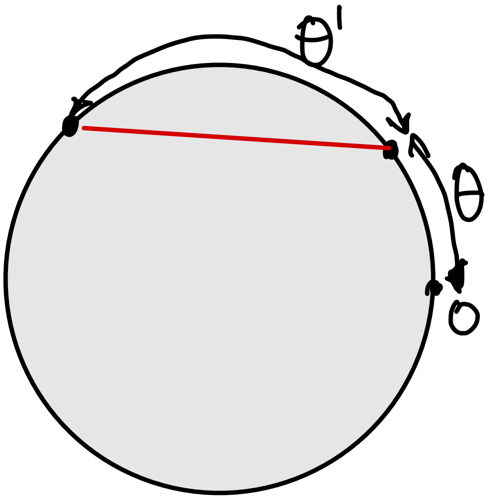
$\Rightarrow$ angle is $\text{Unif}[0,2\pi)$
radius is $\text{Unif}(0,1)$

The length of the chord is $L = 2\sqrt{1-r^2}$

$$\Rightarrow \mathbb{P}(L \geq \sqrt{3}) = \mathbb{P}(2\sqrt{1-R^2} \geq \sqrt{3}) \quad (R \sim \text{Unif}[0,1])$$

$$= \mathbb{P}(R \leq \tfrac{1}{2}) = \tfrac{1}{2}.$$

②



Random end points

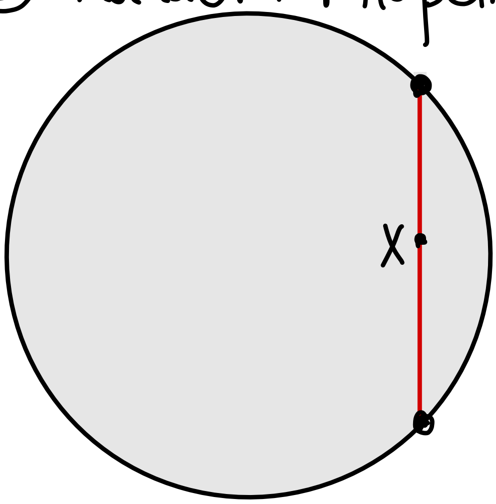
$\theta \sim \mathrm{Unif}[0, 2\pi)$

$\theta' \sim \mathrm{Unif}[0, \pi)$

$\Rightarrow L = 2\sin(\theta'/2)$

$\Rightarrow \mathbb{P}(L \geq \sqrt{3}) = \frac{1}{3}$

③ Random midpoint



$x \sim \mathrm{Unif}(D)$

$\Rightarrow L = 2\sqrt{1 - |x|^2}$

$\Rightarrow \mathbb{P}(L \geq \sqrt{3}) = \frac{1}{4}$.