

1. Compute the average busy period for a $M/M/\infty$ and a $M/M/1$ queue. (The busy period B is the length of time between the arrival of the first customer and the first time afterwards that all servers are free).
2. Consider the $M/M/n$ queue, where the arrival rate is λ and the service rate in each queue is μ . For which values of the parameters is the queue length transient, positive recurrent and null recurrent? Compute the invariant distribution when there exists one.
3. *Queues with balking.* Customers arrive at a single server at rate λ and require an exponential amount of service with rate μ . Customers waiting in line are impatient and if they are not in service they will leave at rate δ , independently of their position in the queue. (a) Show that for any $\delta > 0$ the system has an invariant distribution. (b) Find the invariant distribution when $\delta = \mu$.
4. **Two queues in tandem.** Customers arrive in an $M_\lambda/M_{\mu_1}/1$ queue, and then on departing enter a second $M_\lambda/M_{\mu_2}/1$ queue. When is the paired system reversible in equilibrium? If not can you describe the time-reversal? Find the distribution of the process (D_t) which counts the number of customers whose service is completed by time t .
5. Consider the following queue. Customers arrive at rate $\lambda > 0$ and are served by one server at rate μ . After service, each customer returns to the beginning of the queue with probability $p \in (0, 1)$. Let L_t denote the queue-length at time t . Show that the process $L = (L_t : t \geq 0)$ is a $M/M/1$ queue with modified rates. For which parameters is L transient, and for which is it recurrent?
6. Suppressed.
7. Prove that the traffic equations of an irreducible open Jackson/migration network have a unique solution.
8. Suppressed.
9. Suppressed.
10. Kafkaian Insurances Inc. has a peculiar way of processing claims. Claims arrive at a rate of 10 per day, and are initially randomly assigned to one of two departments, respectively D_1 and D_2 . The service rates in D_1 and D_2 are $\mu_1 = 15$ and $\mu_2 = 20$ per day, respectively. After looking at each claim, the relevant department settles the claim with probability $1/2$, and otherwise finds a pretext to hand it over to the other department to process it. This goes on until the claim is finally settled by one of the two departments. Using a suitable model:
 - (a) what proportion of claims is finally settled by D_1 ?
 - (b) how many claims are settled on average every month by Kafkaian Insurances Inc.?
 - (c) The manager of the company wants to reward the work of his employees based on the number of claims that their department settles. Is that a good idea?
11. Consider a $G/M/1$ queueing system: the n th client arrives at time $A_n = \sum_{i=1}^n \xi_i$, where (ξ_i) is a sequence of nonnegative i.i.d. random variables, and the service times are i.i.d. exponential with rate μ . Let $X_n = L(A_n)$ be the size of the queue just before the n th arrival.
 - (i) Show that (X_n) is a discrete-time Markov chain, and specify its transition matrix.
 - (ii) Show that if $\rho := (\mu \mathbb{E}A)^{-1} < 1$ then the chain (X_n) has a unique equilibrium distribution $\pi = (\pi_i)$ and hence is positive recurrent. Here

$$\pi_i = (1 - \eta)\eta^i, \quad i = 0, 1, \dots$$

and $\eta \in (0, 1)$ is a solution to $\eta = \phi(\mu(\eta - 1))$, where for $\theta \in \mathbb{R}$, $\phi(\theta) = \mathbb{E}(e^{\theta\xi})$.

12. Consider the square lattice \mathbb{Z}^2 , and endow each site $x \in \mathbb{Z}^2$ with a weight W_x , which is an independent exponential random variable of rate μ . An oriented path π between $(1, 1)$ and a point (M, N) , with

$M, N \geq 1$, is called increasing if it only ever goes in the North and East directions. Define the weight of an increasing path π to be $W(\pi) = \sum_{x \in \pi} W_x$, and the passage time from $(1, 1)$ to (M, N) to be

$$T(M, N) = \max_{\pi} W(\pi)$$

where the max is over increasing π 's from $(1, 1)$ to (M, N) . This model is called *Last Passage Percolation*. [Simulations showing optimal paths from $(0,0)$ are interesting.]

The goal of this question is to relate this model to a sequence of N queues operating under the following protocol. At time 0 there are M customers in the first queue, and none at any other queue. Customers are served one at a time at rate μ in each queue, and after service at queue i , a customer moves on to queue $i + 1$. Customers leave the system for good after being served at queue N . Let $\tau(M, N)$ denote the time at which the M th customer completes service in queue N . Show that $\tau(M, N)$ and $T(M, N)$ have the same distribution.

Note for supervisors: The example sheets are essentially the same as those from last year and solutions are available.