Roland Bauerschmidt ⟨rb812@cam.ac.uk⟩ *Lent 2021*

0. Show the superposition and thinning properties of the Poisson process:

Superposition: If N and M are independent Poisson processes of intensities μ and λ then $N + M$ is a Poisson process of intensity $\mu + \lambda$.

Thinning: If N is a Poisson process of intensity μ and X is a sequence of independent Bernoulli random variables with $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0) = p$ then $M(t) = \sum_{n=1}^{N(t)} 1_{X_n=1}$ defines a Poisson process of intensity λp. Also, $N(t) - M(t)$ is a Poisson process of intensity $\lambda(1 - p)$ independent of M.

1. Arrivals of the Number 1 bus form a Poisson process of rate 1 bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate 7 buses per hour.

(a) What is the probability that exactly 5 buses pass by in 1 hour?
(b) What is the probability that exactly 2 Number 7 buses pass by while I am waiting for a Number 1?
(c) When the maintenance depot goes on strike half the buses break down before they reach my stop. What then is the probability that I wait for 30 minutes without seeing a single bus?

2. Let $S_1, S_2, \ldots$ be independent exponential random variables of parameter λ. Show that, for all $n \geq 1$, the sum $T_n = \sum_{i=1}^{n} S_i$ has the probability density function

$$f_{T_n}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \qquad x > 0.$$

This is called the Gamma(n, λ) (or sometimes Gamma(λ, n)) distribution.

Let N be an independent geometric random variable with

$$\mathbb{P}(N = n) = \beta(1 - \beta)^{n-1}, \qquad n = 1, 2, \ldots.$$

Show that $U = \sum_{i=1}^{N} S_i$ has the exponential distribution with parameter $\lambda\beta$. Deduce another proof of the thinning property of Poisson processes.

3. Customers arrive in a supermarket as a Poisson process with rate r. There are r aisles in the supermarket and each customer selects one of them at random, independently of the other customers. Let X_t^r denote the proportion of aisles which are empty at time t and let T^r denote the time until half the aisles are busy (have at least one customer). Show that $X_t^r \to e^{-t}$ and $T^r \to \log 2$ in probability as $r \to \infty$.

4. A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose the passage of vehicles is a Poisson process with rate λ, and suppose it takes time a to walk across the lane. Assuming the pedestrian can foresee correctly the times at which vehicles will pass by, how long on average does it take to cross over safely?

How long on average does it take to cross two similar lanes (a) when one must walk straight across, (b) when an island in the middle of the road makes it safe to stop half way? (Assume the traffic processes in the two lanes are independent Poisson processes with the same rate.)

5. Customers enter a supermarket as a Poisson process with rate 2. There are two salesmen near the door who offer passing customers samples of a new product. Each customer takes an exponential time of parameter 1 to think about the new product, and during this time occupies the full attention of one salesman. Having examined the product, customers proceed into the store and leave by another door. When both salesmen are occupied, customers walk straight in. Assuming that both salesmen are free at time 0, find the probability that both are busy at a later time t.

6. Let $(N_t : t \geq 0)$ be a Poisson process with rate $\lambda > 0$ and let $(X_i : i \geq 0)$ be a sequence of i.i.d. random variables, independent of N. Show that if $g(s, x)$ is a function and T_j are the jump times of N then the following (known as *Campbell's theorem*) holds:

$$\mathbb{E}\left[\exp\left\{\theta\sum_{j=1}^{N_t} g(T_j, X_j)\right\}\right] = \exp\left\{\lambda\int_0^t \left[\mathbb{E}(e^{\theta g(s,X)}) - 1\right] ds\right\}.$$

7. A doubly infinite Poisson process $(N_t : t \in \mathbb{R})$ with $N_0 = 0$ is a process with independent and stationary increments over $\mathbb{R}$. Moreover, for all $-\infty < s \leq t < \infty$, the increment $N_t - N_s$ has the Poisson distribution with parameter $\lambda(t-s)$. Explain how to construct such a process in terms of singly infinite Poisson processes.

8. Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, which then split in the same way but independently. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$. Show that the probability generating function $\phi(t) = \mathbb{E}(z^{X_t})$ satisfies

$$\phi(t) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s}\phi(t-s)^2\, ds$$

and deduce that $\mathbb{P}(X_t = n) = q^{n-1}(1-q)$ for $n = 1, 2, \ldots$, where $q = 1 - e^{-\lambda t}$.

9. Let $\lambda : [0,\infty) \to [0,\infty)$ be a continuous function. An *inhomogeneous Poisson process* with intensity function λ is a process $(N_t : t \geq 0)$ defined as in the infinitesimal definition of a (homogeneous) Poisson process but with

$$\mathbb{P}(N_{t+h} - N_t = 1 \mid N_t = n) = \lambda(t)h + \mathrm{o}(h),$$

(with related changes elsewhere in the definition). Show that N_t has the Poisson distribution with mean $\int_0^t \lambda(u)\, du$.

Particles arrive in the manner of the above inhomogeneous Poisson process. A particle arriving at time t is coloured *green* with probability $\gamma(t)$ (independently of other particles), where g is continuous. Show that the process of green arrivals is an inhomogeneous Poisson process with intensity function $\lambda(t)\gamma(t)$.

10. Let $(q_k : k \geq 1)$ be a sequence of positive numbers such that $q := \sum_k q_k < \infty$. Let $(E_k : k \geq 1)$ be a sequence of independent exponential random variables where the rate (i.e., parameter) of E_k is q_k. Let $E = \inf\{E_k\}$ and let $K = \arg\min\{E_k\}$, with $K = \infty$ if the inf is not attained. Compute $\mathbb{P}(K = k,\ E > t)$ for all $t > 0$ and $k \geq 1$, and hence identify the joint distribution of K and E.

Discuss the relevance of this in the context of constructions of a Markov chain.

11. Let Q be a generator (or Q-matrix) on a finite state space I, and let $f(t) = \det P_t$ where (P_t) is the associated transition semigroup. Show that $f(t+s) = f(t)f(s)$. Deduce that $f(t)$ is of the form e^{tq}, and identify q by taking $t \to 0$.

Note for supervisors: The example sheets are essentially the same as those from last year and solutions are available.