Analysis of Functions (Lent 2023)

Primary references:
C. Warnick, Analysis of Functions, Lecture notes from previous years
E.H. Lieb and M. Loss, Analysis, American Mathematical Society
G.B. Folland, Real Analysis, Wiley

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1. Lebesgue integration theory

1.1. Review of measure theory

<u>Defn</u>. Given a set E, a σ-algebra on E is a collection $\mathcal{E}$ of subsets of E s.t.

(i) $E \in \mathcal{E}$
(ii) $A \in \mathcal{E} \Rightarrow A^c = E \setminus A \in \mathcal{E}$
(iii) $A_n \in \mathcal{E},\ n \in \mathbb{N} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{E}$.

$(E, \mathcal{E})$ is a measurable space, and any $A \in \mathcal{E}$ is called a measurable set.

Given a collection $\mathcal{A}$ of subsets of E, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

<u>Defn</u>. A measure on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$ s.t.

(i) $\mu(\emptyset) = 0$
(ii) $A_n \in \mathcal{E}$, $n \in \mathbb{N}$, disjoint

$$\Rightarrow \mu\Big(\bigcup_n A_n\Big) = \sum_n \mu(A_n).$$

$(E, \mathcal{E}, \mu)$ is called a measure space.

Defn. If (E, τ) is a topological space, the σ-algebra $\sigma(\tau)$ is called Borel algebra, denoted $\mathcal{B}(E)$, and a measure on $(E, \mathcal{B}(E))$ is called a Borel measure.

Example: $E = \mathbb{R}^n$ and μ is the Lebesgue measure satisfying

$$\mu((a_1, b_1) \times \cdots \times (a_n, b_n)) = (b_1 - a_1) \cdots (b_n - a_n).$$

Notation: $\mu(dx) = dx$ and $\mu(A) = |A|$ when μ is the Lebesgue measure.

Defn. Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Then $f: E \to F$ is measurable if $f^{-1}(A) \in \mathcal{E}$ for any $A \in \mathcal{F}$. If $(E, \mathcal{E})$, $(F, \mathcal{F})$ are Borel algebras, a measurable function is called a Borel function.

Special case: $(F, \mathcal{F}) = ([0, +\infty], \mathcal{B}([0, +\infty]))$.
In this case, $f: E \to F$ is called a nonnegative measurable or Borel function.

<u>Fact</u>. The class of measurable functions is closed under addition, multiplication, and taking (pointwise) limits.

<u>Defn</u>. $f: E \to F$ ($F = [0,\infty]$ or $F = \mathbb{R}^n$ or $F = \mathbb{C}^n$) is a simple function if

$$f = \sum_{k=1}^{K} a_k \mathbb{1}_{A_k}$$

for $K \in \mathbb{N}$, $a_k \in F$, $A_k \in \mathcal{E}$. For a nonnegative simple function ($F = [0, +\infty]$), the integral is

$$\int f \, d\mu = \int f(x) \, \mu(dx) = \sum_{k=1}^{K} a_k \, \mu(A_k).$$

$(0 \cdot \infty = 0)$

For a nonnegative measurable function f,

$$\begin{aligned} \int f \, d\mu &= \int f(x) \, \mu(dx) \\ &= \sup \left\{ \int g \, d\mu : g \text{ is simple, } 0 \le g \le f \right\}. \end{aligned}$$

A measurable function $f: E \to \mathbb{R}$ is integrable if $\int |f| \, d\mu < \infty$ and then $f = f_+ - f_-$ with $f_\pm$ nonnegative, measurable, $\int f_\pm \, d\mu < \infty$, and

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu.$$

For $f: E \to \mathbb{R}^n$ this can be applied in any component.

<u>Monotone convergence theorem.</u> Let $(E, \mathcal{E}, \mu)$ be a measure space, and let (f_n) be an increasing sequence of nonnegative meas. functions that converge to f. Then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

<u>Dominated convergence theorem.</u> Let (f_n) be measurable functions on $(E, \mathcal{E}, \mu)$ s.t.

(i) $f_n \to f$ pointwise, a.e.
(ii) $|f_n| \leq g$ a.e. for some integrable g.

Then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

1.2. L^p spaces

Defn. Let $(E,\mathcal{E},\mu)$ be a measure space. For $p\in[1,\infty)$ and $f: E\to \mathbb{R}$ or $\mathbb{C}$ define

$$\|f\|_{L^p} = \left(\int_E |f|^p d\mu\right)^{1/p}$$

and

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}|f| = \inf\{K : |f|\leq K \text{ a.e.}\}.$$

The space L^p, $p\in[0,+\infty]$ is defined by

$$L^p = L^p(E) = L^p(E,\mu) = L^p(E,\mathcal{E},\mu)$$

$$= \{f: E\to\mathbb{R} \text{ measurable} : \|f\|_{L^p}<\infty\}/\sim$$

with $f\sim g$ if $f=g$ a.e.

Riesz–Fisher theorem. L^p is a Banach space, for all $p\in[1,+\infty]$.

Notation: When $E=\mathbb{R}^n$, μ = Lebesgue measure, write $L^p(E,\mu) = L^p(\mathbb{R}^n)$.

Fact. For $p\in[1,\infty)$, the simple functions f with

$$\mu(\{x: f(x)\neq 0\})<\infty \qquad (*)$$

are dense in $L^p(E,\mu)$. For $p=\infty$ drop $(*)$.

Thm. $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $p\in[1,\infty)$.

Exercise: False for $p=\infty$.

Defn. For $f,g:\mathbb{R}^n\to\mathbb{R}$ define $f*g:\mathbb{R}^n\to\mathbb{R}$ by

$$(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy$$

provided the integral exists.

Fact. Let $f,g,h\in C_c^\infty(\mathbb{R}^n)$. Then

$$f*g=g*f,\qquad (f*g)*h=f*(g*h),$$

$$\int_{\mathbb{R}^n} f*g\,dx=\left(\int_{\mathbb{R}^n} f\,dx\right)\left(\int_{\mathbb{R}^n} g\,dx\right)$$

Notation: A multiindex is $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}$.

Set $|\alpha|=\alpha_1+\cdots+\alpha_n$,

$$\alpha!=\alpha_1!\cdots\alpha_n!,$$

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\quad\text{if } x=(x_1,\dots,x_n)\in\mathbb{R}^n,$$

$$\nabla^\alpha f=D^\alpha f=\frac{\partial^\alpha f}{\partial x^\alpha}=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

Defn. $f\in L^p_{loc}(\mathbb{R}^n)$ if $f\mathbb{1}_K\in L^p(\mathbb{R}^n)$ for all $K\subset\mathbb{R}^n$ compact.

Prop. Let $f \in L^1_{loc}(\mathbb{R}^n)$, $g \in C^k_c(\mathbb{R}^n)$, some $k \geq 0$. Then $f * g \in C^k(\mathbb{R}^n)$ and

$$\nabla^\alpha (f*g) = f * (\nabla^\alpha g) \quad \text{for all } |\alpha| \leq k.$$

Proof. First $k=0$. Set $T_z f(x) = f(x-z)$, $z \in \mathbb{R}^n$.

$\Rightarrow$ $T_z(f*g) = f * (T_z g)$ by defn.,

$T_z g(x) \to g(x)$ as $|z| \to 0$ by cont. of g,

$|T_z g(x)| \leq \|g\|_{L^\infty} 1_{B_R(0)}(x)$ if $|x|+1 \leq R$, $|z|<1$,

$$\Rightarrow |f(y) T_z g(x-y)| \leq \underbrace{C |f(y)| 1_{B_R(0)}(x-y)}_{\text{integrable in } y}$$

By DCT, therefore

$$T_z(f*g) = f * T_z g = \int f(y) T_z g(x-y)\, dy$$
$$\to f*g \quad \text{as } |z| \to 0.$$

Thus $f*g \in C^0$.

Now $k=1$. Let $\nabla_i^h g(x) = \dfrac{g(x+he_i) - g(x)}{h}$,

where e_i is the ith unit vector, $h \in \mathbb{R}$, $i = 1, \dots, n$.

$\Rightarrow \nabla_i^h g(x) \longrightarrow \nabla_i g(x)$ as $h \to 0$.

By mean value theorem, there is $t \in [-h,h]$ s.t.

$$\nabla_i^h g(x) = \nabla_i g(x + t e_i).$$

$$\Rightarrow |\nabla_i^h g(x)| \leq \|\nabla_i g\|_{L^\infty} \mathbb{1}_{B_R(0)}(x),$$

Now again, by DCT,

$$\nabla_i^h (f * g) = f * (\nabla_i^h g) \to f * \nabla_i g$$

Thus $f * g \in C^1$.

The case $k>1$ is similar by induction.

<u>Minkowski's integral inequality</u>. Let $p \in [1,\infty)$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ measurable. Then

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x,y)\, dx \right|^p dy \right]^{1/p}$$

$$\leq \int_{\mathbb{R}^n} \left[\int |f(x,y)|^p dy \right]^{1/p} dx$$

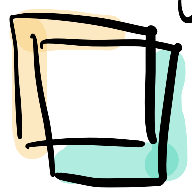
<u>Proof</u>. Ex. Sheet 1.

Prop. Let $p \in [1,\infty)$, $g \in L^p(\mathbb{R}^n)$. Then

$$\| T_z g - g \|_{L^p} \longrightarrow 0 \quad \text{as } |z| \to 0.$$

Proof. Consider first $g = \mathbb{1}_R$, R rectangle. Then the result is clearly true. Hence also true for any finite union of rectangles.

If B is a Borel set, $|B| < \infty$, then for every $\varepsilon > 0$, there is R a finite disjoint union of rectangles s.t.



$$\| \mathbb{1}_B - \mathbb{1}_R \|_{L^p} = |A \triangle B|^{1/p} < \varepsilon.$$

$$\Rightarrow \| T_z \mathbb{1}_B - \mathbb{1}_B \|_{L^p}$$

$$\leq \underbrace{\| T_z \mathbb{1}_B - T_z \mathbb{1}_R \|_{L^p}}_{\substack{= \| \mathbb{1}_B - \mathbb{1}_R \|_{L^p} \\ < \varepsilon}} + \underbrace{\| T_z \mathbb{1}_R - \mathbb{1}_R \|_{L^p}}_{\substack{< \varepsilon \text{ for} \\ |z| \text{ suff. small}}} + \underbrace{\| \mathbb{1}_R - \mathbb{1}_B \|_{L^p}}_{< \varepsilon}$$

$$< 3\varepsilon.$$

Thus the result holds for $g = \mathbb{1}_B$, $B \in \mathcal{B}(\mathbb{R}^n)$.

$\Rightarrow$ The result holds for g simple.

But for any $g \in L^p$ there is $\tilde{g}$ simple such that $\|g-\tilde{g}\|_{L^p} < \varepsilon$. The result follows:

$$\|T_z g - g\|_{L^p} \le \underbrace{\|T_z g - T_z \tilde{g}\|_{L^p}}_{\|g-\tilde{g}\|_{L^p} \le \varepsilon} + \underbrace{\|T_z \tilde{g} - \tilde{g}\|_{L^p}}_{< 3\varepsilon} + \underbrace{\|g-\tilde{g}\|_{L^p}}_{<\varepsilon}$$

$$< 5\varepsilon.$$

<u>Rk</u>. The result is false for $p=\infty$. Let

$$\theta(x) = 1_{x \ge 0} = \begin{cases} 1 & (x \ge 0) \\ 0 & (x<0). \end{cases}$$

$$\Rightarrow T_z\theta - \theta = 1_{x \ge z} - 1_{x \ge 0} \overset{z \ne 0}{\Rightarrow} \|T_z \theta - \theta\|_{L^\infty} = 1.$$

<u>Thm</u>. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be s.t. $\varphi \ge 0$ and $\int_{\mathbb{R}^n} \varphi \, dx = 1$, and set $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$. Then for any $g \in L^p(\mathbb{R}^n)$, $p \in [1,\infty)$, it follows that

$$\varphi_\varepsilon * g \in C^\infty(\mathbb{R}^n)$$

$$\varphi_\varepsilon * g \xrightarrow{\varepsilon \to 0} g \text{ in } L^p.$$

<u>Defn</u>. φ_ε as above is called a (smooth) mollifier.

<u>Cor</u>. $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $p \in [1,\infty)$.

<u>Proof</u> (of corollary). The theorem implies that $C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Since, for any $f \in L^p$, $\|f - f \mathbb{1}_{B_R(0)}\|_{L^p} \to 0$ as $R \to \infty$ by DCT, applying the theorem with $g = f\mathbb{1}_{B_R(0)}$, it also follows that $C_c^\infty(\mathbb{R}^n)$ is dense in L^p.

<u>Proof</u> (of theorem).

$$|\varphi_\varepsilon * g(x) - g(x)| = \left| \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \left[g(x-y) - g(x) \right] dy \right|$$

$\int \varphi_\varepsilon \, dy = 1$

$$= \left| \int_{\mathbb{R}^n} \varphi(z) \underbrace{\left[g(x - \varepsilon z) - g(x) \right]}_{T_{\varepsilon z} g(x) - g(x)} dz \right|$$

$z = y/\varepsilon$

$\varphi \geq 0$

$$\leq \int_{\mathbb{R}^n} \varphi(z) \, |T_{\varepsilon z} g(x) - g(x)| \, dz$$

$$\Rightarrow \|\varphi_\varepsilon * g - g\|_{L^p} = \left(\int_{\mathbb{R}^n} |\varphi_\varepsilon * g(x) - g(x)|^p \, dx \right)^{1/p}$$

$$\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(z)^p \, |T_{\varepsilon z} g(x) - g(x)|^p \, dx \right)^{1/p} dz$$

above + Minkowski

$$= \int_{\mathbb{R}^n} \varphi(z) \, \|T_{\varepsilon z} g - g\|_{L^p} \, dz \xrightarrow{\varepsilon \to 0} 0$$

by DCT since $\varphi(z) \|T_{\varepsilon z} g - g\|_{L^p} \leq 2\varphi(z) \|g\|_{L^p}$ is integrable.

1.3. Lebesgue Differentiation Theorem

<u>Fundamental theorem of calculus</u>. For $f: \mathbb{R} \to \mathbb{R}$ continuous,

$$F(x) = \int_0^x f(t)\, dt$$

is differentiable and $F'(x) = f(x)$.

<u>Lebesgue Differentiation Theorem</u>. For $f: \mathbb{R}^n \to \mathbb{R}$ integrable,

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0, \quad \text{a.e. } x.$$

The x for which this holds are called Lebesgue points.

<u>Cor</u>. If $g \in L^1(\mathbb{R})$ and $G(x) = \int_{-\infty}^{x} g(t)\, dt$ then G is differentiable for a.e. x with

$$G'(x) = g(x).$$

<u>Cor</u>. If φ is a smooth mollifier and $g \in L^p(\mathbb{R}^n)$ then

$$\varphi_\varepsilon * g \xrightarrow{\varepsilon \to 0} g \quad \text{a.e.}$$

Defn. For $f : \mathbb{R}^n \to \mathbb{R}$ integrable, the Hardy-Littlewood Maximal Function $Mf : \mathbb{R}^n \to [0,\infty]$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy.$$

Prop. Let $f \in L^1(\mathbb{R}^n)$. Then Mf is Borel measurable, finite a.e., and

$$|\underbrace{\{Mf > \lambda\}}_{=:A_\lambda}| \leq \frac{3^n}{\lambda} \|f\|_{L^1}.$$

Proof. For each $x \in A_\lambda = \{Mf > \lambda\}$, there is $r_x > 0$ s.t.

$$\frac{1}{|B_{r_x}(x)|} \int_{|B_{r_x}(x)|} |f(y)|\, dy > \lambda.$$

Claim: A_λ is open, i.e., A_λ^c is closed

Suppose $x_k \in A_\lambda^c$, $x_k \to x$. Assume (by contradiction) $x \in A_\lambda$. Then, by DCT,

$$\frac{1}{|B_{r_x}(x_k)|} \int_{B_{r_x}(x_k)} |f(y)|\, dy \xrightarrow{k\to\infty} \frac{1}{|B_{r_x}(x)|} \int_{B_{r_x}(x)} |f(y)|\, dy,$$

same

Since $x_k \in A_\lambda^c$, the LHS is $\leq \lambda$ for each k. Thus the RHS is $\leq \lambda$, a contradiction to $x \in A_\lambda$. So $x \in A_\lambda^c$. Thus A_λ^c is closed, A_λ is open.

Since $A_\lambda = (Mf)^{-1}((\lambda,\infty])$ is open and $(\lambda,\infty]$ generate $\mathcal{B}([0,\infty])$, it follows that Mf is Borel.

Claim: the bound holds.

Let $K \subset A_\lambda$ be compact. Since $\{B_{r_x}(x) : x \in A_\lambda\}$ is an open cover of K, there is a finite subcover

$$K \subset \bigcup_{i=1}^{N} B_i, \qquad B_i = B_{r_x}(x), \text{ some } x \in A_\lambda.$$

Wiener's covering lemma (Ex. Sheet 1): There is a subcollection $(B_{i_k})_k$ of disjoint balls s.t.

$$\left|\bigcup_{i=1}^{N} B_i\right| \leq 3^n \sum_k |B_{i_k}|.$$

Since

$$\frac{1}{|B_i|}\int_{B_i} |f(y)|\,dy > \lambda$$

it follows that

$$|K| \leq 3^n \sum_k |B_{ik}| < \frac{3^n}{\lambda} \sum_k \int_{B_{ik}} |f(y)|\, dy$$

B_{ik} are disjoint $\leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy.$

This holds for all $K \subset A_\lambda$ compact. Since the Lebesgue measure is inner regular, therefore

$$|A_\lambda| \leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy.$$

In particular, $|\{Mf = \infty\}| \leq |\{Mf > \lambda\}| \xrightarrow{\lambda \to \infty} 0$, i.e., $Mf < \infty$ a.e.

<u>Proof</u> of Lebesgue Differentiation Theorem. Let

$$A_\lambda = \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy > 2\lambda \right\}.$$

It suffices to show $|A_\lambda| = 0$. Indeed, the non-Lebesgue points are $\bigcup_n A_{1/n}$, which then also has measure 0.

Given $\varepsilon > 0$, let $g \in C_c^\infty(\mathbb{R}^n)$ be s.t.

$$\|f - g\|_{L^1} < \varepsilon.$$

$$\Rightarrow \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-f(x)|\,dy$$

$$\leq \underbrace{\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-g(y)|\,dy}_{\leq M(f-g)(x)} + |f(x)-g(x)| + \underbrace{\frac{1}{|B_r(x)|}\int_{B_r(x)} |g(y)-g(x)|\,dy}_{\to 0 \text{ by continuity of } g}$$

$$\Rightarrow \limsup_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-f(x)|\,dy \leq M(f-g)(x) + |f(x)-g(x)|$$

If $x \in A_\lambda$ then either $M(f-g)(x) > \lambda$ or $|f(x)-g(x)| > \lambda$.

Previous prop. $\Rightarrow |\{M(f-g) > \lambda\}| \leq \frac{3^n}{\lambda} \|f-g\|_{L^1}$

Markov's ineq. $\Rightarrow |\{|f-g| > \lambda\}| \leq \frac{1}{\lambda} \|f-g\|_{L^1}$

$$\Rightarrow |A_\lambda| \leq \frac{3^n+1}{\lambda} \|f-g\|_{L^1} \leq \frac{3^n+1}{\lambda}\varepsilon \quad \forall \varepsilon$$

$$\Rightarrow |A_\lambda| = 0.$$

1.4. Littlewood's Principles

"Every measurable set is nearly a finite sum of intervals; every function (in L^p) is nearly continuous; every convergent sequence of functions is nearly uniformly convergent."

<u>Thm</u> (Egorov). Let $E \subset \mathbb{R}^n$, $|E| < \infty$, and let $f_k : E \to \mathbb{R}$ be measurable functions s.t. $f_k \to f$ for a.e. $x \in E$. Then for all $\varepsilon > 0$, there is a closed set $A_\varepsilon \subset E$ s.t. $|E \setminus A_\varepsilon| < \varepsilon$ and $f_k \to f$ uniformly on A_ε.

<u>Proof</u>. WLOG $f_k(x) \to f(x)$ for <u>all</u> $x \in E$.

Let $E_k^n = \{x \in E : |f_j(x) - f(x)| < \frac{1}{n} \ \forall j > k\}$

$\Rightarrow E_{k+1}^n \supset E_k^n$, $\bigcup_k E_k^n = E \Rightarrow |E_k^n| \nearrow |E|$.

Let k_n be s.t. $|E \setminus E_{k_n}^n| < 2^{-n}$ and set

$$A_N = \bigcap_{n \geq N} E_{k_n}^n.$$

Then $|E \setminus A_N| \leq \sum_{n \geq N} 2^{-n} \leq 2^{-N+1} < \varepsilon$ for $N = N_\varepsilon$.

Claim: $f_j \to f$ uniformly on A_N.

For $x \in A_N$ and any $n \geq N$, by construction,

$$|f_j(x) - f(x)| < \frac{1}{n} \quad \forall j > k_n$$

$$\Rightarrow \limsup_{j\to\infty} \sup_{A_N} |f_j - f| < \frac{1}{n} \quad \forall n$$

$$\Rightarrow \lim_{j\to\infty} \sup_{A_N} |f_j - f| = 0.$$

<u>Thm</u> (Lusin). Let $f: E \to \mathbb{R}$ be a Borel function, where $E \subset \mathbb{R}^n$, $|E| < \infty$. Then for every $\varepsilon > 0$, there is $F_\varepsilon \subset E$ closed s.t. $|E \setminus F_\varepsilon| < \varepsilon$ and $f|_{F_\varepsilon}$ is continuous.

<u>Careful</u>: This does not mean that f is continuous at $x \in F_\varepsilon$!

<u>Proof</u>. Claim: the statement holds for f simple.

Let $f = \sum_{m=1}^{M} a_m 1_{A_m}$ with the A_m disjoint, and $\bigcup A_m = E$. There are compact $K_m \subset A_m$ with

$$|A_m \setminus K_m| < \frac{\varepsilon}{M}$$

Then $|E \setminus F_\varepsilon| < \varepsilon$ if $F_\varepsilon = \bigcup_m K_m$ and f is constant on each K_m, so continuous.
Since $\text{dist}(K_m, K_{m'}) > 0$ for $m \neq m'$, this means f is continuous on F_ε.

Claim: the statement holds for f measurable.

Let f_n be simple s.t. $f_n \to f$ a.e., and let $C_n \subset E$ be s.t. $|C_n| < 2^{-n}$ and

$$f_n|_{E \setminus C_n} \text{ is continuous.}$$

By Egorov's theorem, there is A_ε s.t. $f_n \to f$ uniformly on A_ε and $|E \setminus A_\varepsilon| < \varepsilon$.

Set $F_\varepsilon' = A_\varepsilon \setminus \bigcup_{n \geq N} C_n$.

$\Rightarrow$ $|E \setminus F_\varepsilon'| < 2\varepsilon$ for $N = N_\varepsilon$.

Since f_n, $n \geq N$ is continuous on F_ε' and $f_n \to f$ uniformly on F_ε', f is continuous on F_ε'. By inner regularity of the Lebesgue measure, there is F_ε closed with $|F_\varepsilon' \setminus F_\varepsilon| < \varepsilon$, so $|E \setminus F_\varepsilon| < 3\varepsilon$ as needed.

2. Banach and Hilbert space analysis

2.1. The Hilbert space L^2

For any measure space $(E, \mathcal{E}, \mu)$, the space $L^2(E,\mu)$ is a Hilbert space with inner product

$$(f,g)_{L^2} = \int_E \bar{f} g \, d\mu.$$

Defn. A set $S = \{u_j\}_{j\in J} \subset H$ of a Hilbert space H is orthogonal if $(u_j, u_k) = 0$ for all $j \neq k$, orthonormal if also $\|u_j\| = 1$ for all j, and complete if $\overline{\operatorname{span}\{u_j\}} = H$. A complete orthonormal set is an orthonormal basis. (Careful: this does not mean it is a basis in the algebraic sense.)

Fact. A Hilbert space is separable (i.e., there is a countable dense set) iff it admits a countable orthonormal basis.

Examples. (i) $L^2([-\pi,\pi])$, $S = \{\frac{1}{\sqrt{2\pi}} e^{-inx}\}_{n\in\mathbb{Z}}$

$\Rightarrow$ S is an orthonormal basis: the Fourier basis. (Completeness follows from Stone-Weierstrass + density of C^∞ in L^2.)

(ii) $L^2(\mathbb{R})$, $S = \{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ where

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k)$$

$$\psi(x) = \begin{cases} 1 & (x \in [0,\tfrac{1}{2}) \\ -1 & (x \in [\tfrac{1}{2}, 1)) \\ 0 & (\text{else} \end{cases}.$$

$\Rightarrow$ S is an orthonormal basis for $L^2(\mathbb{R})$, called the Haar system.

(iii) $L^2(\mathbb{R}, \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx}_{\mu(dx)})$, $\{H_n\}_{n\in\mathbb{Z}_{\geq 0}}$

where the H_n are defined by applying Gram-Schmidt to $\{1, x, x^2, \ldots\}$. The H_n are called Hermite polynomials and form an orthonormal basis of $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx)$.

<u>Riesz representation theorem.</u> For any bounded linear functional $\Lambda : H \to \mathbb{R}$ (resp. $\mathbb{C}$), there is a unique $w \in H$ s.t.

$$\Lambda(u) = (w, u) \quad \forall u \in H.$$

2.2. Radon-Nikodym Theorem

Defn. Let $(E, \mathcal{E})$ be a measurable space and μ and ν two measures on it. Then ν is absolutely continuous with respect to μ, written $\nu \ll \mu$, if for any $A \in \mathcal{E}$,

$$\mu(A) = 0 \Rightarrow \nu(A) = 0.$$

Two measures μ and ν are mutually singular, written $\mu \perp \nu$, if there is $B \in \mathcal{E}$ s.t.

$$\mu(B) = 0 = \nu(B^c).$$

Thm. (Radon-Nikodym). Let μ and ν be finite measures on $(E, \mathcal{E})$ with $\nu \ll \mu$. Then there exists $w \in L^1(E, \mu)$ s.t. for all $A \in \mathcal{E}$,

$$\nu(A) = \int_A w \, d\mu$$

or equivalently, for all $h: E \to [0, \infty]$ Borel,

$$\int h \, d\nu = \int h w \, d\mu.$$

Proof. Set $\alpha = \mu + 2\nu$ and $\beta = 2\mu + \nu$.

Define $\Lambda(f) = \int_E f \, d\beta$.

$$\Rightarrow |\Lambda(f)| \le \int_E |f| \, d\beta \le 2 \int |f| \, d\alpha$$
$$\le 2 \sqrt{\alpha(E)} \, \|f\|_{L^2(E,\alpha)}.$$

Therefore $\Lambda : L^2(E,\alpha) \to \mathbb{R}$ is bounded and linear. By the Riesz representation theorem, there is $g \in L^2(E,\alpha)$ s.t.

$$\Lambda(f) = (g, f)_{L^2(E,\alpha)} \quad \forall f \in L^2(E,\alpha).$$

$$\Leftrightarrow \int f \, d\beta = \int g f \, d\alpha$$

$$\Leftrightarrow \int f (2d\mu + d\nu) = \int g f \, (d\mu + 2d\nu)$$

$$\Leftrightarrow \int f \underbrace{(2-g)}_{\ge 0} \, d\mu = \int f \underbrace{(2g-1)}_{\ge 0} \, d\nu \qquad (*)$$

Claim: $g \in [\frac{1}{2}, 2]$ μ and ν a.e.

and $g \neq \frac{1}{2}$ μ a.e. $\left(\overset{\nu \ll \mu}{\Rightarrow} g \neq \frac{1}{2} \ \nu \text{ a.e.} \right)$

Assuming the claim, the proof is completed as follows: By MCT, $(*)$ can then be extended from $f \in L^2(E,\alpha)$ to all f nonnegative.

Given $h: E \to [0,\infty]$ measurable, set

$$f(x) = \frac{h(x)}{2g(x)-1}, \quad \omega(x) = \frac{2-g(x)}{2g(x)-1}, \quad x \in \underbrace{\{g \neq \tfrac{1}{2}\}}_{|\{\cdots\}|=0}.$$

$$\Rightarrow \int h \, d\nu = \int f(2g-1)\, d\nu$$
$$= \int f(2-g)\, d\mu = \int h\,\omega \, d\mu.$$

In particular, with $h=1$, $\omega \in L^1(E,\mu)$, and with $h = \mathbb{1}_A$, $A \in \mathcal{E}$, the result follows.

Claim: $g \geq \frac{1}{2}$ μ-a.e. and ν-a.e.

Let $f = \mathbb{1}_{A_j}$ for $A_j = \{x \in E : g(x) < \frac{1}{2} - \frac{1}{j}\}$.

$$\Rightarrow \int f(2g-1)\, d\nu \leq -\frac{2}{j}\nu(A_j)$$
$$\int f(2-g)\, d\mu \geq \frac{3}{2}\mu(A_j)$$

$$\Rightarrow \frac{3}{2}\mu(A_j) \leq -\frac{2}{j}\nu(A_j)$$

But $\mu(A_j) \geq 0$ and $\nu(A_j) \geq 0$, so $\mu(A_j) = \nu(A_j) = 0$.

$\Rightarrow g \geq \frac{1}{2}$ μ-a.e. and ν-a.e.

Claim: $g \leq 2$ μ-a.e. and ν-a.e.

Analogous, with $A_j = \{x \in E : g(x) > 2 + \frac{1}{j}\}$.

Claim: $\mu(\{g = \frac{1}{2}\}) = 0$

Set $f = \mathbb{1}_Z$, $Z = \{g = \frac{1}{2}\}$ in $(*)$:

$$\int \mathbb{1}\{g = \tfrac{1}{2}\} \cdot \tfrac{3}{2} \, d\mu = 0 \implies \mu(\{g = \tfrac{1}{2}\}) = 0.$$

2.3. The dual of L^p

A topological vector space (TVS) X is a vector space X with a topology in which $(x,y) \mapsto x+y$ and $(\lambda, x) \mapsto \lambda x$ are continous.

The dual space X' is the linear space of continuous linear maps $\Lambda : X \to \mathbb{R}$ or $\mathbb{C}$.

If X is a normed vector space, then bounded is equivalent to continuous and we define a norm on X' by

$$\|\Lambda\|_{X'} = \sup_{x \in X, \|x\| \leq 1} |\Lambda(x)| .$$

X' is then a Banach space (even if X is not).

Goal: Identify $L^p(\mathbb{R}^n)'$ with $L^q(\mathbb{R}^n)$, $\frac{1}{p}+\frac{1}{q}=1$ if $p \in [1,\infty)$.

Prop. Let $q \in [1,\infty]$. For every $g \in L^q(\mathbb{R}^n)$,

$$\Lambda_g(f) = \int fg \, dx$$

defines $\Lambda_g \in L^p(\mathbb{R}^n)'$ with $\|\Lambda_g\| = \|g\|_{L^q}$.

<u>Proof</u>. By Hölder's inequality,

$\Lambda_g(f) \leq \|f\|_{L^p} \|g\|_{L^q}$.

so $\Lambda_g \in L^p(\mathbb{R}^n)'$ and $\|\Lambda_g\| \leq \|g\|_{L^p}$.

Equality: Ex. Sheet 1.

<u>Cor</u>. The map $J: L^q(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)'$ is a linear isometry (in particular injective).

Thus can identify $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)'$.

<u>Rk</u>. If $p=2$ then $L^2(\mathbb{R}^n)' = L^2(\mathbb{R}^n)$: The map J is surjective.

<u>Thm</u>. Let $p \in [1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$. Then J is surjective, i.e., $L^q(\mathbb{R}^n) = L^p(\mathbb{R}^n)'$.

<u>Rk</u>. $L^1(\mathbb{R}^n)' = L^\infty(\mathbb{R}^n)$ but $L^\infty(\mathbb{R}^n)' \neq L^1(\mathbb{R}^n)$.

<u>Rk</u>. Same true on general measure space if $p \in (1,\infty)$, and σ-finite space for $p=1$. The proof is the same on finite measure spaces but requires different extension argument.

Defn. $\Lambda \in L^p(\mathbb{R}^n)'$ is positive if
$\Lambda(f) \geq 0$ for all $f \in L^p(\mathbb{R}^n)$, $f \geq 0$ a.e.

Lemma. Let $\Lambda \in L^p(\mathbb{R}^n)'$ be positive. Then there is $g \in L^q(\mathbb{R}^n)$ nonnegative with

$$\Lambda(f) = \int g f \, dx \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

Proof. Let $\mu(dx) = e^{-|x|^2} dx$. Then $\mu(\mathbb{R}^n) < \infty$.
Define

$$\nu(A) = \Lambda(e^{-|x|^2/p} \mathbb{1}_A).$$

Claim: ν is a finite measure on $\mathbb{R}^n$

Clearly, $\nu(\emptyset) = 0$ and $\nu(A) \in [0,\infty)$, $A \in \mathcal{B}(\mathbb{R}^n)$.
Let $A_k \in \mathcal{B}(\mathbb{R}^n)$ be disjoint and set $B_m = \bigcup_{k=1}^m A_k$.
Then

$$|\nu(B_\infty) - \nu(B_m)| \leq \|\Lambda\| \cdot \|e^{-|x|^2/p}(\mathbb{1}_{B_\infty} - \mathbb{1}_{B_m})\|_{L^p}$$

$$\leq \|\Lambda\| \, \mu(B_\infty - B_m)^{1/p} \xrightarrow{m\to\infty} 0,$$

so ν is σ-additive, and thus a measure.

Claim: $\nu \ll \mu$

$$\mu(A) = 0 \Rightarrow |\nu(A)| \leq \|\Lambda\| \, \mu(A)^{1/p} = 0$$

Thus, by Radon–Nikodym, there is $\omega \in L^1(\mathbb{R}^n, \mu)$ nonnegative such that

$$\nu(A) = \int_A \omega \, d\mu = \int_A \omega e^{-|x|^2} dx \quad \forall A \in \mathcal{B}.$$

Now let $f = e^{-\frac{|x|^2}{p}} \tilde{f}$ with $\tilde{f}$ simple. Then by linearity of Λ,

$$\Lambda(f) = \int \tilde{f} \, d\nu = \int \tilde{f} \omega e^{-|x|^2} dx$$

$$= \int f \underbrace{\omega e^{-(1-\frac{1}{p})|x|^2}}_{\tilde{\omega} := \omega e^{-\frac{1}{q}|x|^2}} dx$$

Thus $\Lambda(f) = \int f \tilde{\omega} \, dx \quad \forall f$ as above.
Since Λ is bounded, also (for all such f)

$$\int |f \tilde{\omega}| \, dx = \int |f| \tilde{\omega} \, dx = \Lambda(|f|) \leq \|\Lambda\| \, \|f\|_{L^p}$$

Exercise: Functions $f = e^{-\frac{|x|^2}{p}} \tilde{f}$, $\tilde{f}$ simple are dense in $L^p(\mathbb{R}^n)$. So this holds for all $f \in L^p$.

Ex. Sheet 1:

$$\|\tilde{\omega}\|_{L^q} = \sup \left\{ \int |f \tilde{\omega}| \, dx : \|f\|_{L^p} \leq 1 \right\}$$

Thus $\|\tilde{\omega}\|_{L^p} \leq \|\Lambda\|$.

Conversely, $\Lambda(f) \leq \|f\|_{L^p} \|\tilde{\omega}\|_{L^q}$ by Hölder, so

$$\|\tilde{\omega}\|_{L^q} \geq \|\Lambda\|,$$

and thus $\|\tilde{\omega}\|_{L^q} = \|\Lambda\|$. Set $g = \tilde{\omega}$ in statement.

<u>Proof</u> (of theorem, real-valued case).
By Ex. Sheet 2, if $\Lambda \in L^p(\mathbb{R}^n)'$ is $\mathbb{R}$-valued, there are Λ_+ and $\Lambda_- \in L^p(\mathbb{R}^n)'$ positive s.t.

$$\Lambda = \Lambda_+ - \Lambda_- .$$

The claim thus follows from the lemma.

<u>Proof</u> (of theorem, complex-valued case).
If $\Lambda \in L^p(\mathbb{R}^n, \mathbb{C})'$ then

$$\Lambda_r(f) = \operatorname{Re} \Lambda(f), \quad \Lambda_i(f) = \operatorname{Im} \Lambda(f)$$

define two $\mathbb{R}$-linear $\Lambda \in L^p(\mathbb{R}^n)'$ s.t.

$$\Lambda(f_r + i f_i) = \Lambda_r(f_r) - \Lambda_i(f_i) + i \Lambda_r(f_i) + i \Lambda_i(f_r).$$

The claim follows from the real-valued case.

2.4. Riesz-Markov Theorem

Fact. For any finite (positive) regular Borel measure on $\mathbb{R}^n$,

$$\Lambda(f) = \int f \, d\mu$$

defines a positive bounded linear map on $C_c(\mathbb{R}^n)$ (with sup-norm).

Lemma. Λ uniquely determines μ and

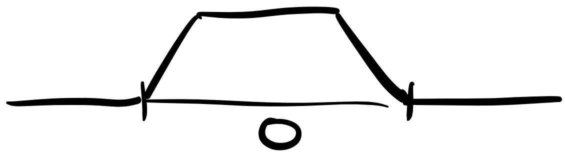
$$\mu(U) = \sup \{ \Lambda(g) : g \in C_c(\mathbb{R}^n), 0 \leq g \leq 1_U \} \quad (*)$$

for any $U \subset \mathbb{R}^n$ open.

Sketch. We would like to take $f = 1_A$, for $A \in \mathcal{B}(\mathbb{R}^n)$, but 1_A is not continuous. Thus approximate: Assume $U \subset \mathbb{R}^n$ is open, set $U_k = U \cap \{ |x| < k \}$, and

$$\chi_k(x) = \begin{cases} 1 & (x \in U_k, \ d(x, U_k^c) \geq \frac{1}{k}) \\ k\, d(x, U_k^c) & (x \in U_k, \ d(x, U_k^c) < \frac{1}{k}) \\ 0 & (x \notin U_k) \end{cases}$$

Then $\chi_k \in C_c(\mathbb{R}^n)$, $\chi_k \uparrow 1_U$, so by MCT,



$$\mu(U) = \lim_{k\to\infty} \int \chi_k \, d\mu = \lim_{k\to\infty} \Lambda(\chi_k).$$

$$\Rightarrow \mu(U) = \sup\{\Lambda(g) : g \in C_c(\mathbb{R}^n), 0 \le g \le 1_U\}$$

Since μ is regular, this determines μ for all $A \in \mathcal{B}(\mathbb{R}^n)$.

<u>Riesz-Markov Thm</u>. Given $\Lambda : C_c(\mathbb{R}^n) \to \mathbb{R}$ positive, bounded, linear, there is a unique finite Borel measure μ on $\mathbb{R}^n$ s.t.

$$\Lambda(f) = \int_{\mathbb{R}} f \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n).$$

The dual space of $C_0(\mathbb{R}^n)$ is the space of signed measures.

<u>Defn</u>. A signed measure is the difference of two mutually singular finite positive measures.

<u>Rk</u>. Boundedness can be dropped.

2.5. Strong, weak, and weak-* topologies

Ex. Sheet 2. Let X be a Banach space. Then the unit ball is compact iff X is finite dimensional.

Goal. Define weaker topologies that give some compactness also in the inf. dimensional case.

Defn. A seminorm p on a vector space X (over $\mathbb{R}$ or $\mathbb{C}$) is a map $p : X \to \mathbb{R}$ s.t.

(i) $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$

(ii) $p(\lambda x) = |\lambda| p(x) \quad \forall x \in X, \lambda \in \mathbb{R}$ or $\mathbb{C}$

(iii) $p(x) \geq 0 \quad \forall x \in X$

A family $\mathcal{P}$ of seminorms is separating if for every $x \in X$ with $x \neq 0$ there is $p \in \mathcal{P}$ s.t. $p(x) \neq 0$.

Defn. The topology $\tau_{\mathcal{P}}$ induced by a family of separating seminorms $\mathcal{P}$ is generated by

$$\mathcal{B} = \{x + B : x \in X, B \in \dot{\mathcal{B}}\}$$

where $\dot{\mathcal{B}}$ consists of finite intersections of

$$V(p, n) = \{x \in X : p(x) < \tfrac{1}{n}\}, \quad p \in \mathcal{P}, n \in \mathbb{N}.$$

$(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space.

Thm. $\mathcal{B}$ is a neighborhood base for $\tau_{\mathcal{P}}$ (every nonempty $U \in \tau_{\mathcal{P}}$ is a union). Moreover, the vector space operations $(x,y) \mapsto x+y$ and $(\lambda, x) \mapsto \lambda x$ as well as each $p \in \mathcal{P}$ are continuous in $\tau_{\mathcal{P}}$.

Ex. Sheet 2: For $(x_k)_{k\in\mathbb{N}} \subset X$, $x_k \to x$ in $\tau_{\mathcal{P}}$ iff $p(x - x_k) \to 0$ for each $p \in \mathcal{P}$.

Fact. If $\mathcal{P} = (p_k)_{k\in\mathbb{N}}$ is countable the topology $\tau_{\mathcal{P}}$ is induced by the metric

$$d_{\mathcal{P}}(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x-y)}{1+p_k(x-y)}.$$

Defn. In the above setting, if the metric d is complete, then $(X, d_{\mathcal{P}})$ is called a Fréchet space.

Examples. (i) X is a Banach space, $\mathcal{P}_s = \{\|\cdot\|\}$. The corresponding topology is the norm topology, also called the strong topology on X.

$$x_k \to x \quad \text{in } \tau_s = \tau_{\mathcal{P}_s}$$
$$\Leftrightarrow \|x_k - x\| \to 0$$

(ii) X is a Banach space, $\mathcal{P}_w = \{p_\Lambda : \Lambda \in X'\}$ where $p_\Lambda(x) = |\Lambda(x)|$ for $\Lambda \in X'$. Each p_Λ is a seminorm and (later) the Hahn-Banach theorem implies that $\mathcal{P}_w$ is separating*. The topology $\tau_w = \tau_{\mathcal{P}_w}$ is called the weak topology.

$x_k \longrightarrow x$ in τ_w

$\Leftrightarrow \Lambda(x_k) \longrightarrow \Lambda(x) \quad \forall \Lambda \in X'$.

* for $X = L^p$ one can show this directly using that $L^q \subset (L^p)'$.

Notation: $x_k \xrightarrow{w} x$

Exercise: $x_k \to x \Rightarrow x_k \xrightarrow{w} x$.

(iii) X is a Banach space. Then X' is a Banach space as well and thus has a strong and weak topology. A third topology is the weak-* topology generated by $\mathcal{P}_{w^*} = \{p_x : x \in X\}$ where $p_x(\Lambda) = |\Lambda(x)|$.

$\Lambda_k \longrightarrow \Lambda$ in τ_{w^*}

$\Leftrightarrow \Lambda_k(x) \longrightarrow \Lambda(x) \quad \forall x \in X$

Notation: $\Lambda_k \xrightarrow{w^*} \Lambda$

Rk. If X is reflexive, $X'' = X$, then $\tau_w = \tau_{w^*}$.

<u>Example</u>. Let $p \in [1,\infty)$ and $f_k \in L^p(\mathbb{R}^n)$.

$$f_i \to f \text{ in } L^p \iff \|f_i - f\|_{L^p} \to 0$$

$$f_i \xrightarrow{w} f \text{ in } L^p \iff \forall g \in L^q : \int g f_i \, dx \to \int g f \, dx$$

$$f_i \xrightarrow{w^*} f \text{ in } L^p \iff \text{same}$$

On the other hand, if $f \in L^\infty(\mathbb{R}^n)$,

$$f_i \to f \text{ in } L^\infty \iff \|f_i - f\|_{L^\infty} \to 0$$

$$f_i \xrightarrow{w^*} f \text{ in } L^\infty \iff \forall g \in L^1 : \int g f_i \, dx \to \int g f \, dx$$

$$\not\Leftrightarrow f_i \xrightarrow{w} f \text{ in } L^\infty$$

2.6. Compactness

Arzelà-Ascoli Thm. Let $I=[0,1]$ (or more generally a compact Hausdorff space). Suppose a sequence of continuous functions $f_k : I \to \mathbb{R}$ is

bounded: $\sup_k \sup_{x\in I} |f_k(x)| < \infty$

equicontinuous: $\forall \varepsilon>0\ \exists \delta>0 : \forall k,\ |x-y|<\delta,$

$$|f_k(x)-f_k(y)| < \varepsilon .$$

Then there is a subsequence (i_k) s.t. f_{i_k} converges to a continuous function f.

Application: $C^{0,\alpha}(I)$ embeds compactly into $C^0(I)$.

Here $C^{0,\alpha}(I) = \{ f \in C^0(I) : \|f\|_{C^{0,\alpha}} < \infty \}$ where

$$\|f\|_{C^{0,\alpha}} = \sup_{x\in I} |f(x)| + \sup_{\substack{x,y\in I\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

The identity map $\mathrm{id} : C^{0,\alpha}(I) \to C^0(I)$ is compact, i.e., any sequence f_i bounded in $C^{0,\alpha}(I)$ has a convergent subsequence in $C^0(I)$.

Later: version where $C^{0,\alpha} \rightsquigarrow$ Sobolev space, $C^0 \rightsquigarrow L^p$

Banach–Alaoglu Thm Let X be a separable Banach space, and let $(\Lambda_j) \subset X'$ be a bounded sequence: WLOG $\sup \|\Lambda_j\|_{X'} \leq 1$. Then there is a subsequence (j_i) and $\Lambda \in X'$ s.t. $\Lambda_{j_i} \xrightarrow{w^*} \Lambda$.

Proof. Step 1. Construction

Let $D = \{x_k\}_{k=1}^{\infty} \subset X$ be dense. Since $(\Lambda_j(x_1))_j$ is bounded, there is a subsequence $J_1 \subset \mathbb{N}$ and $\Lambda(x_1) \in \mathbb{R}$ or $\mathbb{C}$ s.t. $\Lambda_j(x_1) \to \Lambda(x_1)$, for $j \in J_1$, $j \to \infty$, and $|\Lambda_j(x_1)| \leq \|x_1\|$.
Iterating this there are subsequences $J_1 \supset J_2 \supset \cdots$ s.t. $\Lambda_j(x_k) \to \Lambda(x_k)$ for $j \in J_\ell$, $k \leq \ell$.
Now take the 'diagonal subsequence' J of the $J_1 \supset J_2 \supset \cdots$, i.e., the first element is the first in J_1, the second element the second in J_2, etc.
Then $\Lambda_j(x_k) \to \Lambda(x_k)$ $\forall x_k \in D$, $j \in J$, $j \to \infty$.

Step 2. $\Lambda : D \to \mathbb{R}$ is uniformly continuous
($\Rightarrow$ extends to $\Lambda : X \to \mathbb{R}$ continuous)

For each $x, y \in D$ s.t. $\|x-y\| < \varepsilon$ there is $j \in J$ s.t.

$$|\Lambda_j(x) - \Lambda(x)| < \varepsilon, \quad |\Lambda_j(y) - \Lambda(y)| < \varepsilon.$$

$$\Rightarrow |\Lambda(x) - \Lambda(y)| \leq |\Lambda(x) - \Lambda_j(x)| + |\Lambda(y) - \Lambda_j(y)| + |\Lambda_j(x-y)|$$

$$\underbrace{\qquad}_{<\varepsilon} \qquad \underbrace{\qquad}_{<\varepsilon} \qquad \underbrace{\qquad}_{\le \|\Lambda_j\| \varepsilon \le \varepsilon}$$

$$\le 3\varepsilon$$

as needed.

Step 3. $\Lambda : X \to \mathbb{R}$ resp. $\mathbb{C}$ is linear.

For $x, y \in X$, $a \in \mathbb{R}$ or $\mathbb{C}$, let $x', y', z' \in D$ be s.t.

$$\|x - x'\| \le \varepsilon, \quad \|y - y'\| \le \varepsilon, \quad \|x + ay - z'\| \le \varepsilon.$$

Then take j s.t.

$$|\Lambda(x) - \Lambda_j(x')| \le \varepsilon, \quad |\Lambda(y) - \Lambda_j(y')| \le \varepsilon, \quad |\Lambda(z') - \Lambda_j(z')| \le \varepsilon.$$

Then

$$\begin{aligned}
&|\Lambda(x+ay) - \Lambda(x) - a\Lambda(y)| \\
&\le |\underbrace{\Lambda(x+ay) - \Lambda(z')}_{\le \varepsilon}| + |\underbrace{\Lambda(x) - \Lambda(x')}_{\le \varepsilon}| + |a||\underbrace{\Lambda(y) - \Lambda(y')}_{\le \varepsilon}| \\
&\quad + |\underbrace{\Lambda(z') - \Lambda_j(z')}_{\le \varepsilon}| + |\underbrace{\Lambda(x') - \Lambda_j(x')}_{\le \varepsilon}| + |a||\underbrace{\Lambda(y') - \Lambda_j(y')}_{\le \varepsilon}| \\
&\quad + \underbrace{|\Lambda_j(z') - \Lambda_j(x') - a\Lambda_j(y')|}_{\substack{\le \|\Lambda_j\| \|z' - x' - ay'\| \le \|x - x'\| + |a| \|y - y'\| + \varepsilon \\ \le (2 + |a|)\varepsilon}} \\
&\le (6 + 3a)\varepsilon \quad \forall \varepsilon > 0
\end{aligned}$$

$$\Rightarrow \Lambda(x+ay) - \Lambda(x) - a\Lambda(y) = 0$$

Step 4. $\|\Lambda\| \le 1$

$$\|\Lambda\| = \sup_{\substack{x\in X \\ \|x\|\le 1}} \Lambda(x) = \sup_{\substack{x\in D \\ \|x\|\le 1}} \Lambda(x) \le 1 \text{ by density.}$$

Step 5. $\Lambda_j \xrightarrow{w^*} \Lambda$

$$|\Lambda_j(x)-\Lambda(x)| \le \underbrace{|\Lambda_j(x-x')|}_{\le \|x-x'\|\le\varepsilon} + \overbrace{|\Lambda_j(x')-\Lambda(x')|}^{\to 0 \ \forall x'\in D} + \underbrace{|\Lambda(x-x')|}_{\le\|x-x'\|\le\varepsilon}$$

for any $x\in X$, $x'\in D$.

$\Rightarrow |\Lambda_j(x)-\Lambda(x)| \longrightarrow 0$.

Example. Let $p\in(1,\infty]$ and $(f_j)\subset L^p(\mathbb{R}^n)$ s.t. $\|f_j\|_{L^p}\le K$ $\forall j$. Then there is $f\in L^p(\mathbb{R}^n)$ with $\|f\|_{L^p}\le K$ and a subsequence (j_k) s.t.

$$\forall g\in L^q(\mathbb{R}^n): \int f_{j_k} g \, dx \longrightarrow \int fg\, dx.$$

Proof. $L^p(\mathbb{R}^n)' = L^q(\mathbb{R}^n)$ with $q\in[1,\infty)$ and L^q is separable for $q\in[1,\infty)$.

2.7. Hahn-Banach Theorem

Suppose $\tilde{\Lambda}: M \to \mathbb{R}$ is a bounded linear functional defined on a subspace $M \subset X$ of a Banach space.
Goal: Extend $\tilde{\Lambda}$ to $\Lambda: X \to \mathbb{R}$ with $\|\Lambda\| = \|\tilde{\Lambda}\|$.

<u>Defn.</u> Let X be a real vector space. Then $p: X \to \mathbb{R}$ is sublinear if

(i) $p(x+y) \leq p(x) + p(y) \qquad \forall x, y \in X$
(ii) $p(tx) = t\,p(x) \qquad \forall x \in X,\ t \geq 0$

<u>Example.</u> • $p(x) = |\ell(x)|$ for $\ell: X \to \mathbb{R}$ linear.
• any seminorm.

Note: if p is sublinear, ℓ linear, $\ell(x) \leq p(x)\ \forall x$, then
$$-p(-x) \leq \ell(x) \leq p(x)$$

<u>Lemma.</u> (Bounded extension lemma). Let X be a real vector space, $p: X \to \mathbb{R}$ sublinear, and $M \subset X$ a subspace. Assume $\ell: M \to \mathbb{R}$ is linear,
$$\ell(y) \leq p(y) \quad \forall y \in M.$$
For $x \in X \setminus M$, let $\tilde{M} = \operatorname{span}\{M, x\}$. Then there is

$\tilde{\ell}: \tilde{M} \to \mathbb{R}$ linear s.t. $\tilde{\ell}(y) \le p(y)$ $\forall y \in \tilde{M}$ and $\tilde{\ell}(y) = \ell(y)$ $\forall y \in M$.

<u>Proof</u>. If $z \in \tilde{M}$ there are unique $y \in M$ and $\lambda \in \mathbb{R}$ s.t. $z = y + \lambda x$. Define $\tilde{\ell}(x) = a$ for a to be chosen later, $\tilde{\ell}(y) = \ell(y)$ for $y \in M$ and $\ell(z)$ by linearity if $z \in \tilde{M} \setminus M$: $\ell(y + \lambda x) = \ell(y) + \lambda a$.

For each $y, z \in M$,

$$\ell(y) + \ell(z) = \ell(y+z) \le p(y+z) \le p(y-x) + p(x+z).$$

$$\Rightarrow \ell(y) - p(y-x) \le p(z+x) - \ell(z) \quad (*)$$

Let $a = \sup\{\ell(y) - p(y-x) : y \in M\}$ which is finite by $(*)$ and satisfies

$$\ell(y) - a \le p(y-x) \ \forall y \in M. \qquad (*')$$

Also, by $(*)$,

$$\ell(z) + a \le p(z+x) \underbrace{+ p(y-x) - \ell(y)}_{-(\ell(y) - p(y-x))} + a \quad \forall y$$

take $\inf_y$

$$\Rightarrow \ell(z) + a \le p(z+x) - \underbrace{\sup_y(\ell(y) - p(y-x))}_{a} + a$$
$$= p(z+x). \qquad (*'')$$

$$\Rightarrow \quad \ell(y) + a\lambda \le p(y+\lambda x) \qquad \forall y \in M, \lambda > 0$$

(set $z = \lambda^{-1} y$ in $(*'')$, multiply by λ)

and

$$\ell(y) + a\lambda \le p(y+\lambda x) \qquad \forall y \in M, \lambda < 0$$

(set, $y \rightsquigarrow |\lambda|^{-1} y$ in $(*')$ and multiply by $|\lambda|$)

$$\Rightarrow \ell(z) \le p(z) \quad \forall z \in \tilde{M}.$$

<u>Cor.</u> If M has finite codimension in X, then any $\ell : M \to \mathbb{R}$ satisfying $\ell(y) \le p(y)$ $\forall y \in M$ can be extended to $\tilde{\ell} : X \to \mathbb{R}$ linear with $\tilde{\ell}(x) \le p(x)$ $\forall x$.

<u>Proof</u>. Apply lemma repeatedly.

Similarly one can proceed for X separable. For X not separable one needs the Axiom of Choice in the form of Zorn's lemma.

<u>Defn</u>. Let S be a set. A partial order on S is a binary relation $\le$ satisfying

(i) $a \le a \qquad \forall a \in S$ (reflexivity)
(ii) $a \le b,\ b \le a \Rightarrow a = b$ (antisymmetry)
(iii) $a \le b,\ b \le c \Rightarrow a \le c$ (transitivity).

A set with a partial ordering is called a poset (partially ordered set).

Rk. It is not necessary that $a \le b$ or $b \le a$ holds. If either does hold for any $a, b \in S$, then $\le$ is called a total order.

Defn. A totally ordered subset $T \subset S$ of a poset S is called a chain. An element $u \in S$ is an upper bound for $T \subset S$ if $a \le u \ \forall a \in T$. A maximal element $m \in S$ is an element such that $m \le x$ implies $x = m$.

Examples. (i) If A is any set, $S = 2^A$ is a poset ordered by set inclusion.

(ii) $\mathbb{R}$ with standard order is a totally ordered set with no maximal element.

(iii) The collection of open balls in $\mathbb{R}^n$ is a poset ordered by inclusion. The subset

$$T = \{ B_r(0) : 0 < r \le 1 \}$$

is a chain. $B_1(0)$ is a maximal element of T. $B_2(0)$ is an upper bound for T.

<u>Zorn's Lemma</u> Let $(S, \leq)$ be a poset in which every totally ordered subset has an upper bound. Then $(S,\leq)$ contains at least one maximal element.

Zorn's Lemma is an axiom for us.

<u>Thm</u> (Hahn-Banach). Let X be a real vector space, $p: X \to \mathbb{R}$ sublinear and $M \subset X$ a subspace. For every $\ell: M \to \mathbb{R}$ linear s.t. $\ell(x) \leq p(x)$ for all $x \in M$, there is $\tilde{\ell}: X \to \mathbb{R}$ linear s.t. $\hat{\ell}(y) = \ell(y)$ $\forall y \in M$ and

$$\hat{\ell}(x) \leq p(x) \quad \forall x \in X.$$

<u>Proof</u>. Let

$$S = \{ (N, \tilde{\ell}) : \begin{array}{l} X \supseteq N \supseteq M \text{ as subspaces,} \\ \tilde{\ell}: N \to \mathbb{R} \text{ is linear,} \\ \tilde{\ell}(x) \leq p(x) \quad \forall x \in N, \\ \tilde{\ell}(x) = \ell(x) \quad \forall x \in M \end{array} \} \; (\neq \emptyset).$$

and define the partial order

$$(N_1, \tilde{\ell}_1) \leq (N_2, \tilde{\ell}_2) \Leftrightarrow N_1 \subseteq N_2, \; \hat{\ell}_1(z) = \tilde{\ell}_2(z) \; \forall z \in N_1.$$

for every totally ordered $T \subset S$, we obtain an upper bound as follows. Define

$$N_T = \bigcup_{(N,\tilde{\ell}) \in T} N$$

$$\ell_T(x) = \tilde{\ell}(x) \text{ if } x \in N \text{ for some } (N, \tilde{\ell}) \in T.$$

This is well-defined since T is totally ordered. Further, $(N, \tilde{\ell}) \leq (N_T, \ell_T)$ for every $(N, \tilde{\ell}) \in T$. Thus (N_T, ℓ_T) is an upper bound.

Applying Zorn's Lemma, there is a maximal element $(\tilde{N}, \tilde{\ell})$ of S.

Claim: $\tilde{N} = X$.

Suppose not: then there is $x \in X \setminus \tilde{N}$ and the bounded extension lemma produces an extension ℓ^* of $\tilde{\ell}$ to $N^* = \operatorname{span}\{\tilde{N}, x\}$ s.t. $(\tilde{N}, \tilde{\ell}) \leq (N^*, \ell^*)$. Since $(\tilde{N}, \tilde{\ell}) \neq (N^*, \ell^*)$ this is a contradiction to the maximality of $(\tilde{N}, \tilde{\ell})$.

Thus $\tilde{\ell} : X \to \mathbb{R}$ is the required extension.

Cor. Let X be a normed vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $M\subset X$ a subspace. For every bounded linear $\Lambda: M\to\mathbb{K}$ there is a bounded linear $\tilde{\Lambda}: X\to\mathbb{K}$ s.t. $\|\tilde{\Lambda}\|_{X'}=\|\Lambda\|_{M'}$ and $\tilde{\Lambda}|_M=\Lambda$.

Proof. If $\mathbb{K}=\mathbb{R}$, then $p(x)=\|\Lambda\|\,\|x\|$ is sublinear (or $\|\Lambda\|=0$, which is trivial) and the result follows immediately from the theorem.

If $\mathbb{K}=\mathbb{C}$, then $\Lambda(x)=\ell(x)-i\ell(ix)$ with $\ell: X\to\mathbb{R}$, $\ell(x)=\operatorname{Re}\Lambda(x)$ a real-linear functional. Since $|\Lambda(x)|=\ell(e^{i\theta}x)$ for suitable θ,

$$\sup_{\substack{\|x\|\leq 1\\ x\in N}}|\Lambda(x)|=\sup_{\substack{\|x\|\leq 1\\ x\in N}}|\ell(x)| \quad , \quad M\subset X$$

Apply theorem to ℓ and reconstruct Λ.

Cor. Let X be a normed vector space and $x\in X$. Then there is $\Lambda_x\in X'$ s.t. $\|\Lambda_x\|=1$ and $\Lambda_x(x)=\|x\|$.

Λ_x is called a support functional for x.

Proof. Let $M=\{x\}$ and define $\ell\in M'$ by $\ell(tx)=t\|x\|$, $t\in\mathbb{R}$ or $\mathbb{C}$. Clearly $\|\ell\|=1$ and $\ell(x)=\|x\|$. Apply Hahn-Banach.

Ex Sheet 1. For $X = L^p(\mathbb{R}^n)$, $p \in [1,\infty]$, can construct support functionals by hand. ($p=\infty$ uses that $\mathbb{R}^n$ is σ-finite.)

Cor. Let X be a normed vector space and $x \in X$. Then $x = 0 \Leftrightarrow \Lambda(x) = 0 \quad \forall \Lambda \in X'$.

Cor. Let X be a normed vector space and $x, y \in X$. Then there exists $\Lambda \in X'$ s.t. $\Lambda(x) \neq \Lambda(y)$: linear functionals separate.

Proof. Take $\Lambda = \Lambda_{x-y}$.

Cor. The map $\Phi : X \to X''$, $\Phi(x) = \tilde{x}$, where $\tilde{x}(\Lambda) = \Lambda(x)$ is an isometry.

Proof. By defn,

$$\|\Phi(x)\|_{X''} = \sup_{\substack{\Lambda \in X' \\ \|\Lambda\| \leq 1}} |\Phi(x)\Lambda| = \sup_{\substack{\Lambda \in X' \\ \|\Lambda\| \leq 1}} |\Lambda(x)| \leq \|x\|_X .$$

By choosing $\Lambda = \Lambda_x$, there is equality.

Defn. X is reflexive if Φ is surjective: $X = X''$.

Example. $L^p(\mathbb{R}^n)$ is reflexive iff $p \in (1,\infty)$.

Thm. Let $A, B \subset X$ be disjoint, nonempty, convex subsets of a (real or complex) normed space X.

(a) If A is open, there exists $\Lambda \in X'$ and $\gamma \in \mathbb{R}$ s.t.

$$\operatorname{Re} \Lambda(x) < \gamma \leq \operatorname{Re} \Lambda(y) \quad \forall x \in A, y \in B$$

If B is open, the second inequality can be taken strict.

(b) If A is compact, B closed, there exists $\Lambda \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ s.t.

$$\operatorname{Re} \Lambda(x) < \gamma_1 < \gamma_2 < \operatorname{Re} \Lambda(y) \quad \forall x \in A, y \in B.$$

Proof. Assume X is a vector space over $\mathbb{R}$. (Otherwise apply to real part.)

(a) Fix $a_0 \in A$, $b_0 \in B$, and set

$$x_0 = b_0 - a_0, \qquad C = A - B - x_0 \ni 0.$$

Note C is convex (since A and B are), $x_0 \notin C$ (since $A \cap B = \emptyset$).

Let $p(x) = \inf\{t>0 : t^{-1}x \in C\}$.

Ex. Sheet 2. p is sublinear,
$p(x) \leq k\|x\| \ \forall x \in X$,
$p(y) < 1 \Leftrightarrow x \in C$.

Define $M = \{tx_0 : t \in \mathbb{R}\}$, $\ell : M \to \mathbb{R}$, $\ell(tx_0) = t$.

Claim: $\ell(x) \leq p(x) \ \forall x \in M$.

$t>0$: $\ell(tx_0) = t \underset{\uparrow}{\leq} tp(x_0) = tp(x_0)$

$x_0 \notin C \Rightarrow p(x_0) \geq 1$

$t \leq 0$: $\ell(tx_0) = t \leq 0 \leq p(tx_0)$

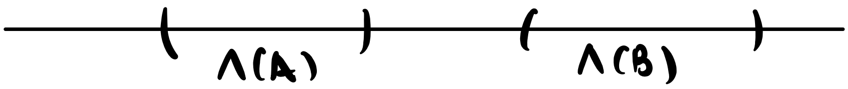
By Hahn-Banach, ℓ extends to $\Lambda : X \to \mathbb{R}$, $\Lambda(x) \leq p(x) \ \forall x \in X$. Moreover,

$$-k\|x\| \leq -p(-x) \leq \Lambda(x) \leq p(x) \leq k\|x\|$$

$\Rightarrow \Lambda \in X'$.

Claim: $\Lambda(a) < \Lambda(b)$ for $a \in A$, $b \in B$.

$$\underbrace{\Lambda(a-b+x_0)}_{\Lambda(a)-\Lambda(b)+1} \leq p(\underbrace{a-b+x_0}_{\in C}) < 1 \Rightarrow \Lambda(a) < \Lambda(b)$$



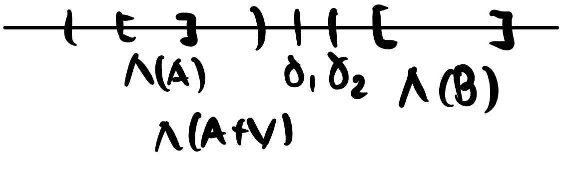
Ex. Sheet 2: $\Lambda \in X' \Rightarrow \Lambda$ is open

Thus $\Lambda(A)$ is an open interval (A is open).
Take γ to be the right endpoint.

If B is also open, the inequality is strict.

(b) Since A is compact, B closed, $A \cap B = \emptyset$
$d = \inf\{\|a-b\| : a \in A, b \in B\} > 0$

Let $V = B_{d/2}(0)$. Then $A+V$ is open and disjoint from B. By (a) there is $\Lambda \in X'$ s.t. $\Lambda(A+V)$ and $\Lambda(B)$ are disjoint convex subsets of $\mathbb{R}$:



The result follows since $\Lambda(A+V)$ is open, $\Lambda(A)$ is a compact subset, and $\Lambda(B)$ is closed.

<u>Cor.</u> Let X be a Banach space and $M \subset X$ a subspace and $x_0 \in X$. If $x_0 \notin \overline{M}$ then there is $\Lambda \in X'$ s.t. $\Lambda(x_0) = 1$ and $\Lambda(x) = 0$ $\forall x \in M$.

Proof. Apply (b) with $A=\{x_0\}$, $B=\overline{M}$. This gives $\Lambda \in X'$ s.t. $\Lambda x_0 \notin \Lambda(\overline{M})$. Thus $\Lambda(\overline{M})$ must be a proper subspace of $\mathbb{K}$, so $\{0\}$. Also $\Lambda x_0 \neq 0$, so $\Lambda / \Lambda(x_0)$ is the required functional.

3. Distributions

Distributions are generalised functions.

Example: $G(x) = \frac{-1}{4\pi|x|}$, $x \in \mathbb{R}^3$ solves

$$-\Delta G = \delta \text{ as distributions.}$$

This means that, for sufficiently nice f,

$$\int (-\Delta f) G \, dx = f(0)$$

$$\text{"} \int f(-\Delta G) \, dx \text{"} \qquad \text{"} \int f(x) \delta(x) \, dx \text{"}$$

3.1. The spaces $\mathcal{D}(U)$ and $\mathcal{D}'(U)$

For $U \subset \mathbb{R}^n$,

$$C_c^\infty(U) = \{\phi : U \to \mathbb{R} \text{ smooth, } \operatorname{supp} f \subset U\}$$

Thm. There is a topology on $C_c^\infty(U)$ s.t.

(i) The vector space operations are continuous.

(ii) $(\phi_j) \subset C_c^\infty(U)$ converges to 0 iff there is $K \subset U$ compact s.t. $\operatorname{supp} \phi_j \subset K \ \forall j$ and

$$\forall \alpha: \quad \sup_K |\nabla^\alpha \phi_j| \longrightarrow 0.$$

(iii) If Y is a LCTVS and $\Lambda : C_c^\infty(U) \to Y$ is linear then Λ is continuous iff it is sequentially continuous.

__Defn.__ $C_c^\infty(U)$ with the above topology is called the space of test functions and denoted $\mathcal{D}(U)$.

__Examples.__ Let $\phi \in C_c^\infty(\mathbb{R})$.

(a) if $\phi_j(x) = e^{-j}\phi(jx)$ then $\phi_j \to 0$ in $\mathcal{D}(\mathbb{R})$

(b) if $\phi_j(x) = j^{-100}\phi(jx)$ then $\phi_j \not\to 0$ in $\mathcal{D}(\mathbb{R})$

(c) if $\phi_j(x) = e^{-j}\phi(x-j)$ then $\phi_j \not\to 0$ in $\mathcal{D}(\mathbb{R})$

__Defn__ The space of distributions $\mathcal{D}'(U)$ is the dual space of $\mathcal{D}(U)$ of continuous linear maps $u : \mathcal{D}(U) \to \mathbb{R}$ with the weak-$*$ topology.

Thus $u \in \mathcal{D}'(U)$ iff

$$u(\phi_j) \longrightarrow u(\phi) \qquad \text{if } \phi_j \to \phi \text{ in } \mathcal{D}(U).$$

and $u_j \longrightarrow u$ in $\mathcal{D}'(U)$ iff

$$u_j(\phi) \to u(\phi) \;\forall \phi \in \mathcal{D}(U).$$

<u>Example</u>. (a) For $x \in U$, define

$$\delta_x : \mathcal{D}(U) \to \mathbb{R}, \quad \delta_x(\phi) = \phi(x)$$

This is the Dirac or delta distribution.

(b) If $f \in L^1_{loc}(\mathbb{R}^n)$ then

$$T_f : \mathcal{D}(U) \to \mathbb{R}, \quad T_f(\phi) = \int_U f \phi \, dx$$

defines $T_f \in \mathcal{D}'(U)$.

<u>Fact</u>. $T_f = T_g \Leftrightarrow \int (f-g)\phi \, dx = 0 \quad \forall \phi \in C_c^\infty(\mathbb{R}^n)$

$\Leftrightarrow f = g$ a.e.

Thus the map $T : L^1_{loc}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, $f \mapsto T_f$ is an injection.

<u>Example</u>. If $\alpha \in C^\infty(U)$ then

$$T_{\alpha f}(\phi) = \int f \alpha \phi \, dx = T_f(\alpha\phi) \quad \forall \phi \in \mathcal{D}(U).$$

<u>Defn</u>. If $u \in \mathcal{D}'(U)$ define $\alpha u \in \mathcal{D}'(U)$ by

$$\alpha u(\phi) = u(\alpha\phi) \qquad \forall \phi \in \mathcal{D}(U).$$

<u>Example</u>. If $f \in C^1(U)$ then

$$T_{\nabla_i f}(\phi) = \int (\nabla_i f)\phi \, dx = -\int f(\nabla_i \phi)\, dx$$
$$= -T_f(\nabla_i \phi) \quad \forall \phi \in C_c^\infty$$

<u>Defn</u>. If $u \in \mathcal{D}'(U)$ define $\nabla^\alpha u \in \mathcal{D}'(U)$ by

$$\nabla^\alpha u(\phi) = (-1)^{|\alpha|} u(\nabla^\alpha \phi) \quad \forall \phi \in C_c^\infty$$

<u>Example</u>. Define $H : \mathbb{R} \to \mathbb{R}$ by $H(x) = \begin{cases} 1 & (x \geq 0) \\ 0 & (x < 0). \end{cases}$
Then for $\phi \in \mathcal{D}(\mathbb{R})$,

$$\nabla T_H(\phi) = -\int_{\mathbb{R}} H \phi' \, dx = -\int_0^\infty \phi'(x)\, dx = \phi(0)$$
$$= \delta_0(\phi)$$

$\Rightarrow \nabla T_H = \delta_0$ or $H' = \delta_0$ in the sense of distributions.

3.2. The spaces $\mathcal{E}(U)$ and $\mathcal{E}'(U)$

Now consider $C^\infty(U) = \{\phi : U \to \mathbb{R} \text{ smooth}\}$.

Let $K_i \subset U$ be compact sets s.t. $K_i \subset \mathring{K}_{i+1}$, $U = \bigcup_i K_i$

For $\phi \in C^\infty$ define

$$p_N(\phi) = \sup_{x \in K_N} \sup_{|\alpha| \le N} |\nabla^\alpha \phi(x)|$$

$\mathcal{P} = \{p_N\}$ is a separating family of seminorms.

<u>Defn</u>. The space $C^\infty(U)$ with the locally convex topology induced by $\mathcal{P}$ is denoted $\mathcal{E}(U)$.

Since $\mathcal{P}$ is countable, $\mathcal{E}(U)$ is a metric space. It is complete, i.e., a Fréchet space.

$(\phi_j) \subset \mathcal{E}(U)$ converges to 0 in $\mathcal{E}(U)$ iff

$$\forall K \subset U \text{ compact} : \forall \alpha : \sup_{x \in K} |\nabla^\alpha \phi_j(x)| \to 0.$$

<u>Example</u>. If $\phi \in C_c^\infty(\mathbb{R})$ then $\phi_j(x) = e^{-j}\phi(x-j)$ converges to 0 in $\mathcal{E}(\mathbb{R})$.

<u>Fact</u>. $\mathcal{D}(U) \subset \mathcal{E}(U)$ continuously

Thus $\mathcal{E}'(U) \subset \mathcal{D}'(U)$.

Lemma. Let $u : \mathcal{E}(U) \to \mathbb{R}$ be linear. Then u is continuous iff

(*) There is $K \subset U$ compact, $N \in \mathbb{N}$, $C > 0$ s.t.

$$|u(\phi)| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |\nabla^\alpha \phi(x)|.$$

Proof. Since $\mathcal{E}(U)$ is a metric space $u \in \mathcal{E}'(U)$ iff $u(\phi_j) \longrightarrow 0$ for all sequences $(\phi_j) \subset \mathcal{E}(U)$ with $\phi_j \to 0$ in $\mathcal{E}(U)$.

Now assume (*) and let $(\phi_j) \subset \mathcal{E}(U)$, $\phi_j \to 0$.

$\Leftrightarrow$ $\forall \tilde{K} \subset U$ compact, $\tilde{N} \in \mathbb{N}$

$$\sup_{\substack{x \in \tilde{K} \\ |\alpha| \leq \tilde{N}}} |\nabla^\alpha \phi_j(x)| \longrightarrow 0.$$

Thus taking $\tilde{K} = K$, $\tilde{N} = N$, (*) implies $u(\phi_j) \to 0$.

Now suppose (*) does not hold. Let $K_j \subset U$ be compact s.t. $K_j \subset \mathring{K}_{j+1}$ and $\bigcup_j K_j = U$. Since (*) is false, for each j there is $\phi_j \in \mathcal{E}(U)$ s.t.

$$|u(\phi_j)| \geq j \sup_{x \in K_j} \sup_{|\alpha| \leq j} |\nabla^\alpha \phi_j(x)|.$$

Then $\psi_j = \phi_j / |u(\phi_j)| \longrightarrow 0$ in $\mathcal{E}(U)$ since $\forall \tilde{K} \subset U$ compact, $\tilde{N} \in \mathbb{N}$, $\exists J > \tilde{N}$ s.t. $\tilde{K} \subset K_j, \forall j \geq J$,

$$\sup_{\substack{x \in \tilde{K} \\ |\alpha| \leq \tilde{N}}} |\nabla^\alpha \psi_j(x)| \leq 1/j \qquad \forall j \geq J.$$

But $|u(\psi_j)| = 1$, so $u(\psi_j) \not\to 0$: a contradiction to continuity.

Recall: $\mathcal{D}(U) \subset \mathcal{E}(U) \Rightarrow \mathcal{E}'(U) \subset \mathcal{D}'(U)$.

<u>Defn</u>. $u \in \mathcal{D}'(U)$ has support in S if

$$u(\phi) = 0 \quad \forall \phi \in C_c^\infty(U \backslash S).$$

If S can be chosen compact then u has compact support.

<u>Cor</u>. $\mathcal{E}'(U) = \{u \in \mathcal{D}'(U) : u \text{ has compact support}\}$.

<u>Proof</u>. If $u \in \mathcal{E}'(U)$ the last lemma implies that u has compact support.

Conversely, if $u \in \mathcal{D}'(U)$ has support in $K \subset U$ compact, define $\tilde{u} \in \mathcal{E}'(U)$ by

$$\tilde{u}(\phi) = u(\chi\phi) \qquad \forall \phi \in \mathcal{E}(U)$$

where $\chi \in C_c^\infty(U)$ satisfies $\chi = 1$ on K. The extension does not depend on χ since for any other such $\tilde{\chi}$ one has $\chi - \tilde{\chi} \in C_c^\infty(U \setminus K)$.

<u>Examples</u>. (a) If $f \in L^1(U)$ vanishes a.e. in $U \setminus K$ for some $K \subset U$ compact. Then $T_f \in \mathcal{E}'(U)$.

(b) For any $x \in U$, $\delta_x \in \mathcal{E}'(U)$.

(c) $u \in \mathcal{D}'(U)$ where $u(\phi) = \sum_{k=-\infty}^{\infty} \phi(n) \notin \mathcal{E}'(U)$.

3.3. The spaces $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$

<u>Defn.</u> $\phi \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing if

$$\sup_{x \in \mathbb{R}^n} |(1+|x|)^N \nabla^\alpha \phi(x)| < \infty$$

for all $N \in \mathbb{N}$ and multiindices α.

<u>Example.</u> $\phi(x) = e^{-|x|^2}$ is rapidly decreasing.

$\phi(x) = |x|^{-2023}$ is not rapidly decreasing.

<u>Defn.</u> The Schwartz space $S(\mathbb{R}^n)$ is the space of rapidly decreasing functions with the l.c. topology generated by the separating family of seminorms

$$P_N(x) = \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N} |(1+|x|)^N \nabla^\alpha \phi(x)|.$$

<u>Rk.</u> Equivalent families of seminorms are

$$\sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N} |(1+|x|^2)^N \nabla^\alpha \phi(x)|$$

$$\sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N} |\nabla^\alpha [(1+|x|^2)^N \phi(x)]|$$

etc.

Fact. $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space,

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n) \text{ continuously.}$$
$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

Defn. $\mathcal{S}'(\mathbb{R}^n)$ is called the space of tempered distributions or Schwartz distributions.

Examples. (a) If $f \in L^1_{loc}(\mathbb{R}^n)$ satisfies
$$\int_{\mathbb{R}^n} (1+|x|)^{-N} |f(x)|\, dx < \infty$$

for some $N \in \mathbb{N}$ then $T_f \in \mathcal{S}'(\mathbb{R}^n)$: indeed, if $\phi \in \mathcal{S}(\mathbb{R}^n)$ then
$$|T_f(\phi)| = \left| \int f(x)\phi(x)\,dx \right| \le \int_{\mathbb{R}^n} (1+|x|)^{-N} |f(x)|\, dx$$
$$\times \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\phi(x)|,$$

so if $\phi_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ then $T_f(\phi_j) \to 0$.

(b) If $f(x) = e^{|x|^2}$ then $T_f \in \mathcal{D}'(\mathbb{R}^n)$, but $T_f \notin \mathcal{S}'(\mathbb{R}^n)$.

(c) $u(\phi) = \sum_{m=-\infty}^{\infty} m^{2023} \phi(m)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ but not to $\mathcal{E}'(\mathbb{R}^n)$.

3.4. Convolutions

Example. Let $f \in L^1_{loc}(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$f * \phi(x) = \int f(y) \underbrace{\phi(x-y)}_{\tau_x \check{\phi}(y)} \, dy = T_f(\tau_x \check{\phi})$$

where $\check{\phi}(y) = \phi(-y)$

$\tau_x g(y) = g(y-x)$

$\Rightarrow \tau_x \check{\phi}(y) = \check{\phi}(y-x)$

$= \phi(x-y)$

Defn. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ define

$$u * \phi(x) = u(\tau_x \check{\phi}).$$

Facts.

- $(u_1 + a u_2) * \phi = u_1 * \phi + a u_2 * \phi$
- $u * (\phi_1 + a\phi_2) = u * \phi_1 + a u * \phi_2$
- $u * \check{\phi}(0) = u(\phi)$ — thus $u * \phi$, $\phi \in \mathcal{D}(\mathbb{R}^n)$ determines u.

Example. $\delta_0 * \phi(x) = \delta_0(\tau_0 \check{\phi}) = \phi(x-y)\big|_{y=0} = \phi(x)$

$$\Rightarrow \delta_0 * \phi = \phi$$

Lemma. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

(i) $u * \phi \in C^\infty(\mathbb{R}^n)$ and $\nabla^\alpha(u*\phi) = \nabla^\alpha u * \phi = u * \nabla^\alpha \phi$

(ii) if $u \in \mathcal{E}'(\mathbb{R}^n)$ then $u*\phi$ has compact support, i.e., $u*\phi \in \mathcal{D}(\mathbb{R}^n)$.

Proof. (i) $\frac{1}{h}\left(u*\phi(x+he_i) - u*\phi(x)\right) = u\Big(\underbrace{\tfrac{1}{h}\left(\tau_{x+he_i}\check{\phi} - \tau_x\check{\phi}\right)}\Big).$

$$\text{(Ex. Sheet 3)} \quad \xrightarrow{\mathcal{D}(\mathbb{R}^n)} \nabla_i \phi(x - \cdot) = \tau_x \widecheck{\nabla_i \phi}$$

$$\Rightarrow \nabla_i u * \phi(x) = u(\tau_x \widecheck{\nabla_i\phi}) = u * \nabla_i \phi(x).$$

By induction then $u * \phi \in C^\infty$ and

$$\nabla^\alpha u * \phi = u * \nabla^\alpha \phi \qquad \forall \alpha$$

Also, $[\nabla^\alpha \tau_x \check{\phi}](y) = \nabla_y^\alpha \phi(x-y) = (-1)^{|\alpha|} \nabla^\alpha \phi(x-y)$
$$= (-1)^{|\alpha|} \tau_x \widecheck{\nabla^\alpha \phi}(y)$$

$$\Rightarrow u * \nabla^\alpha \phi = \nabla^\alpha u * \phi.$$

(ii) Assume $u(\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n \setminus K)$. Then for any $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp} \tau_x \check{\phi} \cap K = \emptyset$ for $|x|$ large enough. Thus $u * \phi(x) = 0$ for $|x|$ large enough, i.e. $u * \phi$ has compact support.

Defn. For $u_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{E}'(\mathbb{R}^n)$, define $u_1 * u_2$ to be the unique distribution s.t.

$$(u_1 * u_2) * \phi = u_1 * \underbrace{(u_2 * \phi)}_{\in \mathcal{D}(\mathbb{R}^n)} \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

Example. Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$. Since $\delta_0 \in \mathcal{E}'(\mathbb{R}^n)$,

$$(u * \delta_0) * \phi = u * (\delta_0 * \phi) = u * \phi$$

$\Rightarrow u * \delta_0 = u.$

Lemma. Let $u_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{E}'(\mathbb{R}^n)$. Then

$$\nabla^\alpha (u_1 * u_2) = u_1 * (\nabla^\alpha u_2) = (\nabla^\alpha u_1) * u_2.$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\nabla^\alpha (u_1 * u_2) * \phi &= (u_1 * u_2) * (\nabla^\alpha \phi) \\
&= u_1 * (u_2 * \nabla^\alpha \phi) \\
&= u_1 * (\nabla^\alpha u_2 * \phi) \\
&= (u_1 * \nabla^\alpha u_2) * \phi
\end{aligned}$$

The other case is similar.

Defn. Let $L = \sum_{|\alpha|\le k} a_\alpha \nabla^\alpha$, $a_\alpha \in \mathbb{R}$ be a const. coefficient differential operator of order k. A fundamental solution of L is a distribution G s.t. $LG = \delta_0$

Thm. If $G \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of L and $f \in \mathcal{E}'(\mathbb{R}^n)$ then $u = G * f$ solves

$$Lu = f.$$

Moreover, if $f \in \mathcal{D}(\mathbb{R}^n)$ then $u = G*f \in C^\infty(\mathbb{R}^n)$ solves $Lu = f$ in the classical sense.

Proof. $L(G*f) = \sum_{|\alpha|\le k} a_\alpha \underbrace{\nabla^\alpha(G*f)}_{(\nabla^\alpha G * f)}$

$$= \Big(\sum_{|\alpha|\le k} a_\alpha \nabla^\alpha G\Big) * f = LG * f$$
$$= \delta_0 * f = f.$$

Example. $L = -\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^3$.

Define $g(x) = \frac{1}{4\pi|x|} \in L^1_{loc}(\mathbb{R}^3)$. Then $G = T_g$ is a fundamental solution for L, i.e., if $f \in C_c^\infty(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^3} \frac{f(y)}{4\pi|x-y|}\,dy \text{ solves } Lu = f.$$

3.5. The Fourier transform

Defn. If $f \in L^1(\mathbb{R}^n)$ the Fourier transform of f is

$$\hat{f} = \mathcal{F}(f) : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx.$$

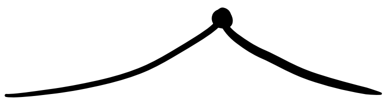
(Since $|f(x)\, e^{-ix\cdot\xi}| \le |f(x)| \in L^1$ the integral exists for all $\xi \in \mathbb{R}^n$.)

Examples. $(n=1)$

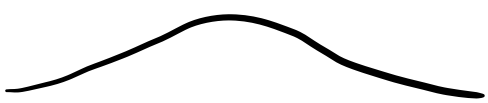
(i) $f(x) = \begin{cases} 1 & (|x|<1) \\ 0 & (|x| \ge 1) \end{cases}$

$$\hat{f}(\xi) = 2\, \frac{\sin \xi}{\xi}$$

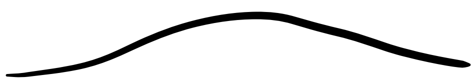
(ii) $f(x) = e^{-|x|}$



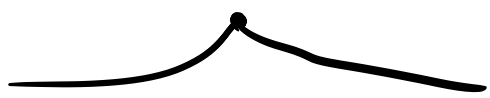
$$\hat{f}(\xi) = \frac{2}{1+\xi^2}$$



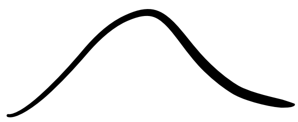
(iii) $f(x) = \dfrac{1}{1+x^2}$



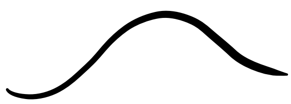
$$\hat{f}(\xi) = \pi e^{-|x|}$$



(iv) $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|x|^2}$



$$\hat{f}(\xi) = e^{-\frac{1}{2}|\xi|^2}$$



Note: f regular $\longleftrightarrow$ $\hat{f}$ decays at ∞ & vice-versa.

<u>Riemann-Lebesgue Lemma</u>. Let $f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^0(\mathbb{R}^n)$ and

$$\sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi)| \leq \|f\|_{L^1}$$

$$\hat{f}(\xi) \to 0 \quad \text{as } |\xi| \to \infty.$$

<u>Proof</u>. Assume $\xi_k \to \xi$. Then for all $x \in \mathbb{R}^n$,

$$f(x) e^{i\xi_k \cdot x} \to f(x) e^{i\xi \cdot x}, \quad |f(x) e^{i\xi_k \cdot x}| \leq |f(x)|$$

so $\hat{f}(\xi_k) \to \hat{f}(\xi)$ by DCT. Hence $\hat{f} \in C^0$.

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, dx \right| \leq \|f\|_{L^1}$$

To show $\hat{f}(\xi) \to 0$ $(|\xi| \to \infty)$, let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be s.t. $\|f - f_\varepsilon\|_{L^1} < \varepsilon$. Then

$$\hat{f}_\varepsilon(\xi) = \int_{\mathbb{R}^n} f_\varepsilon(x) e^{-i\xi \cdot x} \, dx$$

$$= \int_{\mathbb{R}^n} f_\varepsilon(x) \Delta_x \left(\frac{-1}{|\xi|^2} e^{-i\xi \cdot x} \right) dx$$

$$= \frac{-1}{|\xi|^2} \int_{\mathbb{R}^n} \Delta f_\varepsilon(x) e^{-i\xi \cdot x} \, dx \leq \frac{1}{|\xi|^2} \|\Delta f_\varepsilon\|_{L^1}$$

Thus $\limsup_{|\xi| \to \infty} |\hat{f}_\varepsilon(\xi)| = 0$.

$$\Rightarrow |\hat{f}(\xi)| \leq |\hat{f}_\varepsilon(\xi)| + |\hat{f}(\xi) - \hat{f}_\varepsilon(\xi)| \leq |\hat{f}_\varepsilon(\xi)| + \varepsilon$$

$$\Rightarrow \limsup_{|\xi|\to\infty} |\hat{f}(\xi)| \le \varepsilon \ \forall \varepsilon \Rightarrow \lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

Notation: $\tau_y f(x) = f(x-y)$, $e_y(x) = e^{ix\cdot y}$

<u>Prop</u>. (i) Let $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\lambda > 0$, and set

$$f_\lambda(x) = \lambda^{-n} f(x/\lambda).$$

Then $\hat{f_\lambda}(\xi) = \hat{f}(\lambda\xi)$,

$$\widehat{e_y f}(\xi) = \tau_y \hat{f}(\xi), \quad \widehat{\tau_y f}(\xi) = e_{-y}(\xi)\hat{f}(\xi)$$

(ii) Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and

$$\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

<u>Proof</u>. Change of variables and Fubini.

<u>Prop</u>. (i) Let $f \in C^1(\mathbb{R}^n)$ and $f, \nabla_j f \in L^1(\mathbb{R}^n)$, for $j = 1, \dots, n$. Then

$$\widehat{\nabla_j f}(\xi) = i\xi_j \hat{f}(\xi)$$

(ii) Assume $(1+|x|)f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^1(\mathbb{R}^n)$,

$$\nabla_j \hat{f}(\xi) = -i\widehat{x_j f}(\xi)$$

Proof. (i) Let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be s.t.

$$\|f - f_\varepsilon\|_{L^1} + \sum_j \|\nabla_j f - \nabla_j f_\varepsilon\|_{L^1} < \varepsilon.$$

Then

$$\widehat{\nabla_j f_\varepsilon}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \nabla_j f_\varepsilon(x)\,dx$$

$$= i\xi_j \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f_\varepsilon(x)\,dx = i\xi_j \hat{f}_\varepsilon(\xi).$$

$$\Rightarrow |\widehat{\nabla_j f}(\xi) - i\xi_j \hat{f}(\xi)| \le \|\nabla_j f - \nabla_j f_\varepsilon\|_{L^1} + |\xi|\,\|f - f_\varepsilon\|_{L^1}$$
$$\le (1+|\xi|)\,\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

(ii) Since $x_j f \in L^1$, $-i\widehat{x_j f} \in C^0$. Need to prove that $\nabla_j \hat{f}$ exists and equals $-i\widehat{x_j f}$.

$$\frac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} = \int_{\mathbb{R}^n} f(x)\, e^{-i\xi\cdot x} \underbrace{\left(\frac{e^{-ihx_j} - 1}{h}\right)}_{\substack{\to -ix_j \ (h\to 0) \\ \text{pointwise} \\ \le |x_j|}} dx$$

Since $|x_j| f \in L^1$ thus by DCT

$$\frac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} \longrightarrow -ix_j \hat{f}(\xi).$$

<u>Cor.</u> The Fourier transform maps $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ continuously.

<u>Proof.</u> For any $f: \mathbb{R}^n \to \mathbb{C}$,

$$\|f\|_{L^1} \le \sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |f(x)| \underbrace{\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{n+1}}}_{<\infty}$$

$\Rightarrow$ If $f \in S(\mathbb{R}^n)$ then $\nabla^\alpha (x^\beta f(x)) \in L^1(\mathbb{R}^n)$ for any multiindices α and β.

Thus by previous proposition (repeatedly),

$$|\widehat{\nabla^\alpha (x^\beta f)}(\xi)| = |\xi^\alpha \nabla^\beta \hat{f}(\xi)|$$

$$\Rightarrow \sup_{\xi} |\xi^\alpha \nabla^\beta \hat{f}(\xi)| \le C \sup_{\substack{x\in\mathbb{R}^n \\ |\gamma|\le|\alpha|}} \left[(1+|x|)^{|\beta|+n+1} |\nabla^\gamma f(x)| \right]$$

$$\to 0 \text{ if } f \to 0 \text{ in } S(\mathbb{R}^n)$$

$\Rightarrow \hat{f} \to 0$ in $S(\mathbb{R}^n)$.

Thus $\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous.

Thm. (Fourier inversion). Let $f \in L^1(\mathbb{R}^n)$ and assume also $\hat{f} \in L^1(\mathbb{R}^n)$. Then

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi \quad \text{for a.e. } x.$$

(Thus if $\check{f}(x) = f(-x)$ then $\mathcal{F}^2(f) = (2\pi)^n \check{f}$.)

Proof. Let

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\xi}\, d\xi$$

Since $\hat{f} \in L^1$, by DCT,

$$I_\varepsilon(x) \xrightarrow{\varepsilon\to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ix\xi}\, d\xi .$$

On the other hand,

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)\, e^{-i\xi\cdot y}\, dy \right) e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\cdot\xi}\, d\xi$$

$$\overset{\text{Fubini}}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \underbrace{\left(\int_{\mathbb{R}^n} e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{-i(y-x)\cdot\xi}\, d\xi \right)}_{(2\pi)^{n/2}\varepsilon^{-n} e^{-|y-x|^2/2\varepsilon^2}} dy$$

$$= f * \psi_\varepsilon(x), \qquad \begin{aligned} \psi_\varepsilon(x) &= \varepsilon^{-n}\psi(\varepsilon^{-1}x) \\ \psi(x) &= (2\pi)^{-n/2} e^{-|x|^2/2} \end{aligned}$$

Since $\psi \in C^\infty(\mathbb{R}^n)$, $\psi \geq 0$, $\int \psi \, dx = 1$,

$$f * \psi_\varepsilon \to f \text{ in } L^1$$

$$\Rightarrow f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \, e^{ix\cdot\xi} \, d\xi \quad \text{a.e. } x \in \mathbb{R}^n.$$

If f is continuous this holds for all $x \in \mathbb{R}^n$.

<u>Thm</u> (Parseval-Plancherel). Let $f, g \in L^1 \cap L^2(\mathbb{R}^n)$. Then $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$ and

$$(f,g)_{L^2} = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g}).$$

<u>Proof</u>. Suppose $f, g \in S(\mathbb{R}^n)$. Then $\hat{f}, \hat{g} \in S(\mathbb{R}^n)$ and

$$\begin{aligned}
(f,g)_{L^2} &= \int \overline{f(x)} \, g(x) \, dx \\
&= \int \overline{f(x)} \left(\frac{1}{(2\pi)^n} \int \hat{g}(\xi) \, e^{i\xi\cdot x} \, d\xi \right) dx \\
&= \frac{1}{(2\pi)^n} \int \Bigg(\underbrace{\int \overline{f(x)} \, e^{i\xi x} \, dx}_{\overline{\hat{f}(\xi)}} \Bigg) \hat{g}(\xi) \, d\xi \\
&= \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})_{L^2}.
\end{aligned}$$

Given $f,g \in L^1 \cap L^2(\mathbb{R}^n)$, let $f_j, g_j \in S(\mathbb{R}^n)$ be s.t.

$$\|f_j - f\|_{L^1} + \|f_j - f\|_{L^2} + \|g_j - g\|_{L^1} + \|g_j - g\|_{L^2} \to 0.$$

By Riemann-Lebesgue,

$$\sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi) - \hat{f}_j(\xi)| \le \|f - f_j\|_{L^1} \to 0$$

Since $\|\hat{f}_j - \hat{f}_k\|_{L^2}^2 = (2\pi)^n \|f_j - f_k\|_{L^2} \to 0 \quad (j,k \to \infty)$, $(\hat{f}_j)$ is Cauchy in L^2, so $\hat{f} \in L^2$ and $\hat{f}_j \to \hat{f}$ in L^2. Analogously, $\hat{g}_j \to \hat{g}$ in L^2. Thus

$$(f,g)_{L^2} = \lim_{j \to \infty} (f_j, g_j) = \frac{1}{(2\pi)^n} \lim_{j \to \infty} (\hat{f}_j, \hat{g}_j) = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g}).$$

<u>Cor.</u> $f \mapsto (2\pi)^{-n/2} \hat{f}$ is an isometry from $L^1 \cap L^2$ into L^2 and thus extends uniquely to a linear map $(2\pi)^{-n/2} \mathcal{F}$ from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

<u>Defn.</u> For $f \in L^2(\mathbb{R}^n)$, write $\hat{f} = \mathcal{F}(f)$.

<u>Rk.</u> If $f \in L^2(\mathbb{R}^n)$, then $f_R = f \mathbb{1}_{B_R(0)} \in L^1 \cap L^2(\mathbb{R}^n)$ and $f_R \to f$ $(R \to \infty)$ in L^2. Thus $\hat{f}_R \to \hat{f}$ in L^2, i.e.,

$$\left(\xi \mapsto \int_{B_R(0)} f(x) e^{-ix\cdot\xi} dx \right) \xrightarrow{L^2} \hat{f} \qquad (R \to \infty).$$

Example. Let $f \in L^1(\mathbb{R}^n)$, $\phi \in S(\mathbb{R}^n)$. Then

$$T_{\hat f}(\phi) = \int_{\mathbb{R}^n} \hat f(\xi)\,\phi(\xi)\,d\xi$$

$$= \int \left(\int f(x)\, e^{-ix\cdot\xi}\, dx \right) \phi(\xi)\, d\xi$$

$$= \int f(x) \underbrace{\left(\int \phi(\xi)\, e^{-ix\cdot\xi}\, d\xi \right)}_{\hat\phi(x)} dx$$

$$= \int f(x)\, \hat\phi(x)\, dx = T_f(\hat\phi).$$

Defn. For $u \in S'(\mathbb{R}^n)$ define $\hat u \in S'(\mathbb{R}^n)$ by

$$\hat u(\phi) = u(\hat\phi) \qquad \text{for all } \phi \in S(\mathbb{R}^n).$$

Note this makes sense since $f \mapsto \hat f$ is continuous from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$, so $\hat u$ is linear and continuous. For $u \in \mathcal{D}'(\mathbb{R}^n)$ this does not work:

$$\phi \in \mathcal{D}(\mathbb{R}^n) \not\Rightarrow \hat\phi \in \mathcal{D}(\mathbb{R}^n).$$

Examples. (a) Let $\xi \in \mathbb{R}^n$.

$$\hat\delta_\xi(\phi) = \delta_\xi(\hat\phi) = \hat\phi(\xi) = \int_{\mathbb{R}^n} \phi(x)\, e^{-ix\cdot\xi}\, dx$$

$$= T_{e_{-\xi}}(\phi) \quad \forall \phi \in S(\mathbb{R}^n)$$

$\Rightarrow \hat{\delta}_\xi = T_{e_{-\xi}}$ or simply "$\hat{\delta}_\xi = e^{-i(\cdot)\xi}$".

(b) For $x \in \mathbb{R}^n$,

$$\hat{T}_{e_x}(\phi) = T_{e_x}(\hat{\phi}) = \int e^{ix\cdot\xi}\,\hat{\phi}(\xi)\,d\xi = (2\pi)^n \phi(x)$$
$$= (2\pi)^n \delta_x(\phi).$$

$\Rightarrow \hat{T}_{e_x} = (2\pi)^n \delta_x$ or "$\widehat{e^{ix\cdot\xi}} = (2\pi)^n \delta_x$".

<u>Lemma</u>. Let $u \in S'(\mathbb{R}^n)$. Then

$$\widehat{e^{i\xi\cdot(\cdot)}u} = \tau_\xi \hat{u}, \quad \widehat{\tau_x u} = e^{ix\cdot(\cdot)}\hat{u}$$
$$\widehat{\nabla^\alpha u} = (i\xi)^\alpha \hat{u}, \quad \nabla^\alpha \hat{u} = (-1)^{|\alpha|}\widehat{x^\alpha u}$$
$$\hat{\hat{u}} = (2\pi)^n \check{u}.$$

<u>Proof</u>. Let $\phi \in S(\mathbb{R}^n)$. Then

$$\widehat{e^{i\xi\cdot(\,)}u}(\phi) = e^{i\xi\cdot(\cdot)}u(\hat{\phi}) = u(e^{i\xi\cdot(\,)}\hat{\phi})$$
$$= u(\widehat{\tau_{-\xi}\phi})$$
$$= \hat{u}(\tau_{-\xi}\phi)$$
$$= \tau_\xi \hat{u}(\phi).$$

$\Rightarrow \widehat{e^{i\xi\cdot(\,)}u} = \tau_\xi \hat{u}$.

$\widehat{\tau_x u} = e^{ix\cdot(\,)}\hat{u}$ is similar.

$$\widehat{\nabla^\alpha u}(\phi) = \nabla^\alpha u(\hat{\phi}) = (-1)^{|\alpha|} u(\nabla^\alpha \hat{\phi})$$
$$= (-1)^{|\alpha|} u\left((-i)^{|\alpha|} \widehat{\xi^\alpha \phi}\right)$$
$$= i^{|\alpha|} \hat{u}(\xi^\alpha \phi)$$
$$= (i\xi)^\alpha \hat{u}(\phi)$$

$$\Rightarrow \widehat{\nabla^\alpha u} = (i\xi)^\alpha \hat{u}$$

Other similar.

$$\hat{\hat{u}}(\phi) = \hat{u}(\hat{\phi}) = u(\hat{\hat{\phi}}) = u((2\pi)^n \check{\phi}) = (2\pi)^n \check{u}(\phi)$$

<u>Lemma</u>. $\mathcal{F} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is a linear homoemorphism.

<u>Proof</u>. Suppose $u_j \to u$ in $S'(\mathbb{R}^n)$, i.e.,

$$u_j(\phi) \to u(\phi) \quad \forall \phi \in S(\mathbb{R}^n).$$

Then
$$\hat{u}_j(\phi) = u_j(\hat{\phi}) \longrightarrow u(\hat{\phi}) = \hat{u}(\phi) \quad \forall \phi \in S(\mathbb{R}^n)$$

$$\Rightarrow \hat{u}_j \longrightarrow \hat{u} \text{ in } S'(\mathbb{R}^n).$$

Thus $\mathcal{F}$ is continuous. Since $\mathcal{F}^4 = (2\pi)^{2n}\,\mathrm{id}$, $\mathcal{F}$ is invertible with continuous inverse $\mathcal{F}^{-1} = (2\pi)^{2n} \mathcal{F}^3$.

3.6. Periodic distributions

Recall: if $f \in L^2(0,1)$ then

$$f(x) = \sum_{n\in\mathbb{Z}} f_n e^{-2\pi i n x} \quad \text{in } L^2$$

$$f_n = \int_0^1 e^{-2\pi i n x}\, dx.$$

<u>Defn</u>. $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic if for all $g \in \mathbb{Z}^n$,

$$\tau_g u = u.$$

(Recall $\tau_g u(\phi) = u(\tau_{-g}\phi)$).

<u>Example</u>. (a) For $k \in \mathbb{Z}^n$, the distribution $T_{e_{2\pi k}}$ is periodic:

$$\begin{aligned}
\tau_g T_{e_{2\pi k}}(\phi) &= T_{e_{2\pi k}}(\tau_{-g}\phi) \\
&= \int e^{2\pi i k\cdot x} \phi(x+g)\, dx \\
&= \int e^{2\pi i k\cdot(x-g)} \phi(x)\, dx \\
&= e^{-2\pi i k\cdot g}\, T_{e_{2\pi k}}(\phi) = T_{e_{2\pi k}}(\phi).
\end{aligned}$$

(b) Suppose $v \in \mathcal{E}'(\mathbb{R}^n)$. Then

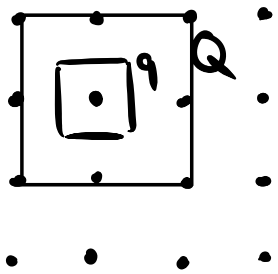
$$u = \sum_{k \in \mathbb{Z}^n} \tau_k v$$

is periodic. Note: u defines a distribution since $u(\phi)$ is a finite sum. For $g \in \mathbb{Z}^n$, $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\tau_g u(\phi) = \sum_{k \in \mathbb{Z}^n} \tau_{g+k} u(\phi) = \sum_{k \in \mathbb{Z}^n} \tau_k u(\phi) = u(\phi).$$

Fundamental cell of lattice:

$$q = \{x \in \mathbb{R}^n : -\tfrac{1}{2} \leq x_i < \tfrac{1}{2},\ i = 1, \ldots, n\}$$



<u>Lemma</u>. Let $Q = \{x \in \mathbb{R}^n : -1 \leq x_i < 1,\ i = 1, \ldots, n\}$. There exists $\psi \in C^\infty(\mathbb{R}^n)$ s.t.

(i) $\psi \geq 0$

(ii) $\operatorname{supp} \psi \subset \mathring{Q}$

(iii) $\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1$.

Such a ψ is called a periodic partition of unity. Suppose ψ and ψ' are both p.p.u. Then if $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic,

$$u(\psi) = u(\psi').$$

Proof. Let $\psi_0 \subset C_c^\infty(\mathbb{R}^n)$, supp $\psi_0 \subset \mathring{Q}$ and $\psi_0(x) = 1$ for $x \in q$ and $\psi_0 \geq 0$. Set

$$S(x) = \sum_{g \in \mathbb{Z}^n} \psi_0(x-g).$$

Then S is smooth and $S(x) \geq 1$ for all $x \in \mathbb{R}^n$. Thus

$$\psi(x) = \frac{\psi_0(x)}{S(x)}$$

satisfies (i)–(iii).

Now let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic and ψ, ψ' p.p.u.

$$\begin{aligned}
\Rightarrow u(\psi) &= u\Big(\psi \sum_{g \in \mathbb{Z}^n} \tau_g \psi'\Big) = \sum_{g \in \mathbb{Z}^n} u(\psi \tau_g \psi') \\
&= \sum_{g \in \mathbb{Z}^n} \underbrace{\tau_{-g} u}_{u}((\tau_{-g}\psi)\psi') \\
&= u\Big(\psi' \underbrace{\sum_{g \in \mathbb{Z}^n} \tau_{-g}\psi}_{1}\Big) \\
&= u(\psi').
\end{aligned}$$

Cor. Let ψ be a p.p.u. Then for $f \in L^1_{loc}(\mathbb{R}^n)$ periodic, $T_f(\psi) = \int_q f(x)\,dx$

Proof. Choose ψ_n p.p.u. s.t. $\psi_n \longrightarrow 1_q$ pointwise and ψ_n bounded.

Defn. For $u \in \mathcal{D}'(\mathbb{R}^n)$ periodic, the average of u over the fundamental cell u is

$$M(u) = u(\psi)$$

where ψ is a p.p.u.

Lemma. Let $v \in \mathcal{E}'(\mathbb{R}^n)$. Then

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v \qquad (*)$$

converges (weak-$*$) in $S'(\mathbb{R}^n)$. Conversely, if $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic there exists $v \in \mathcal{E}'(\mathbb{R}^n)$ s.t. $(*)$ holds. Hence every periodic distribution is tempered.

Proof. Let $K = \operatorname{supp} v$. We have seen that then $\exists N \in \mathbb{N}$, $C > 0$ s.t.

$$|v(\phi)| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |\nabla^\alpha \phi(x)| \quad \forall \phi \in \mathcal{E}(\mathbb{R}^n).$$

Now let $\phi \in S(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$. Then

$$|\tau_g V(\phi)| = |V(\tau_{-g}\phi)| \le C \sup_{\substack{x\in K \\ |\alpha|\le N}} |\nabla^\alpha \phi(x+g)|$$

Since $K \subset B_R(0)$ for some $R>0$,

$$1+|g| \le 1+|x|+|g+x| \le (1+R)(1+|g+x|)$$

$$\Rightarrow 1 \le (1+R)\left(\frac{1+|g+x|}{1+|g|}\right) \quad \forall x\in K$$

Thus for $M\ge 1$,

$$|\tau_g V(\phi)| \le C\left(\frac{1+R}{1+|g|}\right)^M \underbrace{\sup_{\substack{x\in K \\ |\alpha|\le N}} |(1+|x+g|)^M \nabla^\alpha \phi(x+g)|}_{\le \underbrace{\sup_{\substack{y\in \mathbb{R}^n \\ |\alpha|\le N}} |(1+|y|)^M \nabla^\alpha \phi(y)|}_{<\infty}}$$

$$\Rightarrow |\tau_g V(\phi)| \le \frac{\tilde{C}}{(1+|g|)^{n+1}}$$

$\Rightarrow \sum_{g\in\mathbb{Z}^n} \tau_g V(\phi)$ converges for all $\phi \in S(\mathbb{R}^n)$

$\Rightarrow \sum_{g\in\mathbb{Z}^n} \tau_g V$ converges in $S'(\mathbb{R}^n)$.

For the other direction, let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Let Ψ be a p.p.u and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$u(\phi) = \Big(\underbrace{\sum_{g \in \mathbb{Z}^n} \tau_g \Psi}_{1}\Big) u(\phi) = \sum_{g \in \mathbb{Z}^n} \underbrace{u((\tau_g \Psi)\phi)}_{\tau_g u((\tau_g \Psi)\phi)}$$

$$= u(\Psi \tau_{-g} \phi)$$

$$= \Psi u(\tau_{-g}\phi)$$

$$= \tau_g(\Psi u)(\phi)$$

Note that Ψu has compact support:

$$\operatorname{supp} \phi \cap \operatorname{supp} \Psi = \emptyset \Rightarrow \Psi u(\phi) = u(\Psi\phi) = 0.$$

$\Rightarrow$ Ψu extends uniquely to $v \in \mathcal{E}'(\mathbb{R}^n)$.

$$u(\phi) = \sum_{g \in \mathbb{Z}^n} \tau_g v(\phi)$$

By first part, thus $u \in \mathcal{S}'(\mathbb{R}^n)$.

Thm. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Then
$$u = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi g}} \quad \text{(convergence in } S'(\mathbb{R}^n))$$
where $u_g = M(e_{-2\pi g} u) \in \mathbb{C}$ and satisfy
$$|u_g| \leq C(1+|g|)^N \quad \text{for some } C>0, N \in \mathbb{N}.$$

Defn. u_g are the Fourier coefficients of u.

Lemma. Assume $u \in S'(\mathbb{R}^n)$ satisfies
$$(e_{-k} - 1) u = 0 \quad \forall k \in \mathbb{Z}^n. \quad (*)$$
Then
$$u = \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g} \quad \text{(convergence in } S'(\mathbb{R}^n))$$
for $c_g \in \mathbb{C}$ satisfying $|c_g| \leq C(1+|g|)^N$ for some $C>0$ and $N \in \mathbb{N}$.

Proof. Claim: $\operatorname{supp} u \subset \Lambda^* = \{2\pi g : g \in \mathbb{Z}^n\}$

Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \phi \cap \underbrace{\{x \in \mathbb{R}^n : x_1 \in 2\pi\mathbb{Z}\}}_{\supset \Lambda^*} = \emptyset$.

Then for $k = (1,0,\dots,0) \in \mathbb{Z}^n$, $(e_{-k}-1)^{-1}\phi \in S(\mathbb{R}^n)$
since $\phi(x) = 0$ if $x_1 \in 2\pi\mathbb{Z} \Leftrightarrow (e_{-k}-1)(x) = 0$.

Thus by $(*)$,
$$u(\phi) = (e_{-k}-1)u((e_{-k}-1)^{-1}\phi) = 0.$$

Thus $\operatorname{supp} u \subset \Lambda_1^*$. By the same argument, $\operatorname{supp} u \subset \Lambda_i^*$ for $i=2,\dots,n$. Thus $\operatorname{supp} u \subset \Lambda^* = \bigcap \Lambda_i^*$.

Now let ψ be a p.p.u. and set $\tilde{\psi}(x) = \psi(x/2\pi)$:

$$\sum_{g\in\mathbb{Z}^n} \tau_{2\pi g} \tilde{\psi}(x) = 1, \quad \tilde{\psi} \geq 0, \quad \operatorname{supp} \tilde{\psi} \subset \{ |x_i| < 2\pi \}.$$

Let $v_g = (\tau_{2\pi g} \tilde{\psi}) u$. Note $\operatorname{supp} v_g \subset \{2\pi g\}$,

$$\sum_{g\in\mathbb{Z}^n} v_g = u, \quad (e_{-k} - 1) v_g = 0.$$

Take k in the standard basis of $\mathbb{R}^n$:

$$(e^{-ix_j} - 1) v_g = \underbrace{\left(e^{-i(x_j - 2\pi g_j)} - 1\right)}_{(x_j - 2\pi g_j)\cdot K(x_j)} v_g = 0$$

with K smooth, $\neq 0$ near $2\pi g_j$ (Taylor's thm).

$$\Rightarrow (x_j - 2\pi g_j) v_g = 0 \qquad \left(\frac{e^{-it}-1}{t} = -i + O(t) \right)$$

On the other hand, for $\phi \in S(\mathbb{R}^n)$, there are $\phi_j \in C^\infty(\mathbb{R}^n)$ s.t. (again Taylor's thm):

$$\phi(x) = \phi(2\pi g) + \sum_{j=1}^n (x_j - 2\pi g_j) \phi_j(x)$$

Since v_g has compact support it can be extended to $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

$$\Rightarrow v_g(\phi) = v_g(\underbrace{\phi(2\pi g)}_{\text{constant}}) + \sum_{j=1}^{n} \underbrace{(x_j - 2\pi g_j)\, v_g}_{0}(\phi_j)$$

$$= \phi(2\pi g)\; v_g(1)$$

$$= \phi(2\pi g)\, u(\tau_{2\pi g} \tilde{\hat{\psi}})$$

$$= \delta_{2\pi g}(\phi)\, u(\tau_{2\pi g} \tilde{\hat{\psi}})$$

$$\Rightarrow u = \sum_{g \in \mathbb{Z}^n} v_g = \sum_{g \in \mathbb{Z}^n} c_g\, \delta_{2\pi g}, \quad c_g = u(\tau_{2\pi g} \tilde{\hat{\psi}}).$$

Ex. Sheet 3: $\forall u \in S'(\mathbb{R}^n)\ \exists N, k \in \mathbb{N},\ C > 0$ s.t.

$$|u(\phi)| \le C \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le k}} |(1+|x|)^N \nabla^\alpha \phi(x)| \quad \forall \phi \in S(\mathbb{R}^n).$$

$$\Rightarrow |c_g| \le C \sup |(1+|x|)^N \nabla^\alpha \tilde{\psi}(x - 2\pi g)|$$

$$\le C \sup |(1+|x+2\pi g|)^N \nabla^\alpha \tilde{\psi}(x)|$$

$$\le C' \underbrace{\sup |(1+|x|)^N \nabla^\alpha \tilde{\psi}(x)|}_{\le C''} (1+|g|)^N$$

$$\Rightarrow |c_g| \le C(1+|g|)^N.$$

Thm. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Then

$$u = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi g}} \quad \text{(convergence in } S'(\mathbb{R}^n))$$

where $u_g = M(e_{-2\pi g} u) \in \mathbb{C}$ and satisfy

$$|u_g| \le C(1+|g|)^N \quad \text{for some } C>0, N \in \mathbb{N}.$$

Proof. Since u is periodic, $u \in S'(\mathbb{R}^n)$ and the Fourier transform $\hat{u}$ is defined.

$$\tau_k u = u \quad \forall k \in \mathbb{Z}^n$$

$$\Rightarrow e_{-k} \hat{u} = \hat{u} \Rightarrow (e_{-k} - 1)\hat{u} = 0 \quad \forall k \in \mathbb{Z}^n$$

Therefore by the lemma,

$$\hat{u} = (2\pi)^n \sum_{g \in \mathbb{Z}^n} u_g \delta_{2\pi g} \quad \text{in } S'(\mathbb{R}^n).$$

Applying the inverse Fourier transform,

$$u = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi g}} \quad \text{in } S'(\mathbb{R}^n)$$

Since $e_{2\pi g} \in L^1_{loc}$,

$$M(e_{-2\pi k} T_{2\pi k}) = \int_q e^{2\pi i (g-k)\cdot x} dx = \delta_{gk}.$$

Since $u \mapsto M(u)$ is continuous from $S'(\mathbb{R}^n)$ to $\mathbb{C}$, it follows that $M(e_{-2\pi k} u) = u_k$.

Notation: "$u = \sum_{g \in \mathbb{Z}^n} u_g e_{2\pi g}$" or "$u(x) = \sum_{g \in \mathbb{Z}^n} u_g e^{2\pi i g \cdot x}$"

<u>Example</u>. Let $u = \sum_{g \in \mathbb{Z}^n} \delta_g$.

$$\Rightarrow \quad u_k = M(e_{-2\pi k} u)$$
$$= u(\psi e_{-2\pi k})$$
$$= \sum_{g \in \mathbb{Z}^n} \psi(g) \underbrace{e^{-2\pi i k \cdot g}}_{=1} = 1.$$

$$\Rightarrow \quad \sum_{g \in \mathbb{Z}^n} \delta_g = \sum_{g \in \mathbb{Z}^n} T_{e_{2\pi g}} \quad \text{in } S'(\mathbb{R}^n)$$

"$\sum_{g \in \mathbb{Z}^n} \delta(x-g)$" = "$\sum_{g \in \mathbb{Z}^n} e^{2\pi i g \cdot x}$"

<u>Thm</u>. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic with Fourier coefficients u_g. Then

(i) $\nabla^\alpha u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic and
$$\nabla^\alpha u = \sum_{g \in \mathbb{Z}^n} (2\pi i g)^\alpha T_{2\pi g}$$

(ii) If $f \in L^1_{loc}(\mathbb{R}^n)$ is periodic, $u = T_f$, then $|c_g| \leq \|f\|_{L^1}$, $c_g \to 0$ $(|g| \to \infty)$.

(iii) If $f \in C^{n+1}(\mathbb{R}^n)$ is periodic, $u = T_f$, then

$$f(x) = \sum_{g \in \mathbb{Z}^n} u_g e^{2\pi i g \cdot x} \text{ (with uniform conv)}$$

(iv) If $f, h \in L^2_{loc}$ are periodic with Fourier coefficients f_g and h_g then

$$\int_q \bar{f}(x)\, h(x)\, dx = \sum_{g \in \mathbb{Z}^n} \bar{f}_g\, h_g$$

$$f(x) = \sum_{g \in \mathbb{Z}^n} f_g\, e^{2\pi i g \cdot x} \quad \text{in } L^2(q)$$

4. Sobolev spaces and applications

4.1. Sobolev spaces

Defn. Let $U \subset \mathbb{R}^n$ be open, let $k \in \mathbb{Z}_{\geq 0}$, $p \in [1,\infty]$. Then $f \in L^p(\mathbb{R}^n)$ belongs to the Sobolev space $W^{k,p}(U)$ if

$$\forall |\alpha| \leq k \ \exists f^\alpha \in L^p(U) \text{ s.t. } \nabla^\alpha T_f \overset{\mathcal{D}'(\mathbb{R}^n)}{=} T_{f^\alpha} \quad (*)$$

Here f^α is the α-th weak derivative of f. Write $\nabla^\alpha f = f^\alpha$. The Sobolev norm is

$$\|f\|_{W^{k,p}} = \Big(\sum_{|\alpha| \leq k} \|\nabla^\alpha f\|_{L^p}^p \Big)^{1/p} \qquad p \in [1,\infty)$$

$$\|f\|_{W^{k,\infty}} = \max_{|\alpha| \leq k} \|\nabla^\alpha f\|_{L^\infty} \qquad p = \infty.$$

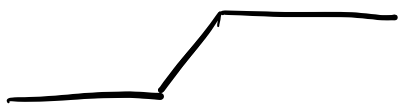
Fact. $W^{k,p}(U)$ is a Banach space for $p \in [1,\infty]$ and a Hilbert space if $p = 2$.

Rk. $(*)$ means $\int_U f^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U f \, \nabla^\alpha \phi \, dx$

for all $\phi \in C_c^\infty(U)$.

Example.

- $f(x) = \begin{cases} -1 & (x < -1) \\ x & (x \in (-1,1)) \\ +1 & (x > 1) \end{cases}$



$\Rightarrow f \in W^{1,\infty}(\mathbb{R})$ with $\nabla f(x) = \begin{cases} 0 & (|x| > 1) \\ 1 & (|x| \leq 1) \end{cases}$

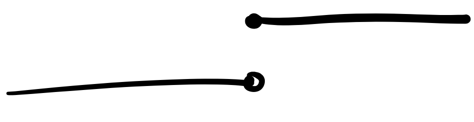
Indeed, for $\phi \in C_c^\infty(\mathbb{R})$,

$$-\int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = \underbrace{-\int_{-\infty}^{-1} \phi'(x)\,dx}_{-\phi(-1)} \underbrace{-\int_{-1}^{1} x\phi'(x)\,dx}_{+\int_{-1}^{1}\phi(x)\,dx - [x\phi(x)]_{-1}^{+1}} \underbrace{-\int_{1}^{\infty} \phi'(x)\,dx}_{+\phi(+1)}$$

$$= \int_{-1}^{1} \phi(x)\,dx$$

$$= \int_{-1}^{1} \nabla f(x)\,\phi(x)\,dx$$

- $H(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases}$



$\Rightarrow H \notin W^{1,p}(\mathbb{R})$ for any p since $\nabla H = \delta_0 \neq T_f$ for any $f \in L^1_{loc}$

Notation: $\nabla^\alpha f \in L^p$ means $\exists f^\alpha \in L^p$ s.t. $\nabla^\alpha T_{f^\alpha} = f^\alpha$

<u>Recall</u>: $\nabla^\alpha f \in L^2(\mathbb{R}^n) \Leftrightarrow \xi^\alpha \hat{f} \in L^2(\mathbb{R}^n)$

<u>Defn</u>. For $s \in \mathbb{R}$, $f \in S'(\mathbb{R}^n)$ belongs to $H^s(\mathbb{R}^n)$ if
$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 \, d\xi < \infty.$$

$H^s(\mathbb{R}^n)$ is a <u>Hilbert space</u> with inner product
$$(f,g)_{H^s} = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \, \hat{g}(\xi) \, (1+|\xi|^2)^s \, d\xi.$$

If $s = k \in \mathbb{Z}_{\geq 0}$ then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.

<u>Thm</u>. (Sobolev embedding). Let $s > \frac{n}{2} + k$ and let $f \in H^s(\mathbb{R}^n)$. Then there exists $f^* \in C^k(\mathbb{R}^n)$ s.t. $f = f^*$ a.e. Write $f = f^*$ and $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

<u>Proof</u>. First assume $f \in S(\mathbb{R}^n)$. Then
$$\nabla^\alpha f(x) = \frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \xi^\alpha \hat{f}(\xi) \, d\xi \qquad (|\alpha| < k)$$

$$\Rightarrow |\nabla^\alpha f(x)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^\alpha \, |\hat{f}(\xi)| \, d\xi$$
$$\leq \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \times \left(\int_{\mathbb{R}^n} |\xi|^{2|\alpha|} (1+|\xi|^2)^{-s} \, d\xi \right)^{1/2}$$

Since $|\alpha| \leq k$,

$$\int |\xi|^{2|\alpha|}(1+|\xi|^2)^{-s}\,d\xi \leq \underbrace{C\int_{\mathbb{R}^n}(1+|\xi|^2)^{s-k}\,d\xi}_{=:C_{n,k,s}^2} < \infty \quad \text{for } s > \tfrac{n}{2}+k.$$

$$\Rightarrow \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq k}} |\nabla^\alpha f(x)| \leq C_{n,k,s}\|f\|_{H^s} \qquad (*)$$

Now for $f \in H^s(\mathbb{R}^n)$, let $(f_i) \subset \mathcal{S}(\mathbb{R}^n)$ be s.t. $f_i \longrightarrow f$ in H^s (Ex. Sheet 4) and $f_i \to f$ a.e.

$\Rightarrow f_i$ is Cauchy in H^s

$\Rightarrow f_i$ is Cauchy in C^k by $(*)$

$\Rightarrow f_i \longrightarrow f^*$ in $C^k(\mathbb{R}^n)$

Since $f_i \longrightarrow f$ a.e. thus $f^* = f$ a.e.

<u>Example</u>. Consider

$$-\Delta u + u = f \qquad \text{on } \mathbb{R}^n \qquad (*)$$

Then if $f \in H^s$ there is a unique $u \in H^{s+2}$ s.t. $(*)$ holds.

__Proof.__ By Fourier transform, (*) is equivalent to

$$(|\xi|^2+1)\,\hat{u}(\xi) = \hat{f}(\xi) \qquad \text{a.e.}$$

$$\Leftrightarrow \quad \hat{u}(\xi) = \frac{\hat{f}(\xi)}{1+|\xi|^2} \qquad \text{a.e.}$$

and $\|u\|_{H^{s+2}} = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s+2} \, |\hat{u}(\xi)|^2$

$$= \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 = \|f\|_{H^s}.$$

__Remarks__ • u is more regular than f, e.g.

$$-\Delta u + u \in L^2 \implies u \in H^2$$

This is an example of elliptic regularity.

• If $s > \frac{n}{2}$ then $f \in C^0(\mathbb{R}^n)$, $u \in C^2(\mathbb{R}^n)$ and (*) holds in the classical sense.

4.2. Traces of Sobolev functions

If $s > \frac{n}{2}$ then $H^s \subset C^0$ and $f|_\Sigma$ makes sense for any $f \in H^s$ and $\Sigma \subset \mathbb{R}^n$ a surface.

What about $s \leq \frac{n}{2}$?

Thm. (Trace theorem). Let $s > \frac{1}{2}$. Then there is a bounded linear operator

$$T: H^s(\mathbb{R}^n) \longrightarrow H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$$

such that

$$Tf = f|_{\mathbb{R}^{n-1} \times \{0\}} \qquad \forall f \in S(\mathbb{R}^n).$$

Tf is the trace of f on $\Sigma = \mathbb{R}^{n-1} \times \{0\}$.

Proof. Ex. Sheet 4.

Rk. By coordinate tranformations, the result can be extended to suff. regular surfaces $\Sigma \subset \mathbb{R}^n$.

4.3. The space $H_0^1(U)$

Let $U \subset \mathbb{R}^n$ be open and let $f \in C_c^\infty(U)$. Extending f by 0 outside U, $f \in H^1(\mathbb{R}^n)$, so
$C_c^\infty(U) \subset H^1(\mathbb{R}^n)$.

<u>Defn.</u> The space $H_0^1(U)$ is the closure of $C_c^\infty(U)$ in $H^1(\mathbb{R}^n)$, i.e., with respect to the norm

$$\|f\|_{H^1} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2) |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$
$$= \left((2\pi)^n \int_U \left(|\nabla f(x)|^2 + |f(x)|^2 \right) dx \right)^{1/2}$$

$H_0^1(U)$ is a Hilbert space with inner product

$$(u,v)_{H_0^1} = \int_U \left(\overline{\nabla u} \cdot \nabla v + \bar{u} v \right) dx .$$

<u>Prop.</u> If $u \in H_0^1(U)$ then $u = 0$ for a.e. $x \notin U$.

<u>Proof.</u> It suffices to show that

$$\int_{\mathbb{R}^n} \phi u \, dx = 0 \quad \forall \phi \in C_c^\infty((\mathring{U}^c)).$$

Let $\phi \in C_c^\infty(\mathring{U}^c)$ and define $\Lambda_\phi(v) = \int \phi v \, dx$.
Then $\Lambda_\phi(v) = 0$ for all $v \in C_c^\infty(U)$.

Since also $|\Lambda_\phi(v)| \leq \|\phi\|_{L^2} \|v\|_{L^2} \leq C\|\phi\|_{L^2} \|v\|_{H^1}$, if $u_n \in C_c^\infty(U)$ satisfies $u_n \longrightarrow u \in H^1$ then

$$\Lambda_\phi(u) = 0.$$

Thus $\int \phi u \, dx = 0$ holds for $u \in H_0^1$, $\phi \in C_c^\infty(\mathring{U}^c)$.

<u>Fact</u>. If ∂U is sufficiently regular, any $u \in H_0^1(U)$ vanishes on ∂U in trace sense.

<u>Proof</u>. $T: H^1(U) \longrightarrow H^{1/2}(U)$ is bounded and clearly $Tu_k = 0$ for every $u_k \in C_c^\infty(U)$. Thus if $u_k \longrightarrow u$ in H^1, also $Tu = 0$.

<u>Example</u> (Elliptic boundary value problem).

Let $U \subset \mathbb{R}^n$ be open and consider

$$\begin{cases} -\Delta u + u = f & \text{in } U \\ \quad\quad\;\; u = 0 & \text{on } \partial U \end{cases}$$

We seek $u \in H_0^1(U)$ s.t. $-\Delta u + u = f$ in the distributional sense (for $f \in L^2(U)$):

$$\int (-\bar{u}\Delta v + \bar{u} v)\, dx = \int_U \bar{f} v \, dx \quad \forall v \in C_c^\infty(U).$$

Since $u \in H^1$, $-\int \bar{u} \nabla v \, dx = \int (\nabla \bar{u}) v \, dx$ for all $v \in C_c^\infty(\mathbb{R}^n)$, so this gives

$$\int_U (\overline{\nabla u} \nabla v + \bar{u} v) \, dx = \int_U \bar{f} v \, dx \quad \forall v \in C_c^\infty(\mathbb{R}^n)$$

<u>Defn</u>. $u \in H_0^1(U)$ is a weak solution of $(*)$, where $f \in L^2(U)$, if

$$(u, v)_{H^1} = (f, v)_{L^2} \qquad \forall v \in H_0^1(U).$$

<u>Prop</u>. Let $f \in L^2(U)$. Then there exists a unique weak solution u of $(*)$ and $\|u\|_{H^1} \leq \|f\|_{L^2}$.

The solution operator $S : L^2(U) \mapsto H_0^1(U)$, $f \mapsto u$, is bounded linear. Interpreted as $S : L^2 \to L^2$, it is also symmetric.

<u>Proof</u>. $\Lambda : H_0^1(U) \longrightarrow \mathbb{C}$, $\Lambda(v) = (f, v)_{L^2}$ is a bounded linear map. Thus by the Riesz representation theorem there is a unique $u \in H_0^1(U)$ s.t. $\Lambda(v) = (u, v)_{H^1}$.

Linearity is clear: if $f_1, f_2 \in L^2(U)$, $a \in \mathbb{C}$, and $u_1 = Sf_1$, $u_2 = Sf_2$ then $u = u_1 + a u_2$ satisfies

$$(u,v)_{H^1} = (f_1 + af_2, v)_{L^2} \quad \forall v \in H_0^1(U).$$

So $S(f_1 + af_2) = Sf_1 + a\,Sf_2$ by uniqueness.

Symmetry:
$$\begin{aligned}(f, Sg)_{L^2} &= (Sf, Sg)_{H^1} \\ &= \overline{(Sg, Sf)_{H^1}} \\ &= \overline{(g, Sf)_{L^2}} = (Sf, g)_{L^2}.\end{aligned}$$

When is u a nice function?

<u>Defn</u>. For $s>0$, define

$$H^s_{loc}(\Omega) = \{u \in L^2_{loc}(U) : \chi u \in H^s(\mathbb{R}^n) \text{ for all } \chi \in C_c^\infty(\Omega)\}.$$

<u>Prop</u>. Let U be open, $\bar{U} \subset \Omega$. Then $u \in H^s_{loc}(\Omega)$ is in $C^k(U)$ if $s > \frac{n}{2} + k$.

<u>Proof</u> If U is open, $\bar{U} \subset \Omega$ then there is $\chi \in C_c^\infty(\Omega)$ s.t. $\chi(x) = 1$ for $x \in U$. Hence if $s > \frac{n}{2} + k$, $u \in H^s_{loc}(\Omega)$, then $\chi u \in C^k(U)$. Since $\chi u = u$ on U, thus $u \in C^k(U)$.

<u>Example</u> (Elliptic boundary value problem cont'd).
If $u \in H_0^1(U)$ is the (unique) weak solution to

$$\begin{cases} -\Delta u + u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \qquad (*)$$

where $f \in L^2(U)$ then $u \in H^2_{loc}(U)$. More generally, if $f \in L^2(U) \cap H^k_{loc}(U)$ then $u \in H^{k+2}_{loc}(U)$.

In particular, if $u \in L^2(U) \cap C^\infty(U)$ then $u \in C^\infty$ and $(*)$ holds classically. This is called elliptic regularity.

<u>Proof</u>. Fix $K \subset U$ compact and let $\chi_K \in C_c^\infty(U)$ be s.t. $\chi_K = 1$ on K. Recall u satisfies

$$\int_U (\overline{\nabla u} \cdot \nabla v + \bar{u} v)\, dx = \int_U \bar{f} v \, dx \quad \forall v \in H_0^1(U)$$

Given $\phi \in S(\mathbb{R}^n)$, let $v(x) = \chi_K(x)\phi(x)$. Then

$$\int_U (\overline{\nabla u} \cdot \nabla v)\, dx = \int_U \overline{\nabla u} \cdot ((\nabla \chi_K)\phi + \chi_K \nabla \phi)\, dx$$

$$= -\int_U u\Big((\Delta \chi_K)\phi \underbrace{+ \nabla \chi_K \cdot \nabla \phi + \nabla \chi_K \cdot \nabla \phi}_{+2\nabla \chi_K \nabla \phi} + \chi_K \Delta \phi\Big) dx$$

$$= -\int_U u\big(-\Delta \chi_K \phi - 2\nabla \chi_K \cdot \nabla u\, \phi + \chi_K \Delta \phi\big)\, dx$$

$$\Rightarrow \int_U \underbrace{\chi_K u}_{w}(-\Delta\phi+\phi)\,dx = \int \overline{\underbrace{(f\chi_K - 2\nabla\chi_K\cdot\nabla u - u\Delta\chi_K)}_{\bar g}}\,\phi\,dx$$

i.e., $\displaystyle\int_{\mathbb{R}^n} w(-\Delta\phi+\phi)\,dx = \int_{\mathbb{R}^n} \bar g\,\phi\,dx \quad \forall \phi\in S(\mathbb{R}^n).$

Note that $g\in L^2(\mathbb{R}^n)$ and w is a weak soln. to

$$-\Delta w + w = g \quad \text{on } \mathbb{R}^n.$$

Therefore $w\in H^2(\mathbb{R}^n)$ as seen before.

For any $\psi\in C_c^\infty(U)$, we can take $K=\operatorname{supp}\psi$. Then

$$\psi u = \psi w \in H^2(\mathbb{R}^n)$$

$\Rightarrow u\in H^2_{loc}(U)$.

More generally, if $f\in L^2(U)\cap H^1_{loc}(U)$ then $g\in H^1(\mathbb{R}^n)$ and thus $u\in H^3_{loc}(U)$. Inductively, if $f\in L^2\cap H^k_{loc}$ then $u\in H^{k+2}_{loc}(U)$.

4.4. Rellich-Kondrachov Theorem

Thm. Let $U \subset \mathbb{R}^n$ be open and bounded. Assume that $(u_j) \subset H_0^1(U)$ satisfies $\|u_j\|_{H^1} \leq K$ and that $u_j \xrightarrow{w^*} u$ in $L^2(U)$, $u \in H_0^1$. Then $u_j \longrightarrow u$ in $L^2(U)$.

Rk. By Banach-Alaoglu, $\|u_j\|_{H^1} \leq K$ implies that there is a subsequence s.t. $u_j \xrightarrow{w^*} u$ in $H^1(U)$ for some u in $H_0^1(U)$ along this subsequence. Note that $u_j \xrightarrow{w^*} u$ in H_0^1 also implies $u_j \xrightarrow{w^*} u$ in L^2. Indeed,

$$
\begin{aligned}
u_j \xrightarrow{w^*} u \text{ in } H_0^1 &\Leftrightarrow (u_j, v) \longrightarrow (u, v) \quad \forall v \in H_0^1 \\
&\Rightarrow (u_j, v) \longrightarrow (u, v) \quad \forall v \in L^2 \\
&\Leftrightarrow u_j \xrightarrow{w^*} u \text{ in } L^2.
\end{aligned}
$$

Proof. By the Parseval identity,

$$
\begin{aligned}
\|u_j - u\|_{L^2}^2 &= \frac{1}{(2\pi)^n} \|\hat{u}_j - \hat{u}\|_{L^2}^2 \\
&= \underbrace{\frac{1}{(2\pi)^n} \int_{|\xi|<R} |\hat{u}_j(\xi) - \hat{u}(\xi)|^2 \, d\xi}_{(I)} \\
&\quad + \underbrace{\frac{1}{(2\pi)^n} \int_{|\xi|\geq R} |\hat{u}_j(\xi) - \hat{u}(\xi)|^2 \, d\xi}_{(II)}
\end{aligned}
$$

$$(\mathrm{II}) \le \frac{2}{(2\pi)^n(1+R^2)} \int (1+|\xi|^2)\left(|\hat{u}_j|^2 + |\hat{u}|^2\right) d\xi$$

$$\le \frac{2}{(1+R^2)}\left(\|u_j\|_{H^1}^2 + \|u\|_{H^1}^2\right) \le \frac{2K^2}{R^2} < \varepsilon \quad (R > R_0)$$

Since $\hat{u}_j(\xi) = (e_\xi, u_j)_{L^2(U)}$, $e_\xi(x) = e^{i\xi\cdot x} \in L^2(U)$

since U bounded,

the assumption $u_j \xrightarrow{w^*} u$ in L^2 implies that

$$\hat{u}_j(\xi) = (e_\xi, u_j) \longrightarrow (e_\xi, u) = \hat{u}(\xi) \quad \forall \xi$$

Also

$$|\hat{u}_j(\xi) - \hat{u}(\xi)|^2 \le 2\left(|\hat{u}_j(\xi)|^2 + |\hat{u}(\xi)|^2\right)$$

$$\le 2\left(\|u_j\|_{L^1}^2 + \|u\|_{L^1}^2\right)$$

$$\le 2|U|\left(\|u_j\|_{L^2}^2 + \|u\|_{L^2}^2\right)$$

$$\le 4|U|K^2$$

Thus by DCT

$(\mathrm{I}) \longrightarrow 0$ as $j \to \infty$.

i.e. $(\mathrm{I}) \le \varepsilon$ for $j \ge j_0$

Thus $(\mathrm{I}) + (\mathrm{II}) \le 2\varepsilon$ for $j \ge j_0$.

Cor. Let $U \subset \mathbb{R}^n$ be open and bounded, and $(u_j) \subset H_0^1(U)$ be bounded. Then there is a subsequence $(j_k)_k$ s.t. $u_{j_k} \xrightarrow{w^*} u$ in $H_0^1(U)$ and $u_j \longrightarrow u$ in $L^2(U)$.

Cor. Let $U \subset \mathbb{R}^n$ be open and bounded, and $A: L^2(U) \longrightarrow H_0^1(U)$ a bounded linear operator. Then $A: L^2(U) \longrightarrow L^2(U)$ is compact.

Example. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $V: U \to \mathbb{R}$ be smooth and bounded. For $f \in L^2(U)$, consider

$$(*) \begin{cases} -\Delta u + Vu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Defn. $u \in H_0^1(U)$ is a weak solution to $(*)$ if

$$(+) \int_U \left(\overline{\nabla u} \cdot \nabla v + V \bar{u} v\right) dx = \int_U \bar{f} v \, dx \quad \forall v \in H_0^1(U).$$

The LHS is not an inner product, so the previous proof to show existence (for $V=1$) does not work.

Prop. Either there exists $w \in H_0^1(U) \cap C^\infty(U)$ s.t.

$$-\Delta w + V w = 0$$

or $\forall f \in L^2(U)\ \exists!\, u \in H_0^1(U)$ solving $(*)$.

Proof.

$$(*) \Leftrightarrow \int_U (\overline{\nabla u} \cdot \nabla v + \bar{u} v)\, dx = \int \overline{(f + (1-V)u)}\, v\, dx \quad \forall v \in H_0^1$$

Let $S : L^2(U) \to H_0^1(U)$ be the solution operator corresponding to $V=1$, i.e., $u = Sf$ is the unique weak solution to

$$\begin{cases} -\Delta u + u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Then $(*) \Leftrightarrow u = S(f + (1-V)u)$

$$\Leftrightarrow (\mathrm{Id} - K) u = Sf, \quad Ku = S((1-V)u)$$

Since $K : L^2(U) \to H_0^1(U)$ is bounded, $K : L^2 \to L^2$ is compact. Thus either:

(a) $\ker(I-K) \neq 0$, i.e., $\exists w \in L^2(U)$ s.t. $(I-K)w = 0$.

(b) $\operatorname{im}(I-K) = L^2(U)$, i.e., $\exists!\, u \in L^2(U)$ s.t. $(I-K)u = Sf$.

In case (a), $\ker(I-K)$ is finite dimensional,

$$(I-K)w=0 \Leftrightarrow w=S((1-V)w) \Rightarrow w\in H_0^1(U).$$

By iteration, $w\in H_0^1(U)\cap C^\infty(U)$.

In case (b), $u=S(f+(1-V)u)\in H_0^1(U)$, so u is a weak solution to $(*)$.

<u>Thm</u>. There exists an orthonormal basis $\{w_k\}$ of $L^2(U)$ s.t. $w_k\in H_0^1\cap C^\infty(U)$ s.t.

$$-\Delta w_k = \lambda_k w_k \quad \text{in } U$$

where $0\le\lambda_1\le\lambda_2\le\cdots$ and $\lambda_k\to\infty$.

<u>Proof</u>. Recall $S: L^2(U)\to H_0^1(U)$ is bounded, linear and self-adjoint. By Rellich-Kondrachov, therefore $S: L^2(U)\to L^2(U)$ is compact.
By the spectral theorem,

$$\sigma(S)=\{0,\mu_1,\mu_2,\dots\}$$

with $\mu_k\in\mathbb{R}$ and only accummulation point at 0, and there is an ONB of eigenvectors w_k for $L^2(U)$.

Assume $Sw_k = \mu_k w_k$. Then $w_k \in H_0^1(U)$ and

$$(w_k, v)_{L^2} = (Sw_k, v)_{H^1} = \mu_k (w_k, v)_{H^1} \quad \forall v \in H_0^1$$

Setting $v = w_k$ we see $w_k = 0$ or $\mu_k \neq 0$. Thus if $w_k \neq 0$ then w_k is a weak soln to

$$(*) \begin{cases} -\Delta w_k + w_k = \frac{1}{\mu_k} w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$$

$$\Leftrightarrow \begin{cases} -\Delta w_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$$

where $\lambda_k = -1 + \frac{1}{\mu_k}$. Since $\{\mu_k\}$ has only accummulation point at 0, $\lambda_k \to \infty$.

Also since $w_k \in H_0^1(U) \subset H_{loc}^1(U)$, elliptic regularity implies $w \in H_0^1(U) \cap H^3(U)$. Iterating, therefore $w \in H_0^1(U) \cap C^\infty(U)$.

4.5. The direct method of the calculus of variations

Given $u \in H_0^1(U)$, $f \in L^2(U)$, consider

$$S(u) = \int_U \left(|\nabla u|^2 + |u|^2 - \bar{f}u - f\bar{u}\right) dx$$
$$= \|u\|_{H^1}^2 - 2\operatorname{Re}(f,u)_{L^2}$$

<u>Thm</u>.

$$\sigma = \inf\{S(u) : u \in H_0^1(U)\} > -\infty$$

and there is a unique $u \in H_0^1(U)$ s.t.

$$\sigma = S(u)$$

This u is the unique weak solution to

$$\begin{cases} -\Delta u + u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

<u>Proof</u>. By Cauchy-Schwarz

$$S(u) \geq \|u\|_{H^1}^2 - 2\|f\|_{L^2}\|u\|_{L^2}$$
$$\geq \|u\|_{H^1}^2 - 2\|f\|_{L^2}^2 - \frac{1}{2}\|u\|_{L^2}^2$$
$$\geq \frac{1}{2}\|u\|_{H^1}^2 - 2\|f\|_{L^2}^2$$

Thus $S(u) \geq -2\|f\|_{L^2}^2$ and $\sigma > -\infty$.

To see the infimum is attained, let $(u_k) \subset H_0^1(U)$ be a minimising sequence:

$$S(u_k) \longrightarrow \sigma$$

In particular, $(S(u_k))_k$ is bounded, and (u_k) is then bounded in $H_0^1(U)$ since

$$\|u_k\|_{H^1}^2 \leq 2S(u_k) + 4\|f\|_{L^2}^2$$

By Banach-Alaoglu there is $\tilde{u} \in H_0^1(U)$ s.t. $u_k \xrightarrow{w^*} \tilde{u}$ in $H_0^1(U)$ along a subsequence.

$$\Rightarrow \|\tilde{u}\|_{H^1} \leq \liminf_{k\to\infty} \|u_k\|_{H^1}.$$

$$(f,u)_{L^2} = \lim_{k\to\infty} (f,u_k)_{L^2}$$

$$\Rightarrow S(\tilde{u}) \leq \liminf_{k\to\infty} S(u_k) = \sigma$$

Since $\sigma = \inf\{\cdots\}$ in fact $S(\tilde{u}) = \sigma$.

To see uniqueness, it suffices to show that $\tilde{u}$ is a weak solution of the PDE.

Let $v \in H_0^1(U)$, $t \in \mathbb{R}$. Then

$$S(\tilde{u} + tv) = S(\tilde{u}) + t^2 \|v\|_{H^1}^2 + 2t \operatorname{Re} \int (\overline{\nabla \tilde{u}} \cdot \nabla v + \bar{\tilde{u}} v - \bar{f} v) dx$$

Since $S(\tilde{u} + tv) \leq S(\tilde{u})$ [$\tilde{u}$ is a minizer],

$$\operatorname{Re} \int_U (\overline{\nabla \tilde{u}} \cdot \nabla v + \bar{\tilde{u}} v - \bar{f} v) dx = 0 \quad \forall v \in H_0^1$$

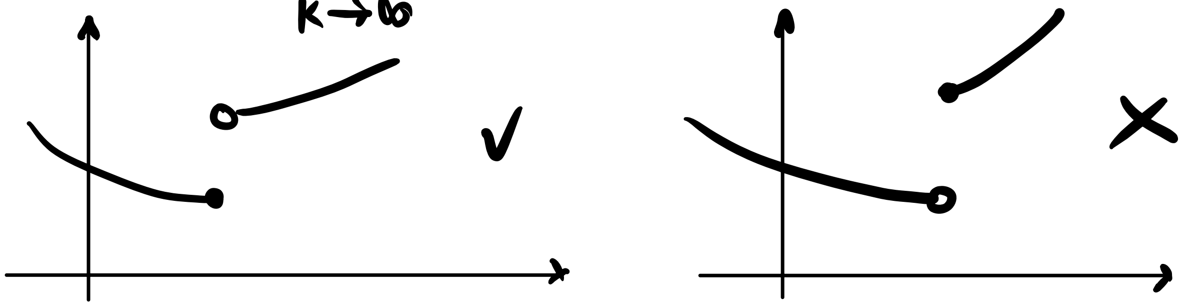
$$\Leftrightarrow \int_U \overline{\nabla \tilde{u}} \cdot \nabla v + \bar{\tilde{u}} v = \int_U \bar{f} v \, dx \quad \forall v \in H_0^1(U),$$

i.e., $\hat{u}$ is a weak solution to the PDE.

The proof can be generalised:

<u>Defn.</u> Let X be a topological space. Then $S: X \to \mathbb{R}$

- is coercive if for any $(u_k) \subset X$ with $S(u_k) \leq K$ (any K) there is a convergent subsequence
- lower semicontinuous if for any (u_k) s.t. $u_k \xrightarrow{X} u$
  $S(u) \leq \liminf_{k \to \infty} S(u_k)$



<u>Thm</u>. Let $S: X \to \mathbb{R}$ be coercive and LSC. Then S achieves its minimum.

<u>Proof</u>. Analogous to last theorem.

From now, let X be a reflexive Banach space.

<u>Prop</u>. If $S(u) \leq K \Rightarrow \|u\| \leq \tilde{K}$ then S is coercive w.r.t. the weak topology on X.

<u>Proof</u>. Banach-Alaoglu (as in proof of theorem).

<u>Lemma</u>. If $U \subset X$ is norm-closed and convex, then U is also closed in weak topology.

<u>Proof</u> Follows from Hahn-Banach separation thm: For $x \notin U$, take $A = \{x\}$ (convex, bounded) and $B = U$ (convex, closed). There is $\Lambda \in X'$ s.t.

$$\operatorname{Re} \Lambda x \leq \gamma_1 < \gamma_2 \leq \operatorname{Re} \Lambda y \quad \forall y \in U$$

$\{\operatorname{Re} \Lambda z < \gamma_1\}$ is open in the weak topology, contains x and is disjoint from U.

$\Rightarrow U^c$ is weakly open $\Rightarrow U$ is weakly closed.

Prop. If $S: X \to \mathbb{R}$ is convex and LSC in the strong topology, then it is also LSC in the weak topology.

Proof. $\{S(u) \leq K\}$ is convex and closed in the strong topology, thus weakly closed. Thus if $u_k \xrightarrow{w} u$ then along a subsequence

$$S(u_{k_j}) \longrightarrow \liminf_{k\to\infty} S(u_k)$$

$$\Rightarrow S(u_{k_j}) \leq K_\varepsilon = \liminf S(u_k) + \varepsilon \quad \forall j \geq j_0(\varepsilon).$$

Since $u_{k_j} \xrightarrow{w} u$ and $\{S(u) \leq K\}$ is weakly closed, it follows that

$$S(u) \leq K_\varepsilon$$

Since ε is arbitrary, $S(u) \leq \liminf S(u_k)$.

Example. Let $U \subset \mathbb{R}^n$ be bounded and open, $L: \mathbb{R}^n \to \mathbb{R}$ convex and

$$L(z) \geq \gamma |z|^2 - C \quad \text{for some } C, \gamma > 0.$$

Then $S: H_0^1(U) \to \mathbb{R}$, $S(u) = \int_U L(\nabla u)\, dx$ has a minimizer.