Corrections to
Introduction to a Renormalisation Group Method
by R. Bauerschmidt, D.C. Brydges and G. Slade

April 18, 2024

1. p.53: Although it is stated that "Periodic boundary conditions are not appropriate for hierarchical fields," the hierarchial formulation in Chapter 4 should in fact be regarded as corresponding to periodic boundary conditions. The distinction between free and periodic boundary conditions in the hierarchical setting is discussed in [1, 2].

2. p.62 (4.2.7): replace $\log L$ by $\log L^2$.

3. p.99 (6.2.13): replace $\log_L m^{-2}$ by $\log_{L^2} m^{-2}$.

4. p.103 (6.2.35) and two lines below (6.2.35): replace $\log L$ by $\log L^2$.

5. p.108 (7.1.3): replace the norm $\|\cdot\|_X$ by absolute values $|\cdot|$.

6. p.131 (8.3.2): replace $-\frac{1}{2}\tilde{g}_j(m^2)$ by $+\frac{1}{2}\tilde{g}_j(m^2)$.

7. p.133 line 9: replace "sequences" by "sequence".

8. p.133 line 10: The claim that the intersection $\cap_{j\geq 1} I_j$ must consist of a single point is not justified. It can be justified as follows:

   Let $I = \cap_{j\geq 1} I_j$. By construction, I is an interval. Any value $\nu \in I$ serves as an initial condition for a flow to all scales $j \in \mathbb{N}$, and in particular it initiates a sequence $\mu_j(\nu)$ with $|\mu_j(\nu)| \leq c_0 \vartheta_j \tilde{g}_j$ for all $j \in \mathbb{N}$. Also, the inductive proof of (8.4.3) applies, so that, for every $\nu \in I$ and for all j,

   $$\frac{\partial \mu_j}{\partial \nu} \geq \frac{1}{2} L^{2j} \Big(\frac{g_j}{g_0}\Big)^{\hat{\gamma}}.$$

   Suppose that $\nu_{0,1} < \nu_{0,2}$ are two elements of I. By the Fundamental Theorem of Calculus,

   $$\mu_j(\nu_{0,2}) - \mu_j(\nu_{0,1}) = \int_{\nu_{0,1}}^{\nu_{0,2}} \frac{\partial \mu_j}{\partial \nu} d\nu \geq \frac{1}{2} L^{2j} \Big(\frac{g_j}{g_0}\Big)^{\hat{\gamma}} (\nu_{0,2} - \nu_{0,1}).$$

   This contradicts the statement that, for both $i = 1$ and $i = 2$, we have $|\mu_j(\nu_{0,i})| \leq c_0 \vartheta_j \tilde{g}_j$ for all j. Therefore I must consist of a single point.

9. pp.168-170: There are errors in Lemma 10.5.3 and its application to prove (10.5.26)–(10.5.27). The statement of Lemma 10.5.3 does not make sense because on the left-hand side of (10.5.12) $T\dot{K}$ is a function of fields which are not constant on blocks in $\mathcal{B}_+$, so we cannot take the $\mathcal{W}_+$ norm. Here is a corrected proof of (10.5.26)–(10.5.27):

Lemma 10.5.3′ (Replacement for Lemma 10.5.3 and (10.5.26)–(10.5.27))**.** Let L be sufficiently large, and let $\tilde{g}$ be sufficiently small depending on L. For $V \in \mathcal{D}$ and $\dot{K} \in \mathcal{F}$,

$$\|\mathbb{E}_+ \theta T\dot{K}\|_{T_0(\ell_+)} \leq O(L^{-2})\|\dot{K}\|_{\mathcal{W}}, \tag{1}$$

$$\|\mathbb{E}_+ \theta T\dot{K}\|_{T_\infty(h_+)} \leq O(L^{-2})\|\dot{K}\|_{T_\infty(h)}. \tag{2}$$

Proof. Let $F_1(b) = \dot{K}(b)$, $F_2(b) = e^{-V(b)}\operatorname{Loc}(e^{V(b)}\dot{K}(b))$. The algebraic manipulations in (10.5.15)–(10.5.17) give

$$T\dot{K} = \sum_{b \in (B)} e^{-V(B \setminus b)}(1 - \operatorname{Loc})(F_1(b) - F_2(b)). \tag{3}$$

By the triangle inequality, by Proposition 7.3.1, and by the product property of the norm,

$$\|\mathbb{E}_+ \theta T\dot{K}\|_{T_\varphi(\mathfrak{h}_+)} \leq \sum_{b \in \mathcal{B}(B)} \mathbb{E}_+ \left[\Big(\prod_{b' \neq b} \|e^{-V(b')}\|_{T_{\varphi+\zeta_{b'}}(\mathfrak{h}_+)} \Big) \sum_{i=1}^{2} \|(1 - \operatorname{Loc})F_i(b)\|_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)} \right]. \tag{4}$$

Let $\hat{\varphi}_b = \varphi + \zeta_b$. By Lemma 10.2.3,

$$\|e^{-V(b')}\|_{T_{\hat{\varphi}_{b'}}(\mathfrak{h}_+)} \leq \Big(2^{1/4} e^{-4c^{\text{st}}|\hat{\varphi}_{b'}/h_+|^4} \Big)^{L^{-d}} \leq (2^{1/4})^{L^{-d}}. \tag{5}$$

Since $\mathfrak{h}_+/\mathfrak{h} = O(L^{-1})$ for both $\mathfrak{h} = \ell$ and $\mathfrak{h} = h$, it follows from (10.5.11) that

$$\|(1 - \operatorname{Loc})F_i(b)\|_{T_{\hat{\varphi}_b}(\mathfrak{h}_+)} \leq O(L^{-6}) P^6_{\mathfrak{h}_+}(\hat{\varphi}_b) \sup_{0 \leq t \leq 1} \|F_i(b)\|_{T_{t\hat{\varphi}_b}(\mathfrak{h})}. \tag{6}$$

Since there are L^4 terms in the sum over b, this gives

$$\|\mathbb{E}_+ \theta T\dot{K}\|_{T_\varphi(\mathfrak{h}_+)} \leq O(L^{-2}) \sum_{i=1}^{2} \sup_{b \in \mathcal{B}(B)} \mathbb{E}_+ P^6_{\mathfrak{h}_+}(\hat{\varphi}_b) \sup_{0 \leq t \leq 1} \|F_i(b)\|_{T_{t\hat{\varphi}_b}(\mathfrak{h})}. \tag{7}$$

For $i = 1$, due to (10.4.5) when $\mathfrak{h} = \ell$,

$$\|F_1(b)\|_{T_{t\hat{\varphi}_b}(\mathfrak{h})} = \|\dot{K}(b)\|_{T_{t\hat{\varphi}_b}(\mathfrak{h})} \leq \begin{cases} P^{10}_\ell(\hat{\varphi}_b)\|\dot{K}(b)\|_{\mathcal{W}} & (\mathfrak{h} = \ell) \\ \|\dot{K}(b)\|_{T_\infty(h)} & (\mathfrak{h} = h). \end{cases} \tag{8}$$

Note that $P_\ell \leq P_{\ell_+}$. By Lemma 10.3.1, with $\mathfrak{h} = \ell$ the expectation $\mathbb{E}_+ P^{16}_{\ell_+}(\zeta_b)$ is bounded, and the corresponding expectation is similarly bounded for $\mathfrak{h} = h$ because $P_{h_+} \leq P_{\ell_+}$. This proves the two desired inequalities for the contribution due to F_1.

For $i = 2$, by Lemma 7.5.1 and Lemma 9.3.1,

$$\|F_2(b)\|_{T_{t\hat{\varphi}_b}(\mathfrak{h})} \leq 2P^4_{\mathfrak{h}}(\hat{\varphi}_b)\|\dot{K}(b)\|_{T_0(\mathfrak{h})}. \tag{9}$$

For $\mathfrak{h} = \ell$, from the above we see that the contribution due to F_2 to the expectation $\|\mathbb{E}_+ \theta T\dot{K}\|_{T_0(\ell_+)}$ is bounded by

$$O(L^{-2}) \sup_{b \in \mathcal{B}(B)} \|\dot{K}(b)\|_{T_0(\ell)} \mathbb{E}_+ P^{10}_\ell(\zeta_b). \tag{10}$$

Since the expectation is bounded due to Lemma 10.3.1, this gives the desired bound for the $T_0(\ell_+)$ norm. Finally, for the $T_\varphi(h_+)$ norm, we have the upper bound

$$O(L^{-2}) \sup_{b \in \mathcal{B}(B)} \|\dot{K}(b)\|_{T_0(h)} \mathbb{E}_+ P_h^4(\hat{\varphi}_b). \tag{11}$$

Since $P_h \leq P_\ell \leq P_{\ell_+}$, the expectation is bounded, and this completes the proof. ■

Acknowledgement. We thank Jiwoon Park for identifying errors.

References

[1] T. Hutchcroft. Critical cluster volumes in hierarchical percolation. Preprint, https://arxiv.org/pdf/2211.05686, (2022).

[2] E. Michta, J. Park, and G. Slade. Boundary conditions and universal finite-size scaling for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher. Preprint, https://arxiv.org/abs/2306.00896, (2023).