Corrections to

*Introduction to a Renormalisation Group Method*

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1. p.53: Although it is stated that “Periodic boundary conditions are not appropriate for hierarchical fields,” the hierarchial formulation in Chapter 4 should in fact be regarded as corresponding to periodic boundary conditions. The distinction between free and periodic boundary conditions in the hierarchical setting is discussed in [1,2].

2. p.62 (4.2.7): replace \( \log L \) by \( \log L^2 \).

3. p.62 (6.2.13): replace \( \log L \) by \( \log L^2 \).

4. p.99 (6.2.13) and two lines below (6.2.13): replace \( \log L \) by \( \log L^2 \).

5. p.108 (7.1.3): replace the norm \( \| \cdot \|_X \) by absolute values \( | \cdot | \).

6. p.131 (8.3.2): replace \(-\frac{1}{2} \tilde{g}_j(m^2)\) by \(+\frac{1}{2} \tilde{g}_j(m^2)\).

7. p.133 line 9: replace “sequences” by “sequence”.

8. p.133 line 10: The claim that the intersection \( \cap_{j \geq 1} I_j \) must consist of a single point is not justified. It can be justified as follows:

   Let \( I = \cap_{j \geq 1} I_j \). By construction, \( I \) is an interval. Any value \( \nu \in I \) serves as an initial condition for a flow to all scales \( j \in \mathbb{N} \), and in particular it initiates a sequence \( \mu_j(\nu) \) with \( |\mu_j(\nu)| \leq c_0 \partial j \tilde{g}_j \) for all \( j \in \mathbb{N} \). Also, the inductive proof of (8.4.3) applies, so that, for every \( \nu \in I \) and for all \( j \),

   \[
   \frac{\partial \mu_j}{\partial \nu} \geq \frac{1}{2} L^{2j} \left( \frac{g_j}{g_0} \right)^{\frac{\gamma}{2}}.
   \]

   Suppose that \( \nu_{0,1} < \nu_{0,2} \) are two elements of \( I \). By the Fundamental Theorem of Calculus,

   \[
   \mu_j(\nu_{0,2}) - \mu_j(\nu_{0,1}) = \int_{\nu_{0,1}}^{\nu_{0,2}} \frac{\partial \mu_j}{\partial \nu} d\nu \geq \frac{1}{2} L^{2j} \left( \frac{g_j}{g_0} \right)^{\frac{\gamma}{2}} (\nu_{0,2} - \nu_{0,1}).
   \]

   This contradicts the statement that, for both \( i = 1 \) and \( i = 2 \), we have \( |\mu_j(\nu_{0,i})| \leq c_0 \partial j \tilde{g}_j \) for all \( j \).

   Therefore \( I \) must consist of a single point.

9. pp.168-170: There are errors in Lemma 10.5.3 and its application to prove (10.5.26)–(10.5.27). The statement of Lemma 10.5.3 does not make sense because on the left-hand side of (10.5.12) \( T \dot{K} \) is a function of fields which are not constant on blocks in \( \mathcal{B}_+ \), so we cannot take the \( \mathcal{W}_+ \) norm. Here is a corrected proof of (10.5.26)–(10.5.27):
\textbf{Lemma 10.5.3'} (Replacement for Lemma 10.5.3 and (10.5.26)–(10.5.27)). Let $L$ be sufficiently large, and let $\hat{g}$ be sufficiently small depending on $L$. For $V \in \mathcal{D}$ and $\hat{K} \in \mathcal{F}$,

$$
\|\mathbb{E}_+ \theta T \hat{K}\|_{T_0(\ell_+)} \leq O(L^{-2})\|\hat{K}\|_{\mathcal{W}},
$$

(1)

$$
\|\mathbb{E}_+ \theta T \hat{K}\|_{T_\infty(h_+)} \leq O(L^{-2})\|\hat{K}\|_{T_\infty(h)}.
$$

(2)

\textbf{Proof.} Let $F_1(b) = \hat{K}(b)$, $F_2(b) = e^{-V(b)} \text{Loc}(e^{V(b)} \hat{K}(b))$. The algebraic manipulations in (10.5.15)–(10.5.17) give

$$
T \hat{K} = \sum_{b \in \mathcal{B}} e^{-V(B \setminus b)(1 - \text{Loc})(F_1(b) - F_2(b)).
$$

(3)

By the triangle inequality, by Proposition 7.3.1, and by the product property of the norm,

$$
\|\mathbb{E}_+ \theta T \hat{K}\|_{T_\varphi(h_+)} \leq \sum_{b \in \mathcal{B}(B)} \mathbb{E}_+ \left[ \left( \prod_{b' \neq b} \|e^{-V(b')\|_{T_{\varphi_b}(b_+)}} \right)^{2} \| (1 - \text{Loc})F_1(b)\|_{T_{\varphi_b}(h_+)} \right].
$$

(4)

Let $\hat{\varphi}_b = \varphi + \zeta_b$. By Lemma 10.2.3,

$$
\|e^{-V(b')}\|_{T_{\varphi_b}(h_+)} \leq \left(2^{1/4}e^{-4c^2\|\hat{\varphi}_b\|_{h_+}}\right)^{L^{-d}} \leq (2^{1/4})^{L^{-d}}.
$$

(5)

Since $h_+/h = O(L^{-1})$ for both $h = \ell$ and $h = h$, it follows from (10.5.11) that

$$
\| (1 - \text{Loc})F_1(b)\|_{T_{\varphi_b}(h_+)} \leq O(L^{-6})P_{h_+}^\varphi(\hat{\varphi}_b) \sup_{0 \leq \ell \leq 1} \|F_1(b)\|_{T_{\varphi_b}(h)}.
$$

(6)

Since there are $L^4$ terms in the sum over $b$, this gives

$$
\|\mathbb{E}_+ \theta T \hat{K}\|_{T_\varphi(h_+)} \leq O(L^{-2}) \sum_{i=1}^{2} \sup_{b \in \mathcal{B}(B)} \mathbb{E}_+ \left[ P_{h_+}^\varphi(\hat{\varphi}_b) \sup_{0 \leq \ell \leq 1} \|F_1(b)\|_{T_{\varphi_b}(h)} \right].
$$

(7)

For $i = 1$, due to (10.4.5) when $h = \ell$,

$$
\|F_1(b)\|_{T_{\varphi_b}(h)} = \|\hat{K}(b)\|_{T_{\varphi_b}(h)} \leq \begin{cases} \|P_{\ell_+}^{10}(\hat{\varphi}_b)\|_{T_{\varphi_b}(h)} & (h = \ell) \\ \|\hat{K}(b)\|_{T_{\varphi_b}(h)} & (h = h). \end{cases}
$$

(8)

Note that $P_{\ell} \leq P_{\ell_+}$. By Lemma 10.3.1, with $h = \ell$ the expectation $\mathbb{E}_+ P_{\ell_+}^{10}(\zeta_b)$ is bounded, and the corresponding expectation is similarly bounded for $h = h$ because $P_{h_+} \leq P_{\ell_+}$. This proves the two desired inequalities for the contribution due to $F_1$.

For $i = 2$, by Lemma 7.5.1 and Lemma 9.3.1,

$$
\|F_2(b)\|_{T_{\varphi_b}(h)} \leq 2P_{h_+}^{10}(\hat{\varphi}_b) \|\hat{K}(b)\|_{T_0(h)}.
$$

(9)

For $h = \ell$, from the above we see that the contribution due to $F_2$ to the expectation $\|\mathbb{E}_+ \theta T \hat{K}\|_{T_0(\ell_+)}$ is bounded by

$$
O(L^{-2}) \sup_{b \in \mathcal{B}(B)} \|\hat{K}(b)\|_{T_0(\ell)} \mathbb{E}_+ P_{\ell_+}^{10}(\zeta_b).
$$

(10)
Since the expectation is bounded due to Lemma 10.3.1, this gives the desired bound for the $T_0(\ell_+)$ norm. Finally, for the $T_\varphi(h_+)$ norm, we have the upper bound

$$O(L^{-2}) \sup_{b \in B(B)} \| \hat{K}(b) \|_{T_0(h)} \mathbb{E}P^4_h(\hat{\varphi}_b).$$

(11)

Since $P_h \leq P_\ell \leq P_{\ell+}$, the expectation is bounded, and this completes the proof. \hfill \blacksquare

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References
