Concentration of Measure: Poincare Inequalities

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Abstract

In this note, I'll discuss the relevance of poincare inequalities to concentration and some basic facts about poincare inequalities. I'll then list some simple examples of spaces with poincare inequalities.

1 Introduction

In the following we will be working with a manifold so that we have some differentiability structure. these results can be extended through the use of dirichlet forms.

Definition 1. Suppose we have a space $(M, g)$ with measure $\mu$ and an operator $L$ defined on a dense subspace of $L_2(\mu)$ with dirichlet form

$$\mathcal{E}(f, g) := (f, -Lg)$$

We say that such a space admits a poincare inequality with spectral gap $\lambda > 0$ if $\forall f \in D(L)$

$$\text{Var}(f) \leq \frac{1}{\lambda} \mathcal{E}(f, f)$$

Remark 2. In many of these spaces we'll have the integration by parts formula

$$\mathcal{E}(f, g) = (\nabla f, \nabla g) = \int \Gamma_1(f, g) d\mu$$

where $\Gamma_1$ is the carr\'e du champs operator.

The main result of this note will to show that in the presence of a spectral gap we have exponential concentration.

Theorem 3. Let $(M, g)$ have a spectral gap. then

$$\alpha_{M, g, \mu}(r) \leq e^{-\frac{r^2}{\lambda}}$$

One of the useful properties of poincare inequalities is that one can generally prove them through fourier type means. Another point is that this inequality, like the log sobolev inequality, tensorizes.

Theorem 4. (Efron Stein) if $(X_i, \mu_i)$ are product measure spaces, then

$$\text{Var}(f) \leq \sum_i \int \text{Var}_{\mu_i}(f) d\mu.$$ 

In particular, the spectral gap constant on the product satisfies $\lambda = \min_i \lambda_i$.
# 2 Examples

The standard examples are compact riemannian manifolds, and gauss space.

**Example 5.** Compact Riemannian Manifold. A compact boundaryless riemannian manifold admits a poincare inequality with spectral gap $\lambda_1$ where $\lambda_1$ is the first nonzero eigenvalue of the laplace beltrami operator with domain $L = C^\infty(M)$ (in the setting with boundary take $C^0_0$ or $H^1_0$) then we can show through fourier means or variational means that

$$\text{Var}(f) \leq \frac{1}{\lambda_1} \mathcal{E}(f,f).$$

For the fourier method, note that $-L$ has compact self adjoint inverse using the fact that $H^1 \subset\subset L^2$, then we have fourier analysis w.r.t the eigen vectors $e_n$so that $f = \sum(f,e_n)e_n$, then $Lf = \sum \lambda_n(f,e_n)e_n$.

In the compact setting you can show that the constants are the eigen vectors with eigen value 0 so that $\mathbb{E}f^2 - (\mathbb{E}f)^2 = \sum_{n=1}^\infty (f,e_n)^2 \leq \frac{1}{\lambda_1} \sum_{n=1}^\infty \lambda_n(f,e_n)^2 = \frac{1}{\lambda_1}(f,-Lf)$ as desired

**Example 6.** Consider a Markov process $P_t$ with infinitesimal generator $L$ and stationary measure $\mu$. Then $-L$ is positive semidefinite. so it has spectrum $0 = \lambda_0 \leq \lambda_1 \leq \ldots$ and we can set up fourier analysis with respect to the eigenvectors of its inverse (which again will be compact self-adjoint when you mod out by the constants but you'll only get $L_2/\{\mathbb{E}f = 0\}$then you add back in the constants to get the full eigenbasis for $-L$ which spans $L_2$) then using the same argument above you'll get spectral gap

# 3 Proofs

**Definition 7.** The expansion coefficient for a metric measure space is defined by

$$\text{Exp}(\epsilon) = \sup \{\epsilon \geq 1 : \mu B_\epsilon \geq \epsilon \mu B, \mu B_\epsilon \leq 1/2\}$$

**Remark 8.** there is an error in the original paper of gromov, as well as in ledoux’s book. they both put an inf, but that’s clearly an unuseful definition by inspection.

**Proposition 9.** Suppose that for some $\epsilon$ we have $\text{Exp}(\epsilon) \geq \epsilon > 1$. Then the space admits exponential type concentration,

$$\alpha_{x,\mu}(r) \leq \frac{1}{2}ce^{-\frac{r^2}{2}\log c}$$

**Proof.** Let $B$ be such that $\mu B_{\epsilon r} \leq 1/2$, then $\mu B_{\epsilon r} \geq \epsilon \mu B_{(k-\epsilon)r} \geq c^k \mu B$ so $\mu B \leq \frac{1}{2^k}$. Now let $\mu A \geq 1/2$ then if we let $B = (A_{\epsilon r})^c$, then $\mu B_{\epsilon r} \leq 1 - \mu A \leq 1/2$ so that $\mu A_{\epsilon r} \leq \frac{1}{2^k}$. if $k \leq \frac{r}{\epsilon} \leq k + 1$, then $1 - k \geq 1 - \frac{r}{\epsilon} \geq -k\mu A_{\epsilon r} \leq \mu A_{\epsilon r} \leq \frac{1}{2}e^{-k\log c} \leq \frac{1}{2}e^{(1-\epsilon^2)\log(c)} = \frac{1}{2}ce^{-\frac{r^2}{2}\log c}$

**Theorem 10.** In the presence of a spectral gap, we get that the expansion coefficient is bounded below by a constant $> 1$. In particular we get exponential type concentration

$$\alpha(r) \leq e^{-r\sqrt{\lambda_{\max}^2}}$$

**Proof.** Let $A$ and $B$ be two sets in our space with $d(A,B) \geq \epsilon$, and let $a,b$ be their measures respectively. Let $f(x) = \frac{1}{a} - (d(x,A) \wedge \epsilon) \frac{1}{b} (\frac{1}{a} + \frac{1}{b})$ which interpolates between $\frac{1}{a}$ and $-\frac{1}{b}$ then $|\nabla f|^2 \leq (\frac{1}{a} + \frac{1}{b})^2 1_{(A \cup B)^c}$. so that $(Df, Df) \leq \frac{1}{a} (\frac{1}{a} + \frac{1}{b})^2 (1 - a - b)$. Also

$$\int (f - \int f)^2 \geq \int_A (f - \int f)^2 + \int_B
$$

$$= a \left(\frac{1}{a} - \int f\right)^2 + b (-\frac{1}{b} - \int f)^2
$$

$$\geq \frac{1}{a} + \frac{1}{b}$$
so that
\[
1 \leq \frac{1}{\lambda \epsilon^2} (1 - a - b) \left(\frac{1}{a} + \frac{1}{b}\right) \leq \frac{1}{\lambda \epsilon^2} \left(\frac{1 - a - b}{ab}\right)
\]
re arranging, this means
\[
b = \frac{1 - a}{1 + \lambda \epsilon^2 a}
\]
Let \( A = \mathcal{B}_\epsilon^c \),
\[
\mu B = b \leq \frac{\mu \mathcal{B}_\epsilon}{1 + \lambda \epsilon^2 a}
\]
so that \( \text{Exp}(\epsilon) \geq 1 + \lambda \epsilon^2 / 2 > 1 \). this works for all epsilon so pick \( \epsilon^2 = \frac{4}{\lambda} \), we get that \( \text{Exp}(\epsilon) \geq 2 \) This means that
\[
\alpha(r) \leq e^{-\sqrt{\lambda} \frac{2}{\log(2)}}
\]