NEW WEIGHTED INEQUALITIES ON TWO MANIFOLDS

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Dedicated to the memory of Kian Pirfalak (2013–2022)

Abstract. We prove a new class of L^2-weighted elliptic estimates on smooth two-manifolds for positive weights ω that formally satisfy $\omega^2\Delta\ln(\omega) = -\kappa(x)\omega^2$. The proof is short and relies on a lemma about symmetric two dimensional matrices.

1. Introduction

We provide L^2-weighted elliptic estimates for a class of positive weights $\omega \in W^{1,2}(\mathcal{M}^2)$ on smooth Riemannian connected two-manifolds $(\mathcal{M}^2, g)$ that weakly satisfy

$$\omega^2\Delta_g \ln(\omega) = -\kappa(x)\omega^2, \tag{1.1}$$

with the weak formulation in Definition 2.1 and where Δ_g is the Laplace-Beltrami operator on $\mathcal{M}^2$. The original motivation of this article is to investigate the **weighted Hodge decomposition** of one-forms in two dimensions and provide estimates on the distance of the *weighted co-exact part* and the standard co-exact part as follows:

Lemma 1.1. Let $(\mathcal{M}^2, g)$ be a Riemannian 2-manifold and let $\Omega \in \mathcal{M}^2$ be a smooth open domain and ω is a weight as in Definition 2.1 with $\kappa = 0$. Any smooth one-form $A \in C_c^\infty(\bigwedge^1 \Omega)$ has a Hodge decomposition and a *weighted Hodge decomposition* as follows:

$$A = \star d\xi_1 + d\xi_2 \text{ and } \omega A = \star\omega d\phi_1 + \omega^{-1} d\phi_2\,,$$

for 4 compactly supported functions $\xi_1, \xi_2, \phi_1, \phi_2$. Moreover for any $0 \le \varepsilon \le C$ we have the estimates:

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\Omega)} \le C\frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2}\|\omega^{-1} d\phi_2\|^2_{L^2(\Omega)}\,.$$

These estimates are instrumental in the quantitative stability of Yang-Mills-Higgs instantons in two dimensions in [3]. We present the results here in a more general setting, in the belief that these inequalities will be useful in other contexts.

In two dimensions, our results improve on Caffarelli-Kohn-Nirenberg inequalities [1] since we prove estimates for a wider class of weights, possibly vanishing on multiple points, with universal constant (e.g. $\omega = |x||x-1|$). There are also weights who satisfy (1.1) (e.g. $\omega = |x|$) which are not in any Muckenhoupt class.

1.1. **Main results.** Let $\Omega \subset \mathcal{M}^2$ be a smooth open connected domain and let λ_1 be the first Dirichlet eigenvalue of the Laplace-Beltrami operator on Ω. First we provide a generalization of Caffarelli-Kohn-Nirenberg interpolation inequalities in two dimensions:

Theorem 1.2. Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω as in Definition 2.1 and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega |\nabla\omega|^2 |f|^2 d\mathrm{vol}_g \le \int_\Omega \omega^2 |\nabla f|^2 d\mathrm{vol}_g \,, \tag{1.2}$$

provided that $\kappa \le \lambda_1$.

In the next theorem we provide homogeneous elliptic estimates:

Theorem 1.3. Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω as in Definition 2.1 and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega \omega^2 |\nabla f|^2 d\mathrm{vol}_g \le \tau^{-1} \int_\Omega 2\frac{\omega^4}{|\nabla\omega|^2} |\Delta_g f|^2 + 5|\nabla\omega|^2 |f|^2 d\mathrm{vol}_g \,, \tag{1.3}$$

provided that $-\frac{\lambda_1}{8}(2-\tau) \le \kappa \le \lambda_1$ for some $0 \le \tau \le 2$.

Theorem 1.4 is the main ingredient used in the proof of the Lemma 1.1 on the weighted Hodge decomposition. We break the homogeneity to remove the term $|\nabla\omega| f$ from the right hand side, thereby introducing a constant on the right hand side as follows:

Theorem 1.4. Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω in Definition 2.1 with $\kappa = 0$ and $\varepsilon \ge 0$ and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega \omega^{2+2\varepsilon} |\nabla f|^2 d\mathrm{vol}_g \le C \frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \int_\Omega \frac{\omega^4}{|\nabla\omega|^2} |\Delta_g f|^2 d\mathrm{vol}_g \,, \tag{1.4}$$

with the bound $C \le \frac{8\varepsilon^2 + 5(1+\varepsilon)^4}{8(1+\varepsilon)^2}$ which is comparable to $\frac{5}{8}$ as $\varepsilon \to 0$.

Note that the Laplace-Beltrami operator Δ_g on functions $u \in W^{1,2}(\mathcal{M}, g)$ is defined by the duality relation below:

$$\int_\Omega -\Delta_g u v \; d\mathrm{vol}_g = \int_\Omega \langle \nabla u, \nabla v \rangle \; d\mathrm{vol}_g \,, \text{ for all } v \in W_0^{1,2}(\Omega).$$

Examples of weights. For any bounded open subset $\Omega \subset \mathbb{R}^2$ and a weight ω as follows:

$$\omega(x) = \Pi_{i=1}^N |x - x_i|^{\alpha_i} \text{ for } x_1, \dots, x_N \in \Omega \subset \mathbb{R}^2 \text{ and } \alpha_1, \dots, \alpha_N > 0 \,. \tag{1.5}$$

For any bounded open domain $\Omega \subset \mathcal{M}^2$ of a smooth two manifold, let $\mathcal{G}_p$ be the green's function for Ω centered on p and ω as follows:

$$\omega(x) = \Pi_{i=1}^N e^{-\alpha_i \mathcal{G}_{p_i}(x)} \text{ for } p_1, \dots, p_N \in \Omega \subset \mathcal{M}^2 \text{ and } \alpha_1, \dots, \alpha_N > 0 \,. \tag{1.6}$$

The weights (1.5) and (1.6) are generalizations of the Caffarelli-Kohn-Nirenberg interpolation results [1], in two dimensions. Moreover Theorem 1.2 and 1.4 provide weighted elliptic estimates for the weight $\omega = |x|^\alpha$:

$$\int_{\mathbb{R}^2} |x|^{2(\alpha-1)}|f|^2 \leq \alpha^{-2} \int_{\mathbb{R}^2} |x|^{2\alpha}|\nabla f|^2 ,$$

$$\int_{\mathbb{R}^2} |x|^{2\alpha}|\nabla f|^2 \leq \alpha^{-2} \int_{\mathbb{R}^2} |x|^{2(\alpha+2)}|\Delta f|^2 + \alpha^2 \int_{\mathbb{R}^2} \frac{5}{2}|x|^{2(\alpha-1)}|f|^2 ,$$

$$\int_{B_1} |x|^{2(\alpha+\varepsilon)}|\nabla f|^2 \leq C(\varepsilon\alpha)^{-2} \int_{B_1} |x|^{2(\alpha+1)}|\Delta f|^2 ,$$

provided that $\alpha > 0$.

The methods throughout the paper are inspired by [1] and [2] and are quiet elementary and only use Stokes theorem. A crucial part of our proof, equation (2.5), uses Lemma 2.2 which is an identity about symmetric matrices in two dimensions which does not hold in other dimensions.

Remark. In the case of unbounded domains (e.g. $\mathcal{M}^2 = \mathbb{R}^2$) we set $\lambda_1 = 0$ in Theorem 1.2 and 1.3.

Remark. Theorem 1.2 and 1.3 also work for the case of closed 2-manifolds $\Omega = \mathcal{M}^2$ with the assumption that $\int_\Omega \omega f \, d\mathrm{vol}_g = 0$. However Theorem 1.4 is a trivial statement for closed manifolds since the assumption $\kappa = 0$ tells us that $\Delta_g \omega^2 = 4|d\omega|^2 \geq 0$ and this means that the only admissible weights are constants.

2. The Proof

Definition 2.1. The weak formulation of (1.1) for a weight $\omega \in W^{1,2}(\mathcal{M}^2)$ is as follows: For any smooth test function $\phi \in C_c^\infty(\mathcal{M}^2)$ we have that:

$$\int_\Omega (4|\nabla\omega|^2 - 2\kappa\omega^2)\phi - \omega^2 \Delta_g \phi \, d\mathrm{vol}_g = 0 .$$

To prove Theorem 1.2 to 1.4 we use Stokes theorem to relate the integral of a carefully chosen positive term, to the difference of the right and the left hand side of (1.2) to (1.4).

Proof of Theorem 1.2. We begin with the identity below:

$$\begin{aligned} 0 \leq \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g &= \int_\Omega |\omega\nabla f + \nabla\omega f|^2 \, d\mathrm{vol}_g \\ &= \int_\Omega \omega^2|\nabla f|^2 + |\nabla\omega|^2|f|^2 + 2\langle\omega\nabla\omega, \nabla f f\rangle \, d\mathrm{vol}_g . \end{aligned}$$

After completing the derivative for the cross term and using Definition 2.1 we see that:

$$\int_\Omega 2\langle\omega\nabla\omega, \nabla f f\rangle \, d\mathrm{vol}_g = \int_\Omega -\frac{\omega^2}{2}\Delta_g(f^2) \, d\mathrm{vol}_g = \int_\Omega (\kappa\omega^2 - 2|\nabla\omega|^2)|f|^2 \, d\mathrm{vol}_g .$$

Then we use $\kappa \le \lambda_1$ to estimate:

$$\int_\Omega \kappa\omega^2|f|^2 \, d\mathrm{vol}_g \le \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g \,.$$

Finally we conclude that:

$$0 \le \int_\Omega \omega^2|\nabla f|^2 - |\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \,.$$

□

Proof of Theorem 1.3. Similarly we begin by integrating a positive term:

$$0 \le \int_\Omega \Big|\frac{\omega^2}{|\nabla\omega|}\Delta_g f + |\nabla\omega|f\Big|^2 \, d\mathrm{vol}_g = \int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 + 2\omega^2 f\Delta_g f + |\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \,.$$

By Stokes theorem for the cross term and Definition 2.1 we get that:

$$\int_\Omega 2\omega^2 f\Delta_g f \, d\mathrm{vol}_g = \int_\Omega -2\omega^2|\nabla f|^2 + (4|\nabla\omega|^2 - 2\kappa\omega^2)|f|^2 \, d\mathrm{vol}_g \,.$$

Since the assumption for an unbounded domain is $\kappa = 0$ the proof follows immediately. Otherwise by the assumption $-\kappa \le \lambda_1(\frac{1}{4} - \frac{\tau}{8})$ we see that:

$$\int_\Omega -2\kappa|\omega f|^2 \, d\mathrm{vol}_g \le \lambda_1(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\omega f|^2 \, d\mathrm{vol}_g \,.$$

By the characterization of the first eigenvalue of the Laplace-Beltrami operator Δ_g we see that:

$$\lambda_1(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega \omega^2|f|^2 \, d\mathrm{vol}_g \le (\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g \,.$$

Since $\kappa \le \lambda_1$, Theorem 1.2 applies and we get that:

$$(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g \le (2-\tau) \int_\Omega \omega^2|\nabla f|^2 \, d\mathrm{vol}_g \,.$$

Finally putting the estimates together, we conclude that:

$$0 \le \int_\Omega 2\frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 + 5|\nabla\omega|^2|f|^2 - \tau\omega^2|\nabla f|^2 d\mathrm{vol}_g \,.$$

□

In the proof of Theorem 1.4 we deal with the weighted hessian matrix $\omega^2\nabla^2\ln(\omega)$ and by the condition (1.1) we know that it is a two dimensional symmetric trace-free matrix. The following lemma uses this structure and it is essential in the proof of Theorem 1.4:

Lemma 2.2. Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, namely $A^T = A$. Then we have that for any two real vectors $b, c \in \mathbb{R}^2$:

$$2\langle A : b\otimes c\rangle\langle b,c\rangle - \langle A : b\otimes b\rangle|c|^2 - (A : c\otimes c)|b|^2 = \mathrm{trace}(A)\langle b, c^\perp\rangle^2 \,, \tag{2.1}$$

where $\langle:\rangle$ is the matrix element-wise inner product and $c^\perp$ is the perpendicular vector to c.

Proof. We first calculate the expression above in dimension n. Since A is symmetric, it has n distinct perpendicular eigen-vectors e_i with real eigen-values μ_i. Then setting $b_i = \langle b, e_i\rangle$ and $c_i = \langle c, e_i\rangle$ we compute:

$$\begin{aligned} &2\langle A : b\otimes c\rangle\langle b,c\rangle - \langle A : b\otimes b\rangle|c|^2 - (A : c\otimes c)|b|^2 \\ =&\sum_{1\leq i,j\leq n} \mu_i(a_ic_j - c_ia_j)^2\,. \end{aligned}$$

In the case $n = 2$:

$$2\langle A : b\otimes c\rangle\langle b,c\rangle - \langle A : b\otimes b\rangle|c|^2 - (A : c\otimes c)|b|^2 = \mathrm{trace}(A)(b_1c_2 - c_1b_2)^2\,.$$

□

Proof of Theorem 1.4. First we integrate a carefully chosen positive term of the form below:

$$\begin{aligned} 0 \leq &\int_\Omega \Big|\frac{\omega^2}{|\nabla\omega|}\Delta_g f + 2\omega\langle\frac{\nabla\omega}{|\nabla\omega|},\nabla f\rangle + 2|\nabla\omega|f\Big|^2\, d\mathrm{vol}_g \\ = &\int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 + 4\omega^2\langle\frac{d\omega}{|\nabla\omega|},\nabla f\rangle^2 + 4|\nabla\omega|^2|f|^2 \qquad (2.2)\\ &+ 4\frac{\omega^3}{|\nabla\omega|^2}\langle\nabla\omega,\nabla f\rangle\Delta_g f + 4\omega^2\Delta_g f f + 8\langle\omega\nabla\omega, f\nabla f\rangle\, d\mathrm{vol}_g\,. \qquad (2.3) \end{aligned}$$

Then for the first cross term in (2.3) we calculate by Stokes theorem and (1.1) (with the weak formulation in Definition 2.1) and the assumption $\kappa \geq 0$ that:

$$\begin{aligned} &\int_\Omega 4\frac{\omega^3}{|\nabla\omega|^2}\langle\nabla\omega,\nabla f\rangle\Delta_g f\, d\mathrm{vol}_g \\ =&\int_\Omega 2\mathrm{div}_g(\frac{\omega^3}{|\nabla\omega|^2}d\omega)|\nabla f|^2 - 4\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega) : \nabla f\otimes\nabla f\, d\mathrm{vol}_g \\ \leq&\int_\Omega 8\omega^2|\nabla f|^2 - 4\langle\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega) : \nabla f\otimes\nabla f\rangle\, d\mathrm{vol}_g\,. \qquad (2.4) \end{aligned}$$

Then we use the identity

$$\omega\nabla^2\omega = \omega^2\nabla^2\ln(\omega) + d\omega\otimes d\omega\,,$$

for the second term in (2.4). We get that:

$$\begin{aligned} &-4\langle\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega) : \nabla f\otimes\nabla f\rangle = \\ =&-8\omega^2\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 + 4\frac{\omega^4}{|\nabla\omega|^4}\big[2\langle\nabla^2\ln(\omega) : \nabla\omega\otimes\nabla f\rangle\langle\nabla\omega,\nabla f\rangle \\ &-\langle\nabla^2\ln(\omega) : \nabla f\otimes\nabla f\rangle|\nabla\omega|^2 - \langle\nabla^2\ln(\omega) : \nabla\omega\otimes\nabla\omega\rangle|\nabla f|^2\big]\,. \qquad (2.5) \end{aligned}$$

We apply Lemma 2.2 with:

$$A = \omega^2 \nabla^2 \ln(\omega)\,, \quad b = \frac{\nabla\omega}{|\nabla\omega|} \quad \text{and } c = \nabla f\,,$$

and $\mathrm{trace}(A) = \omega^2 \Delta_g \ln(\omega) = 0$ to see that:

$$-4\langle \nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega) : \nabla f \otimes \nabla f\rangle = -8\omega^2 \langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle^2\,.$$

For the second and third cross term in (2.3) we see that:

$$\int_\Omega 4\omega^2 \Delta_g f f + 8\langle \omega\nabla\omega, f\nabla f\rangle \, d\mathrm{vol}_g = \int_\Omega -4\omega^2 |\nabla f|^2 \, d\mathrm{vol}_g\,.$$

Then putting the estimates together we see that:

$$4\int_\Omega \omega^2 \langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - |\nabla\omega|^2 |f|^2 \, d\mathrm{vol}_g \le \int_\Omega \frac{\omega^4}{|\nabla\omega|^2} |\Delta_g f|^2 \, d\mathrm{vol}_g\,. \tag{2.6}$$

Using (1.1) with $\kappa = 0$ we get that for (2.6):

$$\begin{aligned}
&\int_\Omega \omega^2 \langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - |\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega |\omega\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle + |\nabla\omega| f|^2 \, d\mathrm{vol}_g \\
\ge&(\sup_\Omega \omega)^{-2\varepsilon} \int_\Omega \omega^{2\varepsilon} |\omega \langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle + |\nabla\omega| f|^2 \, d\mathrm{vol}_g\,. \qquad (2.7)
\end{aligned}$$

Notice that $\omega^{1+\varepsilon}$ also satisfies (1.1) weakly in the case of $\kappa = 0$, so we compute (2.7) as follows:

$$\begin{aligned}
&\int_\Omega \omega^{2\varepsilon} |\omega\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle + |\nabla\omega| f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega \omega^{2+2\varepsilon}\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle^2 + \omega^{2\varepsilon}|\nabla\omega|^2|f|^2 + 2\omega^{1+2\varepsilon}\langle \nabla\omega, \nabla f\rangle f \, d\mathrm{vol}_g \\
=&\int_\Omega \omega^{2+2\varepsilon}\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle^2 + \omega^{2\varepsilon}|\nabla\omega|^2|f|^2 - \Delta_g(\frac{\omega^{2+2\varepsilon}}{2+2\varepsilon})|f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega \langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - (1+2\varepsilon)\omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g\,. \qquad (2.8)
\end{aligned}$$

Notice that for $\omega^{1+\varepsilon}$ we have:

$$\begin{aligned}
0 \le& \int_\Omega \omega^{2\varepsilon}|\omega\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle + (1+\varepsilon)|\nabla\omega| f|^2 \, d\mathrm{vol}_g \\
=& \int_\Omega \omega^{2+2\varepsilon}\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle + (1+\varepsilon)^2\omega^{2\varepsilon}|\nabla\omega|^2|f|^2 + 2(1+\varepsilon)\omega^{1+2\varepsilon}\langle \nabla\omega, \nabla f\rangle f \, d\mathrm{vol}_g \\
=& \int_\Omega \omega^{2+2\varepsilon}\langle \nabla f, \frac{\nabla\omega}{|\nabla\omega|}\rangle - (1+\varepsilon)^2\omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g\,.
\end{aligned}$$

We expand the square $(1+\varepsilon)^2)$ to get a lower bound for (2.8):

$$\int_\Omega \langle \nabla f, \frac{\nabla \omega}{|\nabla \omega|}\rangle^2 - (1+2\varepsilon)\omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \geq \varepsilon^2 \int_\Omega \omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \,.$$

and we get a preliminary inequality as follows:

$$\int_\Omega \omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \leq \frac{(\sup_\Omega \omega)^{2\varepsilon}}{4\varepsilon^2} \int_\Omega \frac{|\omega|^4}{|\nabla\omega|^2}|\Delta_g f|^2 \, d\mathrm{vol}_g \,. \tag{2.9}$$

Then we use Theorem 1.3 for $\omega^{1+\varepsilon}$ and $\kappa = 0$ and $\tau = 2$ to see that:

$$\int_\Omega 2\omega^{2+2\varepsilon}|\nabla f|^2 \, d\mathrm{vol}_g \leq \int_\Omega 2\frac{\omega^{4+2\varepsilon}}{(1+\varepsilon)^2|\nabla\omega|^2}|\Delta_g f|^2 + 5(1+\varepsilon)^2\omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \,.$$

Finally we use (2.9) to conclude that:

$$\int_\Omega \omega^{2+2\varepsilon}|\nabla f|^2 \, d\mathrm{vol}_g \leq (\frac{8\varepsilon^2 + 5(1+\varepsilon)^4}{8(1+\varepsilon)^2})\frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 \, d\mathrm{vol}_g \,.$$

□

Remark 2.3. In the case of $\mathcal{M}^2 = B_1^2(0) \subset \mathbb{R}^2$ and $\omega = |x|$ after the log-polar transformation $B_1^2 \to \mathbb{R}^+ \times S^1 = \mathcal{C}$ by the map $t = -\ln(|x|)$ and $\theta = \arctan(\frac{y}{x})$ or equivalently a conformal change of metric with the factor $\frac{1}{|x|^2}$ and defining $f = |x|^{-1}u$ for $f \in C_1^\infty(B_1^2(0))$ we can see that:

$$\int_{B_1^2(0)} |\nabla\omega|^2|f|^2 = \int_{\mathcal{C}} |u|^2 d\mathrm{vol}_{\mathcal{C}} \,, \tag{2.10}$$

$$\int_{B_1^2(0)} \omega^2|\nabla f|^2 = \int_{\mathcal{C}} |\nabla u|^2 + |u|^2 d\mathrm{vol}_{\mathcal{C}} \,,$$

$$\int_{B_1^2(0)} \frac{\omega^4}{|\nabla\omega|^2}|\nabla f|^2 = \int_{\mathcal{C}} |\Delta u + 2\partial_t u + u|^2 d\mathrm{vol}_{\mathcal{C}} \,. \tag{2.11}$$

After squaring and integrating by parts we see that (2.11) becomes:

$$\int_{B_1^2(0)} \frac{\omega^4}{|\nabla\omega|^2}|\nabla f|^2 = \int_{\mathcal{C}} |\partial_{tt}u|^2 + |\partial_{t\theta}u|^2 + 2|\partial_t u|^2 + |\partial_{\theta\theta}u + u|^2 \,.$$

We can see that if $u(t,\theta) = \sin(\theta)$ then (2.11) vanishes however (2.10) does not vanish so the term $|\nabla\omega|f$ on the right hand side of (1.3) is necessary. However the extra ε in the power

$$\int_{B_1^2(0)} \omega^{2+2\varepsilon}|\nabla f|^2 = \int_{\mathcal{C}} (|\nabla u|^2 + |u|^2)e^{-2\varepsilon t} d\mathrm{vol}_{\mathcal{C}} \,,$$

compactifies the domain $\mathbb{R}^+ \times S^1$ with a total measure of ε^{-2}. This provides some insight on Theorem 1.4 and the constants in (1.4).

We conclude the paper with the proof of the weighted Hodge decomposition estimates:

Proof of Lemma 1.1. We consider the two variational problems below:

$$\inf_{\xi \in C_c^\infty(\Omega)} \int_\Omega |A - \star d\xi|^2 \, d\mathrm{vol}_g \text{ and } \inf_{\phi \in C_c^\infty(\Omega)} \int_\Omega \omega^2 |A - \star d\phi|^2 \, d\mathrm{vol}_g \,. \tag{2.12}$$

Let $W_0^{1,2}(\omega^2, \Omega)$ be the completion of $C_c^\infty(\Omega)$ under the ω^2-weighted norm

$$\|u\|_{W_0^{1,2}(\omega^2,\Omega)} = \int_\Omega \omega^2 (|u|^2 + |du|^2) \,.$$

By Theorem 1.2 we see that

$$C^{-1} \|u\|_{W^{1,2}(\omega^2,\Omega)} \leq \|\omega u\|_{W^{1,2}(\Omega)} \leq C \|u\|_{W^{1,2}(\omega^2,\Omega)} \,,$$

and by the equivalence of the norms, the family of functions $\{u : \omega u \in W_0^{1,2}(\Omega)\}$ is equivalent to $W_0^{1,2}(\omega^2, \Omega)$ the existence of minimizers of (2.12) follows from convexity and the direct method in the calculus of variations. The Euler Lagrange equations for minimizers tell us that

$$\star d(A - \star d\xi_1) = 0 \Rightarrow \text{ there exists } \xi_2 \text{ such that } A - \star d\xi_1 = d\xi_2 \text{ and}$$
$$\star d(\omega^2 (A - \star d\phi_1)) = 0 \Rightarrow \text{ there exists } \phi_2 \text{ such that } \omega^2 (A - \star d\phi_1) = d\phi_2 \,.$$

in the sense of distributions. Then with a direct application of Theorem 1.4

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)} \leq C \frac{(\sup_{\mathcal{M}^2} \omega)^{2\varepsilon}}{\varepsilon^2} \|\frac{\omega^2}{|d\omega|} \Delta_g (\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)}$$

and

$$\|\frac{\omega^2}{|d\omega|} \Delta_g (\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)} = \|\frac{\omega^2}{|d\omega|} d(\omega^{-2} d\phi_2 - d\xi_1)\|^2_{L^2(\mathcal{M}^2)} = 4 \|\omega^{-1} d\phi_2\|^2_{L^2(\mathcal{M}^2)} \,.$$

we conclude the proof. □

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