Quantitative stability in the 2D Abelian Higgs model

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2/14/2024

The plan

- What is a Hermitian line bundles?
- The Abelian Higgs model in dimension 2 and the ”*vortex equations*”
- Previous results.
- Motivating the question of quantitative stability
- Core ideas of proof: Weighted hodge decomposition and a selection principle

What is a line bundle?

- Informally, a line bundle over a smooth manifold is a 'twisted' Cartesian product of the manifold with the complex numbers.
- Think about the Mobius strip. Locally it is a Cartesian product, however globally it has a "twist", which gives the Mobius bands it nontrivial topology! (thus interesting :D)

What is a line bundle, Formally?

Definition [7, p.23-27]

A line bundle over a smooth manifold M is a triple (L, π, M) where:

1. L is a smooth manifold and π is a smooth map of L to M.
2. For any $m \in M$ the set $L_m = \pi^{-1}(m)$ has the structure of a one dimensional complex vector space. (L_m is called the fiber over m)
3. There is an open cover U_j of M and a collection of $s_j : U_j \to L$ such that $\forall j$, $\pi \circ s_j = 1_{U_j}$ and the map $\psi_j : U_j \times \mathbb{C} \to \pi^{-1}(U_j) : (m, z) \to z.s_j(m)$ is a diffeomorphism.

On line bundles, functions are replaced by the notion of "sections" and are usually denoted by the symbol $\Gamma(L)$.

What is a Hermitian line bundle?

A Hermitian line bundle is a line bundle $L \to M$ with two additional structures:

- A **Hermitian metric**: On each fiber over $m \in M$ there is a Hilbert space metric $\langle ., . \rangle_m$ which varies smoothly with m.
- A **connection**: There is a map ∇ which acts like a gradient in a twisted sense: It assigns to any vector field ξ, an endomorphism $\nabla_\xi : \Gamma(L) \to \Gamma(L)$ and it is linear in ξ and obeys the Leibniz rule (like a gradient).
- These two structures are **compatible**:

$$\xi\langle s, t\rangle = \langle \nabla_\xi s, t\rangle + \langle s, \nabla_\xi t\rangle\,, \qquad (1)$$

for any smooth vector field $\xi \in \mathcal{U}(M)$ and smooth sections $s, t \in \Gamma(L)$.

The connection one-form

Given any connection ∇ on a Hermitian line bundle, we can construct another one by picking a "real" one-form $A \in \Omega^1(M)$ and defining:

$$\nabla_1 = \nabla - iA\,. \tag{2}$$

Intuitively, this means that we are rotating the base point of a fiber in the direction of any vector field ξ with the speed $A(\xi)$.
This is not a coincidence, since all connections are of this form.
Basically, after choosing a reference connection ∇_0 all other connection can be written as:

$$\nabla = \nabla_0 - iA\,. \tag{3}$$

Curvature

In general, for two vector fields ξ, η with $[\xi, \eta] = 0$ the operators ∇_ξ and ∇_η might not commute, which hints at the presence of curvature. For any connection ∇ we define the curvature to be the two form:

$$F_\nabla(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}\,. \tag{4}$$

In the formulation of the connection one-form $\nabla = \nabla_0 - iA$, if ∇_0 is flat, we can see that:

$$F_\nabla := -idA\,. \tag{5}$$

A crucial result is that this two-form is always integral, meaning that integrating this two-form always gives us an integer.

The Abelian Higgs model

Let $L \to M$ be a hermitian line bundle over M. Then for any section and connection (u, ∇) we define the Yang-Mills-Higgs energy as:

$$E(u,\nabla) = \int_M |\nabla u|^2 + |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4}\,. \tag{6}$$

In an equivalent formulation, for any complex valued function and a one-form (u, A) we define the energy to be:

$$E(u,A) = \int_M |du - iu\otimes A|^2 + |dA|^2 + \frac{(1-|u|^2)^2}{4}\,. \tag{7}$$

Intuitively, minimizers of this energy want a section with non trivial topology close to 1 that does not vary too much w.r.t. a connection that does not curve too much.

Gauge invariance

For any function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ the transformation:

$$(u, \nabla) \to (ue^{i\gamma}, \nabla + id\gamma)\,.$$

leaves the energy unchanged! This is called "Gauge invariance". This means that the Abelian Higgs functional has $U(1)$ as its group of symmetries.
We see later on that the phase completely decouples from the modulus and the energy.

- But these energies are special in two dimensions!

Our setting in 2D

For the rest of the talk we take $L := \mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}^2$ to be the trivial line bundle over the Euclidean space and we work with the Yang Mills Higgs energy in two dimensions:

$$E(u, \nabla) = \int_{\mathbb{R}^2} |\nabla u|^2 + |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4}. \tag{8}$$

Equivalently, for any function $u : \mathbb{R}^2 \to \mathbb{C}$ and real one-form (vector field) $A : \mathbb{R}^2 \to \mathbb{R}^2$:

$$E(u, A) = \int_{\mathbb{R}^2} |du - iuA|^2 + |\mathrm{curl}(A)|^2 + \frac{(1-|u|^2)^2}{4} \tag{9}$$

Two dimensions is special!

With a slight abuse of notation assume that $u = re^{i\theta}$ where $r = |u|$ and $\theta : \mathbb{R}^2 \to S^1$. Then we can rewrite the energy as:

$$E(re^{i\theta}, A) = \int_{\mathbb{R}^2} |dr|^2 + r^2|A - d\theta|^2 + |dA|^2 + \frac{(1 - r^2)^2}{4} . \quad (10)$$

Notice that $|dr|^2 = |\star dr|^2$ where $\star$ is the Hodge star operator and in two dimensions $\star$ of a vector-field is a one-forms.
Then by a completion of squares and integrating by parts:

$$E(re^{i\theta}, A) = \pm 2\pi N + \int_{\mathbb{R}^2} |\star dr \pm r(A - d\theta)|^2 + |\star dA \mp \frac{1 - r^2}{2}|^2.$$

Magic! Here N is the rotation number of θ at infinity or equivalently:

$$\int_{\mathbb{R}^2} \star dA = 2\pi N .$$

The vortex equations

After fixing an orientation we see that

$$E(u, \nabla) \geq 2\pi N$$

and the equality happens if and only if:

$$\star dr + r(A - d\theta) = 0 \text{ and } \star dA = \frac{1 - r^2}{2} \tag{11}$$

These are called the first order vortex equations! Dividing the first equation by r and substituting in the second equation, we see that:

$$-\Delta \ln(r) + \frac{1}{2}(r^2 - 1) = -2\pi \sum_{r(x_k)=0} \delta_{x_k} \, ,$$

where δ_x is a point mass centered on x.

Existence and uniqueness

In [8] C.H.Taubes proved that, after fixing the location of zeros of u the solution of these equations exists and are unique:

Theorem ([8] and [9] C.H.Taubes 1980)

Given any set of points $\{a_1, \ldots, a_N\} \subset \mathbb{R}^2$ counted with multiplicity, there exists a unique smooth solution to the vortex equation which vanishes on $\{a_k\}_{k=1}^{n}$ with degree equal to the multiplicity. Moreover all stationary points are actually minimizers.

In other words, solutions of this model behaves like non interacting vortices in two dimensions with a fixed mass.

Methods of Taubes

Given the vortex set (zero set) $\{a_1, \ldots, a_N\}$, look at

$$v = \ln(r) - \frac{1}{2}\sum_{k=1}^{N} \ln(1 + \frac{\lambda}{(x-a_k)^2}) \tag{12}$$

$\rightsquigarrow -\ln(|x-a_k|)$

for some positive $\lambda > 0$. Taubes showed that v is a stationary point for a convex functional:

$W^{1,2}(\mathbb{R}^2) \subset BMO$

$$\int_{\mathbb{R}^2} \frac{|dv|^2}{2} - v(1-g_0) + e^{u_0}(e^v - 1)\,. \tag{13}$$

where $u_0 = \frac{1}{2}\sum_{k=1}^{N} \ln(1 + \frac{\lambda}{(x-a_k)^2})$ and $g_0 = -\Delta u_0 - 2\pi \sum_k \delta_{x_k}$.

- Existence and uniqueness follows by standard methods.

Minimal sub-manifolds of co-dimension two

In higher dimensions and on any closed manifold M, this model can also be used to produce stationary sub-manifolds of co-dimension two. First we define what is a stationary submanifold:

Definition

A stationary sub-manifold $\Sigma^k \subset M^n$ is a stationary point for the k-dimensional area functional with respect to domain variations, precisely for any family of diffeomorphisms $\phi_t : M^n \times [0, \delta) \to M^n$ with $\phi_0 = Id$:

$$\frac{d}{dt}\mathrm{vol}_k(\Sigma_t) = 0\,, \tag{14}$$

where Σ_t is the pushforward of Σ w.r.t. ϕ_t.

Minimal sub-manifolds of co-dimension two

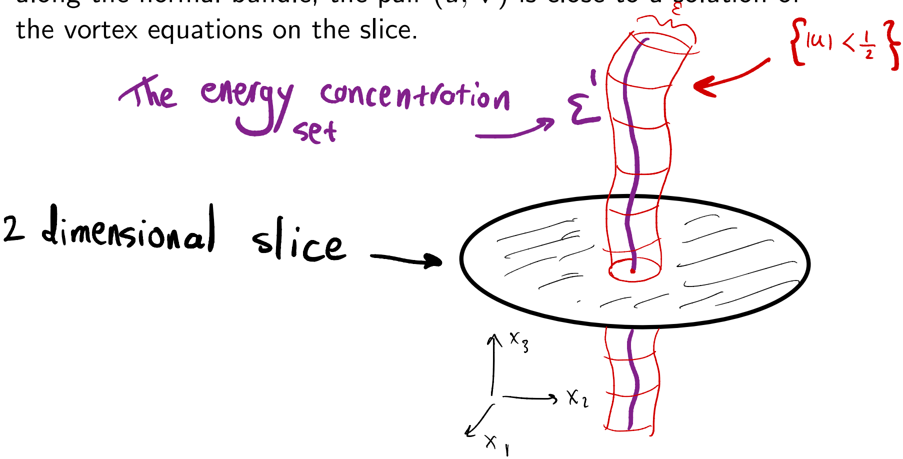
For any $\varepsilon > 0$ consider the re-scaled energy for any pair (u, ∇):

$$E_\varepsilon(u, \nabla) = \int_M |\nabla u|^2 + \varepsilon^2 |F_\nabla|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2}\,. \qquad (15)$$

In 2019 [6] A.Pigati and D.Stern showed that given a sequence of stationary points $(u_\varepsilon, \nabla_\varepsilon)$ with vanishing $\varepsilon \to 0$ and bounded energy $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda$, the energy measures $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)dx$ concentrate along a Integeral stationary sub-manifold Σ of dimension $n - 2$ (in a weak sense).
In particular the analysis of these energies closely resembles Allan-Cahn. (Modica bounds & equi-distribution of energy & Integrality)

Slices

Vaguely, if one looks at perpendicular two dimensional slices of Σ along the normal bundle, the pair (u, ∇) is close to a solution of the vortex equations on the slice.



Motivating the study of stability

- For any uniqueness result one might expect some stability.
- Having a quantitative version of the inequality $E(u, \nabla) \geq 2\pi N$.
- (The original motivation) The regularity theory of stationary sub-manifolds (in particular Allard's [1]) and stability of slices combined with the work of Pigati and Stern [6] gives us a De-Giorgi type result (as in the Allan Cahn model) for global solutions (This is a paper in preperation with Guido De philippis and Alessandro Pigati)
- It is some nice mathematics by itself :D

Quantitative stability

What I was able to prove:

Theorem ([5] H 2023)

For any $N > 0$ there exists $C_N, \eta_0 > 0$ with the following property: Let $L := \mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the trivial line bundle. Then any N-vortex section and connection (u, ∇) with small enough discrepancy $E(u, \nabla) - 2\pi N = \eta^2 \leq \eta_0^2$ satisfies:

$$\min_{(u_0, \nabla_0) \in \mathcal{F}} \|u - u_0\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} \leq C_N \eta^2 \,, \tag{16}$$

where $\mathcal{F}$ is the moduli space of all solutions of the vortex equations.

The discrepancy

First notice that by completion of squares and integrating by parts we saw (up to a change of sign):

$$E(u, \nabla) - 2\pi N = \int_{\mathbb{R}^2} r^2 |\star d \ln(r) + A - d\theta|^2 + |\star dA - \frac{1-r^2}{2}|^2 .$$

We aim to prove that this discrepancy bounds the distance with some solution.

Question

What happens if we linearize the discrepancy?

Answer: Good things!

Linearizing

For the sake of simplicity assume that $N = 1$ and (u, A) is smooth and $|u|$ vanishes on the origin comparable to $|x| \sim |u_0|$ and it is a compact perturbation of the one-vortex solution (u_0, A_0) of Taubes:

$$|u| = |u_0|e^h \text{ for } h \in C_c^\infty(B_1)\,. \tag{17}$$

Then we gauge fix such that they have equal phase:

$$\frac{u}{|u|} = \frac{u_0}{|u_0|}$$

(this is possible since they are comparable) Then:

$$E(u, \nabla) - 2\pi \sim \int_{B_1(0)} |x|^2 |\star dh + B|^2 + |\star dB + |u_0|^2 h|^2\,. \tag{18}$$

Notice that $(|u| - |u_0|) \sim |x|h$. So L^2 stability is a question of weighted L^2 bounds on h.

A proof without weight?

Since the problem is linear now, we can approach by a compactness argument, First we give a try without the presence of the weight $|x|^2$. Imagine a sequence of compactly supported functions:

$$\int_{B_1} |h_k|^2 = 1 \text{ and } \int_{B_1} |\star dh_k + B_k|^2 + |\star dB_k + r_0^2 h_k|^2 \to 0\,.$$

$\inf_g \int |B - \star dg|^2$

By Hodge decomposition:

$$B_k = \star dg_k + df_k\,. \tag{19}$$

We see that:

$$\int_{B_1} |d(h_k + g_k)|^2 + |\Delta g_k + r_0^2 h_k|^2 \to 0\,. \tag{20}$$

By standard elliptic estimates we see that $\|h_k\|_{W^{1,2}(B_1)} \leq C$, from which compactness follows immediately.

Weighted Hodge decomposition

We adapt this proof to the weighted case, First we define a weighted Hodge decomposition inpired by the **variational formulation of Hodge decomposition**: For any compactly supported vector field $A : B_1 \to \mathbb{R}^2$ we consider the minimization problem:

$$\inf_{g \in C_c^\infty(B_1)} \int |x|^2 |A - \star dg|^2 . \tag{21}$$

To see that:

$$|x|A = \star |x| dg + |x|^{-1} df \tag{22}$$

and the orthogonality:

$$\int_{B_1} |x|^2 |A|^2 = \int_{B_1} |x|^2 |dg|^2 + |x|^{-2} |df|^2 . \tag{23}$$

This turns out to be just the right decomposition for our problem!

Weighted Hodge decomposition

Now again by a compactness argument imagine a sequence:

$$\int_{B_1} |x|^2 |h_k|^2 = 1 \text{ and } \int_{B_1} |x|^2 |\star dh_k + B_k|^2 + |\star dB_k + r_0^2 h_k|^2 \to 0\,.$$

Then we perform the weighted Hodge decomposition and the standard Hodge decomposition:

$$|x| B_k = \star |x| dg_k + |x|^{-1} df_k \text{ and } B_k = \star dp_k + dq_k$$

and rewrite the discrepancy:

$$\int_{B_1} |x|^2 |d(h_k + g_k)|^2 + |x|^{-2} |df_k|^2 + |\Delta p_k + r_0^2 h_k|^2 \to 0$$

Compactness

$$\int_{B_1} |x|^2 |d(h_k + g_k)|^2 + |x|^{-2}|df_k|^2 + |\Delta p_k + r_0^2 h_k|^2 \to 0$$

To gain compactness from the quantity above we use a result of H [4] to see that we can quantify the distance of the standard and weighted Hodge decomposition:

$$|x|B_k = \star|x|dg_k + |x|^{-1}df_k \text{ and } B_k = dp_k + dq_k$$

$$\int_{B_1} |x|^{2+2\varepsilon}|d(g_k - p_k)|^2 \leq C\varepsilon^{-2}\int_{B_1} |x|^{-2}|df_k|^2$$

This gives us the desired compactness for the discrepancy and the conclusion!

Weighted inequalities

The tool we used is:

Theorem ([4] H. 2023)

For any function $f \in C_c^\infty(B_1)$:

$$\int_{B_1} |x|^{2+2\varepsilon}|df|^2 \leq C\varepsilon^{-2}\int_{B_1} |x|^4|\Delta f|^2$$

Moreover in [4] the inequality above is generalized with **uniform constants** for all weight ω on a 2-manifold that satisfy:

$$-\omega^2\Delta\ln(\omega) = 0\,. \tag{24}$$

This covers all weights $\Pi_{k=1}^N|x-x_k|^{\alpha_k}$ for $x_k \in \mathbb{R}^2$ and $\alpha_k > 0$. These inequalities can be thought of as generalizations to the interpolation results of Cafarelli-Kohn-Nirenberg [2].

Stability of comparable pairs

Using Theorem 1.4 of [4] we can show the same stability for linearized equation in the case of many vortices:

Theorem ([5] H 2023)

For any Λ, N there exists $C_{N,\Lambda}, \eta_0$ with the following property: Let (u, ∇) be an N-vortex pair with small enough discrepancy $E(u, \nabla) - 2\pi N = \eta^2 \leq \eta_0^2$ such that $\Lambda^{-1}|u| \leq |u_0| \leq \Lambda|u|$, where (u_0, ∇_0) is an N-vortex solution of Taubes. Then:

$$\|u - u_0\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} \leq C_{N,\Lambda}\eta^2 ,$$

Generalizing?

We showed that if $|u|$ is comparable to a vortex solution, we can linearize the problem and deduce quantitative stability.

Question

Given a pair (u, ∇) with small enough discrepancy, can we find a comparable pair nearby?

The answer is Yes!

Generalizing to non-regular pairs

To generalize to all pairs with small discrepancy:

- We first show that any N vortex pair with small enough discrepancy can be uniformly approximated by $C^{N,\alpha}$ pairs with sharp estimates.
- Then we show that a pair with uniform C^N estimates and small enough discrepancy is comparable to some Taubes solution.
- We inspire from techniques of the proof the quantitative isoperimetric inequality in [3].

Selection principle

The idea is that for any (u, A) we solve a new variational problem:

$$\min_{(u_1,\nabla_1)} E(u_1, A_1) + \|u_1 - u\|^2_{L^2(\mathbb{R}^2)} + \|A_1 - A\|^2_{L^2(\mathbb{R}^2)} \tag{25}$$

to find a new pair (u_1, A_1) with more regularity. The remarkable fact is that, since the original pair (u, A) is a competitor, then:

$$E(u_1, A_1) + \|u_1 - u\|^2_{L^2(\mathbb{R}^2)} + \|A_1 - A\|^2_{L^2(\mathbb{R}^2)} \leq E(u, A)\,. \tag{26}$$

iterating this process, we get a final pair (u_M, A_M) with $|u_M|_{C^{N,\alpha}} \leq C_N$ in some local comloumb gauge.

Perturbation of complex polynomials

- It is not hard to show that functions $f :\to B_1 \subset \mathbb{C} \to \mathbb{C}$ that are close enough to z in C^1 topology are comparable to $|z - b|$ for some $b \in \mathbb{C}$.
- This can be generalized: Function that are close enough to $\Pi_{k=1}^{N}(z - a_k)$ in C^N topology are uniformly comparable to some $\Pi_{k=1}^{N}(z - b_k)$.
- Since vortex solutions are comparable to modulus complex polynomials, we can see that for any pair with small enough discrepancy, we can find a pair quantitatively close to our original pair that satisfies stability.
- We are Done!

Thank you

Thank you for your attention!

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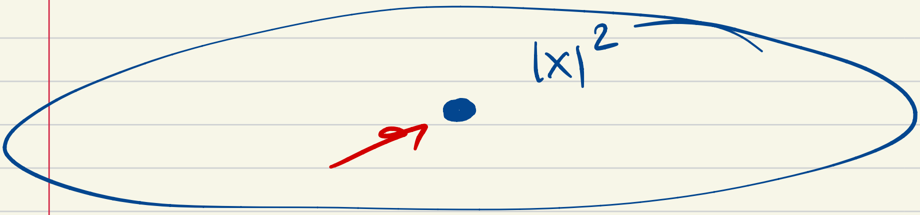

$$x \longrightarrow -\ln(|x|), \theta$$

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