Diffuse approximations of minimal surfaces

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Interfaces and concentration sets

Several physical and mathematical phenomena leads to the understanding and description of "transition interfaces" and of "concentration" sets.

These sets are typically of lower dimension and represents the first non trivial contribution to various physical energies.

Allen Cahn energy

Allen Cahn type energy describes the behavior of an order parameter $u : \Omega \to \mathbb{R}$ which undergoes a phase transition.

Features of the model are

- "Pure states" $u = \pm 1$ are absolute minima of the energy.
- The energy should favor some "ordered transition" between the two sets.

Allen Cahn energy

The energy then takes the following form:

$$E(u) = \int \underbrace{|du|^2}_{\text{ordered transition}} + \underbrace{\frac{(1-|u|^2)^2}{4}}_{\text{favors pure state}}$$

Critical points are solutions of

$$-\Delta u = \frac{u - u^3}{2}$$

The potential $\frac{(1-|u|^2)^2}{4}$ can be replaced by any other "two well" potential W.

Allen Cahn energy: rescaling and convergence

Let us then consider the rescaling $u_\epsilon(x) = u(\frac{x}{\epsilon})$, then u_ϵ is a minimizer of the energy

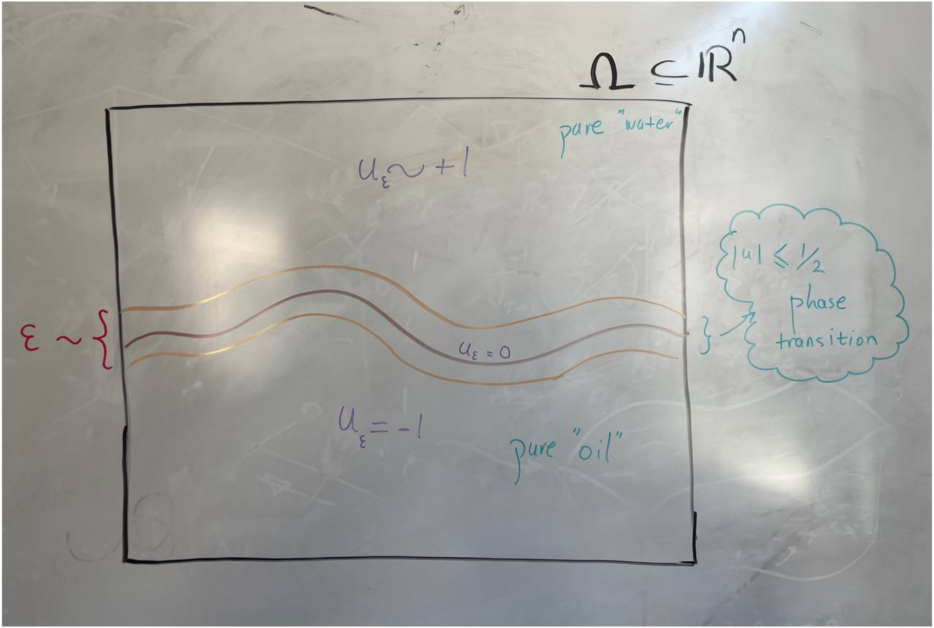
$$E_\epsilon(u) = \int \epsilon|du|^2 + \frac{(1-|u|^2)^2}{4\epsilon}$$

Moreover it can be shown that

$$u_\epsilon \to \pm 1.$$

The expected picture is to have $u = \pm 1$ outside a strip of thickness $\approx \epsilon$ around $\{u = 0\}$.

Allan Cahn: A picture



Allen Cahn: convergence to minimal surfaces heuristic

In particular:

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This points to us that this energy should "somehow" approximate the area of the "zero level set" of u_ϵ.

Allen Cahn: convergence to minimal surfaces

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Moreover:

- Convergence of critical points (Hutchinson-Tonegawa) and gradient flows (Ilmanen)
- Most *non-degenerate* minimal surfaces can be recovered as limit of critical points of Allen-Cahn energies (Pacard-Ritorè, Del Pino-Wei, D.-Pigati, ...)
- The above techniques can be used to construct minimal surfaces on manifolds, via min-max for AC (Guaraco, Chodosh-Mantoulidis, Bellettini-Wickramasekera,...)

Thank you for your attention!