Decay of excess for the abelian Higgs model

Aria Halavati

(joint work with G. De Philippis and A. Pigati)

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Interfaces and concentration sets

Several physical phenomena and their mathematical models lead to the understanding of "interfaces" and "concentration sets".

It's the set where the phase changes or the "phase transition" happens. It typically has a fixed co-dimension (the dimension of the states).

Many of these models are set up to prefer some type of "ordered" transition and studying these interfaces illuminates beautiful and interesting connections to geometry. If lucky some even yield interesting consequences which are harder to obtain by pure geometric methods.

The story of co-dimension 1
The Allen-Cahn Model

Allan-Cahn: The model

This model has a "phase" parameter $u : \mathbb{R}^n \supset \Omega \to [-1, +1]$:

- The values $u = \pm 1$ correspond to *pure* states; i.e. *water* or *oil*.
- The set $\{u = 0\}$ represents the interface between the states (Note that it has codimension 1). More precisely, one should think about $\{|u| \leq 1/2\}$ as a diffuse interface between the two phases.

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The energy of the model has the following form:

$$E(u) = \int_{\Omega} \underbrace{|du|^2}_{\text{favors ordered transition}} + \underbrace{\frac{(1-u^2)^2}{4}}_{\text{likes pure states}}$$

The stationary points satisfy the following semilinear PDE:

$$-\Delta u = \frac{u - u^3}{2}.$$

Allan-Cahn: The picture

One should imagine the domain Ω to be very large. Then we have the exponential decay away from the transition layer:

$$|du(x)| + |1 - |u(x)|| \lesssim e^{-C\,\mathrm{dist}(x,\{u=0\})}\,.$$

Then the expected picture is that $u \sim \pm 1$ outside a strip of thickness ≈ 1. Moreover the energy concentrates on the transition layer.

Allan-Cahn: The rescaled picture

We can rescale the picture by considering $u_\epsilon(x) = u(x/\epsilon)$, which means looking at the following rescaled energy:

$$E_\epsilon(u_\epsilon) = \int_\Omega \epsilon |du_\epsilon|^2 + \frac{(1 - u_\epsilon^2)^2}{4\epsilon} \,.$$

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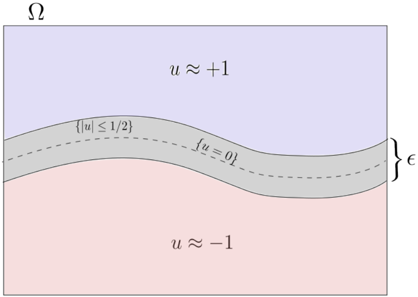

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$$=_{\text{Coarea formula}} \int_{-1}^{+1} (1-t^2)\mathcal{H}^{n-1}\left(\{u_\epsilon = t\}\right) dt \sim c_1 \mathcal{H}^{n-1}\left(\{u_\epsilon = 0\}\right)$$

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If the above achieves equality, it means that:

$$u_\epsilon(x) = g\left(\frac{\text{signed-dist}(x, \{u=0\})}{\epsilon}\right).$$

where g is the one dimensional solution $g' = 1 - g^2$.

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This suggests this energy is related to minimal surfaces.

Allan-Cahn: convergence to minimal surfaces

Theorem: Modica-Mortola

As $\epsilon \to 0$ the Allan-Cahn energy E_ϵ Γ-converges to the functional:

$$u \to \mathrm{Per}(\{u = 1\}),$$

for $u \in \mathrm{BV}(\Omega; \{-1, 1\})$.

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More results:

- Convergence of stationary points (Hutchinson-Tonegawa) and Gradient flow (Ilmanen).
- Most *Non-degenerate* minimal submanifolds can be recovered as limits of critical points (Pacard-Ritorè, Del Pino-Wei, De Philippis-Pigati, . . .)
- Minimal surfaces can be constucted via minMax for Allan-Cahn (Guaraco, Chodosh-Mantoulidis, Bellettini-Wickramasekera,. . .)

Overture

It is well known that large scale behavior of the set $\{u = 0\}$ is described by minimal surfaces.

Question

Do level sets of Allen-Cahn inherit more "interesting" behavior from minimal surfaces?

Rigidity results for minimal surfaces: Allard

Theorem: Allard

There exists $\tau(k, n) > 0$ such that if Σ is a k-dimensional minimal surface such that $0 \in \Sigma$ and:

$$\lim_{R\to\infty} \frac{\mathrm{Area}_k(\Sigma \cap B_R)}{\omega_k R^k} \leq 1 + \tau \, .$$

Then Σ is a flat k-plane.

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The above theorem is in fact a consequence of the following local result:

Theorem: Allard's ϵ-regularity

There exists $\epsilon(k,n) > 0$ such that if $\Sigma \subset B_1$ is a k-dimensional minimal surface (without boundary inside B_1) such that $0 \in \Sigma$ and:

$$\mathrm{Area}_k(\Sigma \cap B_1) \leq \omega_k(1+\epsilon)\,,$$

then (up to a rotation) $\Sigma \cap B_{1/2}$ is the graph of a $C^{1,\alpha}$ function f with $\|f\|_{C^{1,\alpha}} \lesssim \epsilon$.

Rigidity results for minimal surfaces: Bernstein theorems

For the case of Hypersurfaces more can be said:

Bernstein theorems

Let $\Sigma \in \mathbb{R}^n$ be a complete immersed co-dimension 1 minimal hypersurface. Then Σ is a plane if one of the following is true:

- Σ is a graph and $n \leq 8$. (Bernstein, Almgren, De Griogi, Simons)
- Σ is stable and $n \leq 6$. (Chodosh-Li, Chodosh-Li-Minter-Stryker, Catino-Mastrolia-Roncoroni, Mazet)

Does a "Bernstein" theorem holds for level-sets of Allan-Cahn?

De Giorgi's Conjecture 78'

Let $u : \mathbb{R}^n \to [-1, +1]$ be an entire critical point of the Allan-Cahn energy such that:

$$\partial_n u > 0\,.$$

Then u is one-dimensional, meaning after a possible rotation

$$u(x', x_n) = g(x_n)$$

where g is the one-dimensional profile (provided $n \leq 8$).

Savin's result

In 2009 Savin proved the following version of De-Giorgi's conjecture:

Theorem: Savin 09'

Let $u : \mathbb{R}^n \to [-1, +1]$ be an entire critical point of the Allan-Cahn energy such that:

$$\partial_n u > 0 \quad \text{and} \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \,.$$

Then u is one-dimensional (provided $n \leq 8$).

Wang's result

Wang also discovered a variational proof which implies Savin's result:

Theorem: Wang 15'

There is a constant τ such that if u is an entire solution of AC with:

$$\frac{E_{\mathbf{AC}}(u)(B_R)}{R^{n-1}} \leq c_1 + \tau \,,$$

then u is one-dimensional.

Savin and Wang: Idea of the proof

- First with a simple compactness argument and Allard, we can see that the configuration is flat on large scales with respect to a (possibly changing) plane.

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- The main idea is then an "improvement of flatness":

If the configuration is close to be flat at scale 1, then it is much closer to be flat at scale $1/2$.

- Here closeness can be measured in different ways which depends on the problem.

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The intuition is that the area functional linearizes to the Laplace equation, which enjoys good decay estimates. Take the surface as $\mathrm{graph}(f)$:

$$\mathrm{Area}(graph(f)) = \int \sqrt{1 + |\nabla f|^2} \sim \int 1 + \frac{|\nabla f|^2}{2} = 1 + \mathrm{Dir}(f)$$

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The main (interesting) difficulty is to make this linearization rigorous.

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- His proof relies on the tool-box of maximum principle type arguments and comparison functions which are "scalar" in nature.
- Co-dimension 1 is essential for this toolbox.

The story of co-dimension 2 Abelian Higgs (Ginzburg Landau)

Co-dimension 2: The Ginzburg Landau model

For $u : \Omega \subset \mathbb{R}^3 \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^3$, the Ginzburg energy takes the following form:

$$E(u, A) = \int_\Omega |du - iAu|^2 + |\mathrm{curl}(A)|^2 + \kappa \frac{(1 - |u|^2)^2}{4}.$$

Here u is the order parameter and $|u| = 1$ reflects pure states; A is the magnetic vector potential and $\mathrm{curl}(A)$ is the magnetic field.

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Note the following *gauge invariance* of the energy:

$$(u, A) \rightsquigarrow (ue^{i\theta}, A + d\theta)$$

Co-dimension 2: The Abelian Higgs model

Let $L \to M$ be a complex line bundle over M, u a section and ∇ a metric connection, then the Yang-Mills-Higgs energy take the following form:

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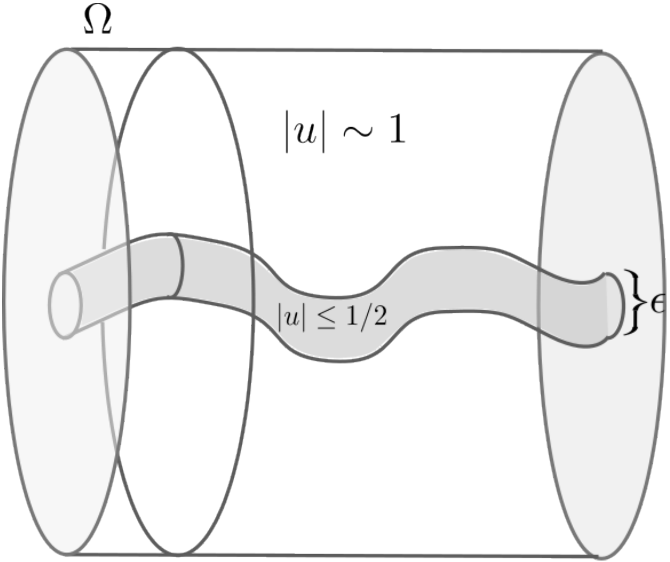
Here the gauge invariant *vortex set* $\{|u| \leq 1/2\}$ plays the role of transition layer for AC and is of codimension-2.

Co-dim 2: A picture



Ginzburg-Landau: Background

The case $\alpha = 0$ has been studied by many mathematicians (Bethuel, Brezis, Orlandi, Serfaty, Lin, Rivere, Pacard, Smets, ...) and it is quiet difficult to analyze.

- The energy localize very slowly. (energy grows like $|\log \epsilon|$), more precisely on the set $\{|u| \geq \frac{1}{2}\}$:

$$|du(x)|^2 \sim |d(\frac{u}{|u|})|^2 \sim \frac{1}{\mathrm{dist}^2(x, u=0)} .$$

$$\text{so on a transversal 2-dim slice } \int_{B_1^2 \backslash B_\epsilon^2} |du(x)|^2 \sim |\log \epsilon|$$

- Vortices repulse each other with energy of order $|\log(\mathrm{distance})|$.
- Because of this interaction, integrality of the limit sub-manifold is not guaranteed (Pigati-Stern, Dávila-del Pino-Medina-Rodiac).

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- First we see that if $u = re^{i\theta}$ and $\nabla : d - i\alpha$:

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- Indeed on $\mathbb{R}^2$ it's even better: we can see that (Bogomolny):

$$\begin{aligned} E(u,\nabla) &= \int_{\mathbb{R}^2} |\nabla u|^2 + |F_\nabla|^2 + \frac{1}{4}(1-|u|^2)^2 \\ &= 2\pi|N| + \int_{\mathbb{R}^2} |\nabla_{\partial_1} u \pm i\nabla_{\partial_2} u|^2 + |\star F_\nabla \mp \frac{1-|u|^2}{2}|^2 \,. \end{aligned}$$

  Here N is the vortex number or the winding number of u at ∞ and is a topological constant.

Abelian Higgs: 2 dimensions

Minimizers satisfy a system of first order equations (up to a conjugation) called *the vortex equations*:

$$\nabla_{\partial_1} u + i\nabla_{\partial_2} u = 0 \text{ and } \star F_{\nabla} = \frac{1 - |u|^2}{2} .$$

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Taubes, in his PhD thesis, showed that:

- On $\mathbb{R}^2$ all stationary points are minimizers. (Equivalence of first and second order equations)
- After prescribing the zero set $u = 0$ to be $\{a_1, \dots, a_N\}$, counting with multiplicity, the solution is unique (up to a change of gauge).

Abelian Higgs: stability in 2 dimensions (A necessary tool)

The uniqueness result can be strengthened as follows:

Theorem: H. 23'

For any N there exists $C_{|N|}$ such that any N-vortex pair (u, ∇) satisfies:

$$\min_{(u_0,\nabla_0)\in\mathcal{F}} \|u - u_0\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} \leq C_{|N|}\left[E(u,\nabla) - 2\pi|N|\right] .$$

provided that $E(u,\nabla) - 2\pi|N|$ is small enough. Here $\mathcal{F}$ is the moduli space of all solutions to the vortex equations.

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Ideas of proof:

- If $(u, \nabla) \rightsquigarrow (re^{i\theta}, A)$ the discrepancy becomes:

$$E(u, \nabla) - 2\pi|N| = \int_{\mathbb{R}^2} r^2 |d\log(r) + \star(A - d\theta)|^2 + |\star dA - \frac{1 - r^2}{2}|^2$$

- New weighted CKN-type inequalities on two-manifolds needed (H.)
- A smoothing method using a penalized functional (inspired by the quantitative isoperimetric inequality Cicalese-Leonardi)

A glimpse of the inequalities

H. 23'

Let ω be a positive weight on a two-manifold M (with boundary) such that:

$$\omega^2 \Delta \log \omega = 0$$

Then for any $f \in C_c^\infty(M)$ the following holds for $\epsilon \leq 1$:

$$\int_M |\omega|^{2+2\epsilon} |df|^2 \leq \frac{3 \sup_M \omega^{2\epsilon}}{\epsilon^2} \int_M \frac{\omega^4}{|d\omega|^2} |\Delta f|^2 .$$

- As a special case:

$$\int_{B_1^2} |x|^{2+2\epsilon} |df|^2 \leq \frac{3}{\epsilon^2} \int_{B_1^2} |x|^4 |\Delta f|^2 .$$

- All weights of the form

$$\omega = \Pi_{k=1}^n |x - x_k|^{\alpha_k}$$

with $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$ and $\alpha_k > 0$ are admissible.

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Abelian Higgs: The rescaled picture

in the case $\kappa = 1$, the energy decays exponentially away from the vortex set:

$$|\nabla_\epsilon u_\epsilon| + \epsilon|F_{\nabla_\epsilon}| + \epsilon^{-1}|1 - |u_\epsilon|| \lesssim e^{-C\,\mathrm{dist}(.,\{|u|=0\})/\epsilon}$$

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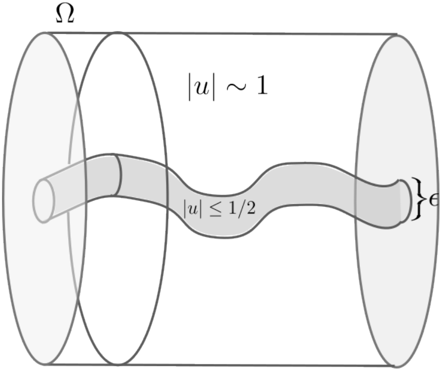

Abelian Higgs $\to$ minimal submanifolds of co-dim 2

Analogous to Allan-Cahn we have the following result:

Thoerem: Pigati-Stern, Parise-Pigati-Stern

As $\epsilon \to 0$, the YMH functional E_ϵ converges (in a suitable sense) to the $n-2$ area of the zero level set (the only gauge invariant one):

$$\mathcal{H}^{n-2}(\{u=0\})\,.$$

- In fact the energy measures $\frac{1}{2\pi}e_\epsilon(u,\nabla)$ converge to a stationary co-dim 2 integral varifold V.
- the Currents dual to the Jacobian $J(u,\nabla)=d\langle iu,\nabla u\rangle$ converge weakly to a cycle Γ with $|\Gamma|\leq\mu_V$.

Question

We see that $\{u = 0\}$ behaves like a minimal submanifold in the large scale. As before we can ask:

Question

Does $\{u = 0\}$ inherit any *rigidity* from minimal surfaces?

The answer is Yes!

Theorem 1: De Philippis-H.-Pigati 24'

There is τ such that for $2 \leq n \leq 4$ an entire **stationary** pair (u, ∇) for the Yang-Mills-Higgs functional E_1 with:

$$\lim_{R\to\infty} \frac{E_1(u,\nabla)(B_R)}{\omega_{n-2}R^{n-2}} \leq 2\pi + \tau$$

is necessarilly two dimensional; Meaning there is a projection $P : \mathbb{R}^n \to \mathbb{R}^2$ such that $(u, \nabla) = P^*(u_0, \nabla_0)$, where (u_0, ∇_0) is a one-vortex solution.

Abelian Higgs: Rigidity of local minimizers

For minimizers we can remove the dimension restriction:

Theorem 2: De Philippis-H.-Pigati 24'

For any $n \geq 2$ there is $\tau(n) > 0$ such that an entire **local minimizing** pair (u, ∇) for the Yang-Mills-Higgs functional E_1 with:

$$\lim_{R\to\infty} \frac{E_1(u,\nabla)(B_R)}{\omega_{n-2}R^{n-2}} \leq 2\pi + \tau$$

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Ideas of proof: Excess

We measure flatness in two ways:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2\,,$$

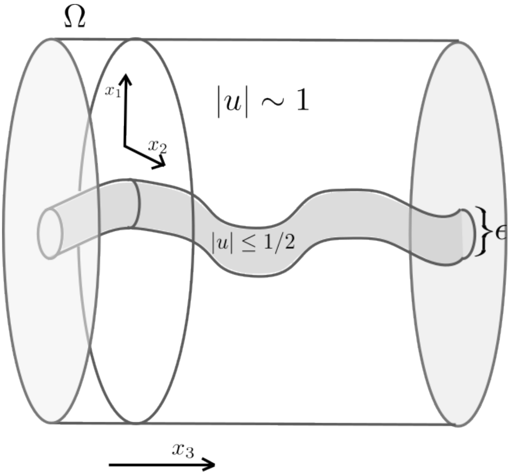
$$\mathbf{E}_1(u, \nabla, B_R) = \frac{1}{R^{n-2}} \int_{B_R} \sum_{k=3}^{n} |\nabla_{\partial_k} u|^2 + \sum_{(j,k)\neq(1,2)} |F_\nabla(\partial_j, \partial_k)|^2\,,$$

$$\mathbf{E}_2(u, \nabla, B_R) = \frac{1}{R^{n-2}} \int_{B_R} |\nabla_{\partial_1} u - i\nabla_{\partial_2} u|^2 + |F_\nabla(\partial_1, \partial_2) - \frac{1-|u|^2}{2}|^2\,.$$

- $\mathbf{E}_1$ measures how flat (u, ∇) is and does not depend on orientation. (parallel to varifold excess)
- $\mathbf{E}_2$ measures how far (u, ∇) to be a solution of the vortex equation (on the slice) and depends on the orientation.

In particular:

$$\int_{B_1^2\times B_1^{n-2}} e_\epsilon(u,\nabla) = 2\pi\omega_{n-2} + \mathbf{E}(u,\nabla,B_1) + O(e^{-\frac{K}{\epsilon}})\,.$$



Ideas of proof: Excess decay for solutions

The main ingredient is the following:

Theorem 3: De Philippis-H.-Pigati 24'

For any $n \geq 2$, there exists $\tau(n), R_0(n)$ such that if (u, ∇) is an entire critical points of YMH energy such that:

$$\frac{E_1(u,\nabla)(B_R)}{\omega_{n-2}R^{n-2}} \leq 2\pi + \tau\,,$$

with $R \geq R_0$. Then the **first excess** decays (after a possible rotation):

$$\mathbf{E}_1(u,\nabla,B_{\frac{R}{2}}) \leq \frac{1}{2}\mathbf{E}_1(u,\nabla,B_R)\,,$$

or it is already small:

$$\mathbf{E}_1 \lesssim \frac{|\log \mathbf{E}|^2\sqrt{\mathbf{E}}}{R^2} + e^{-CR}\,.$$

Unfortunately, for critical pairs, only $\mathbf{E}_1$ decays.

For minimizers, we have comparison arguments, hence we can do better:

Theorem 4: De Philippis-H.-Pigati 24'

For any $n \geq 2$ and $\beta > 0$ there is $\tau(\beta, n), R_0(\beta, n)$ such that if (u, ∇) is an entire local minimizer of YMH such that:

$$\frac{E_1(u,\nabla)(B_R)}{\omega_{n-2}R^{n-2}} \leq 2\pi + \tau \,,$$

with $R \geq R_0$. Then the **full excess** decays (after a possible rotation)

$$\mathbf{E}(u, \nabla, B_{\frac{R}{2}}) \leq \frac{1}{2}\mathbf{E}(u, \nabla, B_R) \,.$$

or it is already small:

$$\mathbf{E}(u, \nabla, B_R) \leq \frac{1}{R^\beta} \,.$$

Idea of proof

- It is not hard to see that (By Allard) the configuration is flat on large scales with respect to a (possibly changing) plane.
- We then aim to linearize in the regime where excess $\mathbf{E}_1$ vanishes and radius R becomes large.
- Equivalently in the rescaled picture we linearize the equation in the regime $\mathbf{E}_1 \to 0$ and $\epsilon \to 0$.

Ideas of proof of Theorem 3: Lipschitz approximation

Lipschitz approximation $\rightsquigarrow$ Gauge invariance means a generic level set of u might be irregular

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$$J(u,\nabla) = d\langle iu, \nabla u\rangle \rightsquigarrow J(u,\nabla)_{1,2} = J_x : \mathbb{R}^{n-2} \to \mathcal{M}(B_1^2)\,.$$

and take a Lipschitz approximation of the *barycenter*

$$\langle J_x, (x_1, x_2)\rangle := \int_{B_1^2 \times x} J(u,\nabla)_{1,2}.(x_1, x_2)$$

to be

$$\Phi(x) : \mathbb{R}^{n-2} \to \mathbb{R}^2.$$

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We also get L^2 bounds:

$$\int_{B_R^{n-2}} |d\Phi|^2 \leq C\mathbf{E}_1\,.$$

Ideas of proof of Theorem 3: Harmonic approximation

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- The stress energy tensor (obtained by inner variations)

$$T(u,\nabla) = e(u,\nabla)Id - 2\nabla u^{*}\nabla u - 2\omega^{*}\omega$$

  is (row-wise) divergence free (for smooth solutions).

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- It is also closely related to the Jacobian $J(u, \nabla)$ with via $\mathbf{E}_2$. In fact:

$$\|J(u, \nabla)_{1,k} - T(u, \nabla)_{2,k}\|_{L^2}^2 \lesssim \sqrt{\mathbf{E}_1 \mathbf{E}} \text{ for } k = 3, \ldots, n\,.$$

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- We test $\mathrm{div}(T(u, \nabla)) = 0$ with an appropriate vector field to see:

$$\left| \int d\Phi . d\xi \right| \lesssim \sqrt{\mathbf{E}_1 \mathbf{E}} \|d\xi\|_\infty$$

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  for any test function $\xi : \mathbb{R}^{n-2} \to \mathbb{R}^2$.
- This and $\int |d\Phi|^2 \lesssim \mathbf{E}_1$ gives us harmonic approximation for some h:

$$\int |\Phi - h|^2 \lesssim o(\mathbf{E}_1)\,.$$

  with $\Delta h = 0$.

Ideas of proof of Theorem 3: Caccioppoli and decay

Then with a Caccioppoli type inequality we get an excess-height bound

$\rightsquigarrow$ decay properties of harmonic functions means height decays

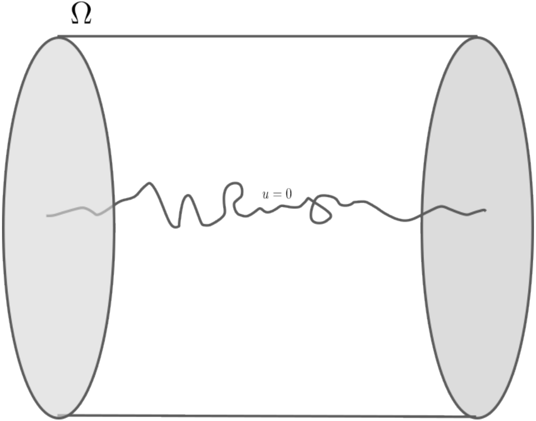
$\rightsquigarrow$ excess decays.

$\rightsquigarrow$ The obstruction in dimension comes from estimating the "variance" of slice measures $\rightsquigarrow$ accurate up to order $o(\epsilon^2 \sim \frac{1}{R^2})$.

Decay of the full excess for local minimizers
A visual guide

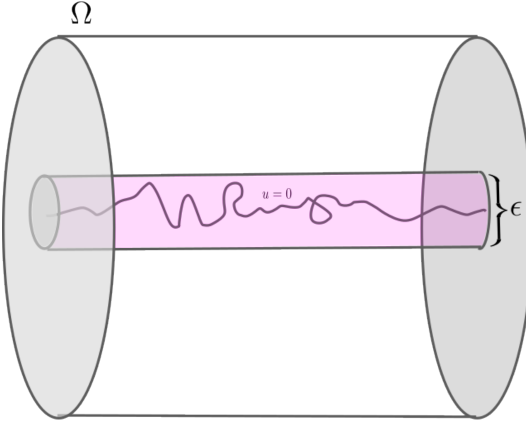
Ideas of proof of Theorem 4 (local minimizers)

Apriori the picture looks like this:



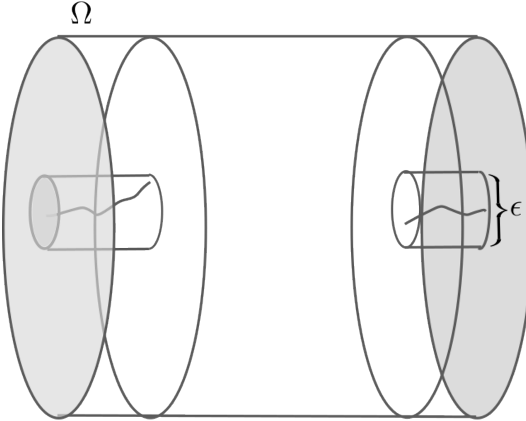
Ideas of proof of Theorem 4 (local minimizers)

Iterating theorem 3 tells us that the vortex set lies ϵ near a line:



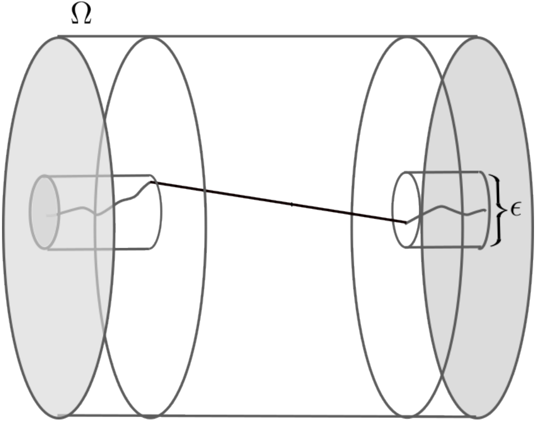
Ideas of proof of Theorem 4 (local minimizers)

We find a good radius with small excess $\mathbf{E}_1 + \mathbf{E}_2$ on the boundary, to cut:



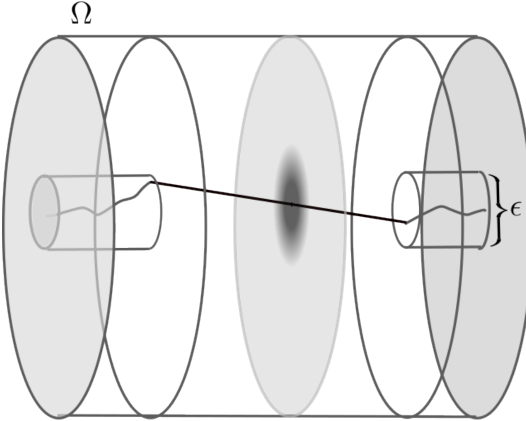
Ideas of proof of Theorem 4 (local minimizers)

We replace inside with a line (harmonic function), **length decays**!



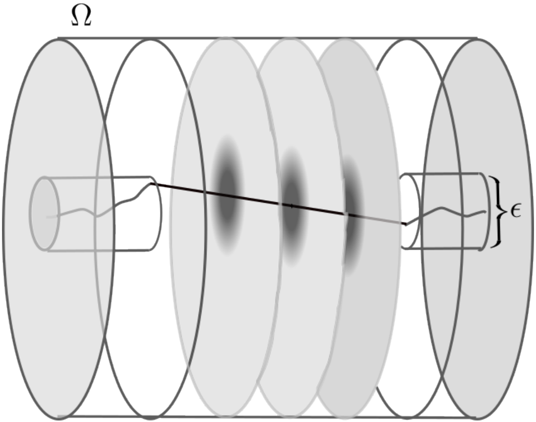
We want to mimick this on the energy level to contradict minimality.

We pull-back a one-vortex solutions with zero as this line.



Ideas of proof of Theorem 4 (local minimizers)

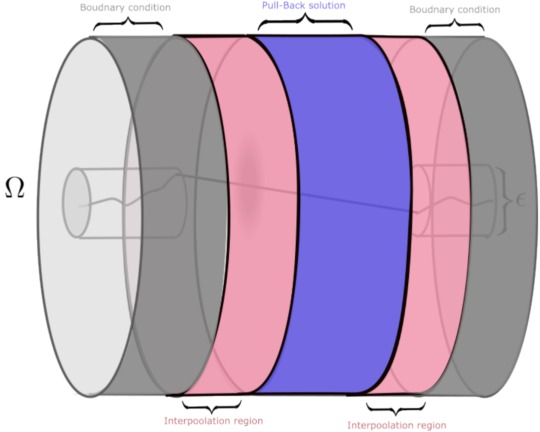
Energy $\sim$ length inside.



However we need to attach to boundary conditions to have a competitor.

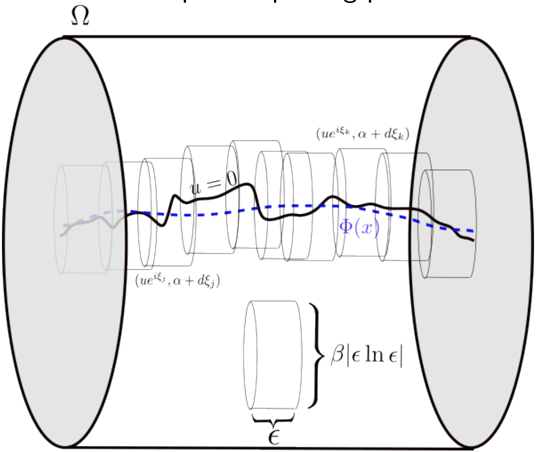
Ideas of proof of Theorem 4 (local minimizers)

We need to interpolate with the boundary conditions $\rightsquigarrow$ Quantitative stability in some gauge, but which one? $\rightsquigarrow$ a very delicate gauge fixing has to be done $\rightsquigarrow$ A crucial tool $\rightarrow$ the zero set is ϵ near a line (C^1 graph).



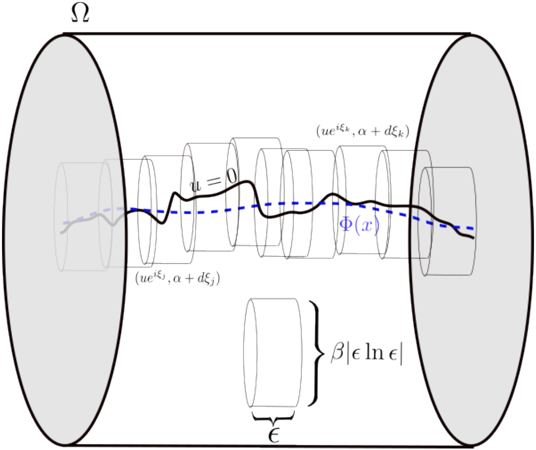
Idea of proof of Theorem 4: The crazy gauge

Cover the vortex set with cylinders like $B^2_{C\epsilon} \times B^{n-2}_{C\beta|\epsilon \log \epsilon|}$. Using the structure theorem 3 gives us $\rightsquigarrow$ no two cylinders are on top of each other.
$\rightsquigarrow$ Gauge fix in each and then patch up using partition of unity and stability.



ϵ^β comes from the decay away from

$$e^{-\beta|\epsilon \log \epsilon|/\epsilon} \lesssim \epsilon^\beta \sim \frac{1}{R^\beta} .$$



$\rightsquigarrow$ with a comparison and using

$$\text{Length} \sim \int_{B_1^2 \times B_1^{n-2}} e_\epsilon(u, \nabla) \sim 2\pi\omega_{n-2} + \mathbf{E}(u, \nabla, B_1)\,.$$

we conclude the decay.

Summary

In the multiplicity one regime:

- We were able to obtain rigidity for solution up to $n \leq 4$.
- and rigidity for local minimizers for all dimensions $n \geq 2$.
- the case of solutions for $n > 4$ remains open (There are some slight of possible ways to push further but it is not clear at the moment).

Discussion

- It's interesting to see if we can push the classification to all dimensions for stationary points (In the multiplicity one regime)?
- Applying this pipeline to Ginzburg Landau without magnetic field (In the works).
- This pipeline applies to diffuse energies (blowing down to minimal sub-manifolds) who carry a *self dual structure* (or equivalently an equi-partition of energy) like the Abelian Higgs and Allan Cahn.

Thank you for your attention!