CARLEMAN ESTIMATES FOR STATIONARY Q-VALUED MAPS: A VARIATIONAL APPROACH

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Abstract. We prove a Carleman-type estimate for Dirichlet-stationary multivalued functions and apply it to give a different proof of the optimal dimension of the singular set of Dir-minimizing multivalued functions, originally due to Almgren and to De Lellis–Spadaro.

Contents

1. Introduction

Recall that a Q-valued map $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$, $\Omega \subset \mathbb{R}^n$ open, is *Dir-stationary (with respect to outer and inner variations of the Dirichlet energy)* if it is a critical point of the Dirichlet energy, that is it satisfies an *outer variation formula*

$$\mathcal{O}(f,\psi) := \int \sum_i \left[\langle Df_i(x) : D_x\psi(x, f_i(x))\rangle + \langle Df_i(x) : D_u\psi(x, f_i(x)) \cdot Df_i(x)\rangle\right] dx = 0\,, \tag{1.1}$$

for every $\psi(x,u) \in C^\infty(\Omega \times \mathbb{R}^m; \mathbb{R}^m)$ with *compact support in* x and

$$|D_u\psi| \le C < \infty \qquad \text{and} \qquad |\psi| + |D_x\psi| \le C\,(1+|u|)\,,$$

and an *inner variation formula*

$$\mathcal{I}(f,\phi) := 2\int \sum_{i=1}^{Q} \langle Df_i : Df_i \cdot D\phi\rangle - \int |Df|^2 \operatorname{div}\phi = 0\,, \qquad \forall \phi \in C^\infty_c(\Omega, \mathbb{R}^n)\,. \tag{1.2}$$

We will say that a Q-valued map $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ is *weakly stationary* if $\mathcal{O}(f,\cdot) = 0$, i.e. stationary with respect to outer variations only.

For the derivation of these formulas as the Euler–Lagrange equations of the Dirichlet energy and the related Sobolev theory of Q-valued maps, see [5, Sections 2 and 3], whose notations we follow.

Multivalued maps that minimize an appropriate Dirichlet energy were introduced by Almgren in [1] in his celebrated proof of the optimal bound on the dimension of the singular set of area minimizing currents in high codimension, as the appropriate linearized problem. More recently, De Lellis and Spadaro revisited this theory with metric techniques in [5, 7, 6, 8, 9] (see also [11] for minimizers taking values in a smooth compact Riemannian manifold, and [3] for minimizers of the p-Dirichlet energy).

The main difficulty in proving such optimal bound is the presence of branch points: points at which the blow-up is regular but the presence of multiplicity causes singularity to appear. Almgren's innovative insight was to understand how such branch points can be studied with the same techniques that are used in the study of unique continuation properties for elliptic PDEs. However multivalued functions do not satisfy a PDE in the usual sense, and so he had to find a variational approach to unique continuation which gave birth to the so-called frequency function. However, another technique that has been extensively used in the context of standard PDEs to study unique continuation type question is the so called Carleman estimates technique (see for instance [4, 2, 14, 16, 13, 18]). In this paper we give the first variational proof of such an estimate for Q-valued functions and we use it to recover Almgren's optimal bound on their singular set. We point out that Carleman estimates have been used in a similar setting for J-holomorphic maps by Riviere-Tian (see [17]) where they take advantage of the complex structure to turn the problem into a first order elliptic system, our approach is different (partially inspired by the proofs in [10]).

We remark that, although our methods differ, the information obtained through Carleman estimates is essentially captured by the frequency function approach. The main purpose of this note is to make the community aware of this technique (mainly the two inner and outer variations needed to find the estimates).

1.1. Main results.

The main result of the paper is the following Carleman-type estimate.

Theorem 1.1 (Carleman estimate). *Let $f \in W^{1,2}(B_1; \mathcal{A}_Q(\mathbb{R}^m))$ be a Dir-stationary Q-valued function, $B_1 \subset \mathbb{R}^n$, and let $\tau > 0$ and $\eta = \frac{2\tau-n+2}{2}$. Then the following estimate holds*

$$\int_{B_1} \chi \sum_{i=1}^{Q} \left(\varepsilon^2 \frac{|f_i|^2}{|x|^{2\tau+2-\varepsilon}} + \frac{|Df_i \cdot x - \eta f_i|^2}{|x|^{2\tau+2}} \right) \leq C \int_{B_1} |D\chi| \sum_{i=1}^{Q} \left(\frac{|Df_i|^2}{|x|^{2\tau-1}} + \frac{|f|^2}{|x|^{2\tau+1}} \right) . \tag{1.3}$$

for any compactly supported function $\chi \in C_c^\infty(B_1 \setminus \{0\})$.

We remark that in the literature the name Carleman estimate is usually associated to estimates of the form

$$\left\| |x|^{-\tau} Du \right\|_{L^2(\Omega)} \leq \left\| |x|^{-\tau+1} \Delta u \right\|_{L^2(\Omega)} , \tag{1.4}$$

(for large $\tau > 0$) which are then used to derive expressions of the form Eq. (1.3). However in our case the Laplacian is replaced by inner and outer variations for the Dirichlet energy, so Eq. (1.4) doesn't make sense in our setting and we have to give a variational proof of Eq. (1.3), that is by testing inner and outer variations with proper vector fields..

A straightforward consequence of Theorem 1.1 is the following strong unique continuation result

Theorem 1.2 (Strong unique continuation). *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^m))$, with $B_1 \subset \mathbb{R}^n$, be a Dir-stationary map and suppose that*

$$\lim_{r\to 0} \frac{1}{r^N} \int_{B_r} |f|^2 = 0 \qquad \forall N \in \mathbb{N} .$$

Then $f \equiv Q [\![0]\!]$ in B_1.

Moreover, we are also able to recover the optimal bound on the singular set of Dir-minimizing multivalued functions. We recall that a point $x \in \Omega$ *is regular* if there exists a neighborhood $B \subset \Omega$ of x and Q analytic functions $f_i\colon B \to \mathbb{R}^n$ such that

$$f(y) = \sum_{i=1}^{Q} [\![f_i(y)]\!] \qquad \text{for almost every } y \in B\,,$$

and either $f_i(x) \neq f_j(x)$ for every $x \in B$ or $f_i \equiv f_j$. The *singular set* Σ_f *of* f is the complement in Ω of the set of regular points.

Theorem 1.3 (Dimension of the singular set)**.** *Let $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$, with $\Omega \subset \mathbb{R}^n$, be a Dir-minimizing map. Then $\dim_{\mathcal{H}}(\Sigma_f) \leq n-2$. Moreover if $n=2$, then Σ_f is locally finite.*

We remark that Theorem 1.3 is already known, see [1, 5]. Our main contribution is the use of Eq. (1.3) in place of the monotonicity formula for the frequency function to prove them.

Finally, as an instructive remark, we note that by adapting ideas from [12], one can reprove the result therein for two-dimensional Dir-stationary multivalued functions, replacing frequency function techniques with our Carleman estimate in combination with Weiss' energy.

2. Proof of Carleman estimate: Theorem 1.1

For a compactly supported positive test function $\chi \in C_c^\infty(B_1 \setminus \{0\})$ we test the outer variation Eq. (1.1) with the admissible test function $\psi(x,u) = \chi \dfrac{u}{|x|^{2\tau}}$ to obtain

$$\begin{aligned} 0 &= \int_{B_1} \sum_{i=1}^{Q} \left[\chi \frac{|Df_i|^2}{|x|^{2\tau}} + \frac{\langle D\chi : Df_i \rangle\, f_i}{|x|^{2\tau}} - 2\,\tau\,\chi\, \frac{\partial_r f_i\, f_i}{|x|^{2\tau+1}} \right] \\ &= \int_{B_1} \sum_{i=1}^{Q} \left[\chi \frac{|Df_i|^2}{|x|^{2\tau}} + \frac{\langle D\chi : Df_i \rangle\, f_i}{|x|^{2\tau}} + \tau(n-2-2\tau)\chi \frac{|f_i|^2}{|x|^{2\tau+2}} + \tau \partial_r \chi \frac{|f_i|^2}{|x|^{2\tau+1}} \right], \end{aligned} \tag{2.1}$$

where the last equality is obtained by an integration by parts. Next we want to test the inner variation Eq. (1.2) with the admissible vector field $\phi = \chi \frac{x}{|x|^{2\tau}}$. We compute

$$D\phi = \frac{1}{|x|^{2\tau}}\, D\chi \otimes x + \frac{\chi}{|x|^{2\tau}}\, I - 2\tau\, \frac{\chi}{|x|^{2\tau+2}}\, x \otimes x\,, \tag{2.2}$$

$$\operatorname{div}(\phi) = \frac{\partial_r \chi}{|x|^{2\tau-1}} + (n-2\tau)\, \frac{\chi}{|x|^{2\tau}}\,. \tag{2.3}$$

This implies that

$$\int_{B_1} \sum_{i=1}^{Q} \left[\frac{(n-2\tau-2)}{2} \chi \frac{|Df_i|^2}{|x|^{2\tau}} + 2\tau\chi \frac{|\partial_r f_i|^2}{|x|^{2\tau}} + \frac{\partial_r \chi}{2} \frac{|Df_i|^2}{|x|^{2\tau-1}} - \frac{\langle Df_i : D\chi \rangle \langle Df_i : x \rangle}{|x|^{2\tau}} \right] = 0\,. \tag{2.4}$$

Now we name $\eta = \frac{2\tau-n+2}{2}$, multiply Eq. (2.1) by η and add it to Eq. (2.4) to see that:

$$\int_{B_1} \chi \sum_{i=1}^{Q} \left[\frac{|\partial_r f_i|^2}{|x|^{2\tau}} - \eta^2 \frac{|f_i|^2}{|x|^{2\tau+2}} \right] \leq C\left(\frac{\eta}{\tau}\right) \int_{B_1} |D\chi| \sum_{i=1}^{Q} \left[\frac{|Df_i|^2}{|x|^{2\tau-1}} + \frac{|f_i|^2}{|x|^{2\tau+1}} \right]. \tag{2.5}$$

This is a first Carleman estimate: to conclude we need to complete the square on the left hand side. We calculate as follows:

$$\begin{aligned}\int_{B_1} 2\eta^2\chi\sum_{i=1}^{Q}\frac{|f_i|^2}{|x|^{2\tau+2}} &= \int_{S^{n-1}}\left(\int_0^1 2\eta^2\chi\sum_{i=1}^{Q}\frac{|f_i|^2}{r^{2\tau-n+3}}\,dr\right)d\theta\\ &= \int_{S^{n-1}}\left(\int_0^1 -\eta\chi\partial_r(r^{-2\tau+n+2})\sum_{i=1}^{Q}|f_i|^2\,dr\right)d\theta\\ &= \int_{S^{n-1}}\left(\int_0^1 2\eta\,\chi\, r^{-2\tau+n-2}\sum_{i=1}^{Q} f_i\partial_r f_i\,dr\right)d\theta\\ &\quad+\int_{S^{n-1}}\left(\int_0^1 \eta\,\partial_r\chi\, r^{-2\tau+n-2}\sum_{i=1}^{Q}|f_i|^2\,dr\right)d\theta\,,\end{aligned}$$

that is

$$\int_{B_1} 2\eta^2\chi\sum_{i=1}^{Q}\frac{|f_i|^2}{|x|^{2\tau+2}} - \int_{B_1} 2\eta\,\chi\sum_{i=1}^{Q}\frac{f_i\,Df_i\cdot x}{|x|^{2\tau+2}} = \eta\int_{B_1} D\chi\cdot x\sum_{i=1}^{Q}\frac{|f_i|^2}{|x|^{2\tau+2}} \tag{2.6}$$

Then, combining Eq. (2.5) and Eq. (2.6), we see that:

$$\int_{B_1}\chi\sum_{i=1}^{Q}\frac{|Df_i\cdot x-\eta f_i|^2}{|x|^{2\tau+2}} \le C\int_{B_1}|D\chi|\sum_{i=1}^{Q}\left(\frac{|Df_i|^2}{|x|^{2\tau-1}}+\frac{|f_i|^2}{|x|^{2\tau+1}}\right). \tag{2.7}$$

To estimate the L^2 term on the left hand side of Eq. (1.3), we can see Eq. (2.6) for $\tau-\varepsilon$ in place of τ and the obvious modification for η:

$$\int_{B_1}\chi\sum_{i=1}^{Q}\left(|\partial_r f_i|^2-(\eta-\varepsilon)^2|f_i|^2\right) \ge -C\int_{B_1}|D\chi|\sum_{i=1}^{Q}\left(\frac{|Df_i|^2}{|x^{2\tau-1}}+\frac{|f_i|^2}{|x|^{2\tau+1}}\right). \tag{2.8}$$

Combining this, with Eq. (2.7) and the following computation, we conclude:

$$\begin{aligned}\int_{B_1}\chi\sum_{i=1}^{Q}\frac{|Df_i.x-\eta f_i|^2}{|x|^{2\tau+2-2\varepsilon}} &\ge \int_{B_1}\chi\sum_{i=1}^{Q}\frac{|\partial_r f_i|^2}{|x|^{2\tau+2-2\varepsilon}}-\eta\frac{\partial_r|f_i|^2}{|x|^{2\tau+1-2\varepsilon}}+\eta^2\frac{|f_i|^2}{|x|^{2\tau-2\varepsilon}}\\ &\ge \int_{B_1}\chi\sum_{i=1}^{Q}\frac{|\partial_r f_i|^2}{|x|^{2\tau+2-2\varepsilon}}+(\eta^2-2\eta(\eta-\varepsilon))\frac{|f_i|^2}{|x|^{2\tau-2\varepsilon}}\\ &\overset{Eq.\ (2.8)}{\ge} \int_{B_1}\chi\sum_{i=1}^{Q}\varepsilon^2\frac{|f_i|^2}{|x|^{2\tau-2\varepsilon}}\\ &\quad- C\int_{B_1}|D\chi|\sum_{i=1}^{Q}\left(\frac{|Df_i|^2}{|x|^{2\tau-1}}+\frac{|f_i|^2}{|x|^{2\tau+1}}\right).\end{aligned}$$

□

Corollary 2.1 (Three sphere inequality)**.** *There exists a constant $C>0$, possibly depending on the dimension, with the following property. For any constant $\tau>0$ (possibly large), any Dir-stationary multifunction $f\in W^{1,2}(B_1;\mathcal{A}_Q)$ and three radii $r_1<r_2<r_3<\frac{1-|x|}{2}$ with*

$\min\left(\frac{r_3}{r_2}, \frac{r_2}{r_1}\right) > 2$ *and* $x \in B_1$, *the following estimate is true:*

$$\left[\frac{1}{1+\log(r_3/r_2)^2} + \frac{1}{1+\log(r_2/r_1)^2}\right] \frac{\|f\|^2_{L^2(B_{2r_2}\setminus B_{r_2}(x))}}{r_2^{2\tau}}$$
$$\leq C\frac{\|f\|^2_{L^2(B_{2r_1}\setminus B_{r_1}(x))}}{r_1^{2\tau}} + C\frac{\|f\|^2_{L^2(B_{2r_3}\setminus B_{r_3}(x))}}{r_3^{2\tau}}\,.$$

Proof. First testing the outer variation Eq. (1.1) with $\Psi(x,u) = \phi^2(x)u$ we get the Caccioppoli inequality:

$$\int_{B_1} \phi^2 |Df|^2 \leq C \int_{B_1} |D\phi|^2 |f|^2\,. \tag{2.9}$$

Without loss of generality we assume that x is the origin. Then we distinguish two cases.

Case I: If $\log(r_3/r_2) > \log(r_2/r_1)$ (meaning that r_2 is closer to r_1 than r_3) we rescale such that r_1 becomes the unit radius. Then we choose $\chi = 1$ on $B_{1.1r_3/r_1} \setminus B_{1.9}$, supported in $B_{1.9r_3\setminus r_1} \setminus B_{1.1}$, and to decay linearly to zero in $B_{1.1}$ and outside $B_{1.1r_3/r_1}$ so that $|D\chi| \leq C^{-1}$ in $B_{1.9} \setminus B_{1.1}$ and $|D\chi| \leq C(r_3/r_1)^{-1}$ in $B_{1.9r_3/r_1} \setminus B_{1.1r_3/r_1}$. Then we use Theorem 1.1 and rescale back with $r_1 > 0$ to see that for any $\varepsilon > 0$:

$$\varepsilon^2 \left(\frac{r_2}{r_1}\right)^{2\varepsilon} r_2^{-2\tau} \int_{B_{2r_2}\setminus B_{r_2}(x)} |f|^2 \leq C r_1^{-2\tau} \int_{B_{1.9r_1}\setminus B_{1.1r_1}(x)} |f|^2 + r_1^2 |Df|^2$$
$$+ C r_3^{-2\tau} \int_{B_{1.9r_3}\setminus B_{1.1r_3}(x)} |f|^2 + r_3^2 |Df|^2\,.$$

Taking a smooth test function $\phi = 1$ on $B_{1.9r_1} \setminus B_{1.1r_1}(x)$ and $\phi = 0$ outside $B_{2r_1} \setminus B_{r_1}(x)$ with $|d\phi| \leq C r_1^{-1}$ and similarly for r_3 and using the Cacciopoli inequality we can bound the gradient terms to get that:

$$\varepsilon^2 \left(\frac{r_2}{r_1}\right)^{2\varepsilon} r_2^{-2\tau} \int_{B_{2r_2}\setminus B_{r_2}(x)} |f|^2 \leq C r_1^{-2\tau} \int_{B_{2r_1}\setminus B_{r_1}(x)} |f|^2 + C r_3^{-2\tau} \int_{B_{2r_3}\setminus B_{r_3}(x)} |f|^2\,.$$

The desired estimate comes from optimizing $\varepsilon^2 \left(\frac{r_2}{r_1}\right)^{2\varepsilon}$ and plugging in $\varepsilon = \frac{1}{\sqrt{\ln(r_2/r_1)^2+1}}$.

Case II: If $\log(r_3/r_2) \leq \log(r_2/r_1)$ Then we rescale so that r_3 radius becomes unit scale and perform the same analysis. The result follows by adding the two possibilities. □

3. The vanishing order and its properties

This part of our paper is inspired by the work of [15], and we follow their strategy. We start by defining the vanishing order of a Dir-minimizing multivalued function as follows:

Definition 3.1 (Vanishing order)**.** Let $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ be a Dir-stationary multi-valued function. Then around any point $x \in B_1$ we define the vanishing degree κ_x as follows:

$$\kappa_{x,f} = \limsup_{r\to 0} \frac{\log\left(\fint_{B_{2r}\setminus B_r(x)} |f|^2\right)}{2\log(r)} = \lim_{r\to 0} \frac{\log\left(\fint_{B_{2r}\setminus B_r(x)} |f|^2\right)}{2\log(r)}$$

When it's clear from the context we will drop the subindex f.

Remark 3.2. We define the vanishing order around all points, however only collapsed points are relevant since non-collapsed points have 0 as their vanishing order. Also notice that if $f(x) = Q\,[\![y]\!]$, for some $y \in \mathbb{R}^n$, then a more meaningful definition of vanishing order could be obtained by replacing $|f|^2$ with $|f \ominus y|^2$ in the integral. However, later we will consider function with zero average, and so we will not change the definition here. Finally we observe that, if $f \equiv Q\,[\![0]\!]$ in a neighborhood of a point x, then clearly $\kappa_x = \infty$.

In the next two subsections we will prove that the vanishing order is well defined (i.e., the limit exists) and we will show some of its properties, namely upper semicontinuity and homogeneity of suitable sequences of blow-ups.

3.1. **The vanishing order and strong unique continuation.** In order to prove that the limit above is well defined we will need the following immediate corollary of Theorem 1.1.

Using the three sphere inequality first we show that:

Lemma 3.3 (Strong unique continuation)**.** *Let f be an average free, Dir-stationary functions on an open and connected domain Ω. If there exists a point $x \in \Omega$ such that $\kappa_x = +\infty$, then $f = Q\,[\![0]\!]$ in Ω. In Particular Theorem 1.2 is true.*

Proof. Take the set $S = \{x \in \Omega : \kappa_x = \infty\}$ and assume it is non-empty. We aim to show that S is either empty or $S = \Omega$. By contradiction, assume that $\Omega \neq S$. Then for any point $x \in S$ we have:

$$\limsup_{r \to 0} \frac{\log\left(\fint_{B_{2r}\setminus B_r(x)} |f|^2\right)}{2\log(r)} = \infty\,.$$

This means that the L^2-norm of f vanishes faster than any polynomial. Hence we can take $r_1 \to 0$ in the Carleman estimate of Theorem 1.1 to see that for any $\tau > 0$ and $r_1 \geq r_2$:

$$\int_{B_{2r_2}\setminus B_{r_2}(x)} |f|^2 \leq (\log(r_2)^2 + 1)\left(\frac{r_2}{r_1}\right)^{2\tau} \int_{B_{2r_1}\setminus B_{r_1}(x)} |f|^2\,.$$

We can take $\tau \to \infty$ to see that for all $r < \frac{\mathrm{dist}(x,\partial\Omega)}{2}$:

$$\int_{B_{2r}\setminus B_r(x)} |f|^2 = 0\,.$$

Hence we see that $f(x) = Q[\![0]\!]$ and $\kappa_x = \infty$ for all $x \in B_{\mathrm{dist}(x,\partial\Omega)}(x)$. Hence we have just shown that:

$$\text{for all } x \in S \text{ we have } B_{\mathrm{dist}(x,\partial\Omega)}(x) \subset S\,.$$

Since Ω is connected, we can find a continuous path $\gamma : [0,1] \to \Omega$ between any two points $\gamma(0) = x \in S$ and $\gamma(1) = y \in \Omega \setminus S$. Now take $r_0 := \inf_{0\leq t\leq 1} \mathrm{dist}(\gamma(t), \partial\Omega)$. Note that since $\gamma([0,1])$ is closed, we have $r_0 > 0$. Consider $T = \sup\{0 \leq t \leq 1 : \gamma(t) \in S\}$ and note that $T < 1$ since $\gamma(1) \notin S$. But since we know that $B_{r_0}(\gamma(T)) \subset B_{\mathrm{dist}(\gamma(T),\partial\Omega)}(\gamma(T)) \subset S$ and by continuity we can verify that for small enough ε we have $\gamma(T+\varepsilon) \in B_{r_0}(\gamma(T))$, so that $f \equiv Q\,[\![0]\!]$ in a neighborhood of $\gamma(T+\varepsilon)$ and therefore $\gamma(T+\varepsilon) \in S$. This is in contradiction with the definition of T and our claim follows. Hence $S = \Omega$ and $f = Q[\![0]\!]$ in Ω.

Notice that Theorem 1.2 follows since the assumption therein guarantees that $f = Q\,[\![0]\!]$ and moreover that $\kappa_{0,f} = \infty$. □

Having ruled out the case $\kappa_x = \infty$, we can show that for a nontrivial Dir-stationary multivalued function κ_x is well defined.

Lemma 3.4. *Let $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ be a nontrivial, Dir-stationary multivalued function. Then for every $x \in \Omega$, the vanishing order κ_x is well defined as*

$$\kappa_x = \lim_{r \to 0} \frac{\log\left(\fint_{B_{2r}\setminus B_r(x)} |f|^2\right)}{2\log(r)}$$

Proof. For any point x take a sequence of radii $\{r_j\}_{j=1}^{\infty}$ realizing the lim-sup $\kappa_x < \infty$. Taking j large enough so that

$$\frac{\log\left(\fint_{B_{2r_j}\setminus B_{r_j}(x)} |f|^2\right)}{2\log(r_j)} \geq \kappa_x - \frac{\varepsilon}{2}$$

we see, since $\log(r_j) \leq 0$, that

$$\fint_{B_{2r_j}\setminus B_{r_j}(x)} |f|^2 \leq r_j^{2\kappa_x - \varepsilon}\,.$$

Let $r > 0$ be such that $r_{j-1} \geq r \geq r_j$. Then the three sphere inequality for $\tau = \kappa_x + \frac{n}{2} - \varepsilon$ states

$$\frac{\fint_{B_{2r}\setminus B_r(x)} |f|^2}{(\log(r)^2+1)r^{2\kappa_x-2\varepsilon}} \leq C\frac{\fint_{B_{2r_j}\setminus B_{r_j}(x)} |f|^2}{r_j^{2\kappa_x-2\varepsilon}} + C\frac{\fint_{B_{2r_{j-1}}\setminus B_{r_{j-1}}(x)} |f|^2}{r_{j-1}^{2\kappa_x-2\varepsilon}}\,.$$

Hence we see that for all $j > j_0(\varepsilon)$ large enough and r as above:

$$\frac{\fint_{B_{2r}\setminus B_r(x)} |f|^2}{(\log(r)^2+1)r^{2\kappa_x-2\varepsilon}} \leq C\, r_{j-1}^{\varepsilon} \leq C\, r^{\varepsilon}\,.$$

Taking logarithm, for r sufficiently small we have

$$\frac{\log\left(\fint_{B_{2r}\setminus B_r(x)} |f|^2\right)}{2\log(r)} \geq \kappa_x - \varepsilon - C\frac{\log(\log(r))}{\log(r)} \geq \kappa_x - 2\varepsilon\,.$$

This means that

$$\kappa_x = \lim_{r\to 0} \frac{\log\left(\fint_{B_{2r}\setminus B_r(x)} |f|^2\right)}{2\log(r)}\,.$$

and we are done. □

Remark 3.5 (Frequency and vanishing order coincide)**.** As a consequence of [5, EQ (3.42)] and Lemma 3.4, for Dir-stationary multi-valued functions, the vanishing order and the frequency (defined via a linear cut-off as in [9, Definition 3.1]) coincide:

$$\kappa_{x,f} = I_{x,f}\,. \tag{3.1}$$

Note that the same conclusion of [5, EQ (3.42)] applies, hence integrating from s to t yields:

$$\log\left(\frac{H(r)}{r^{n-1}}\right) - \log\left(\frac{H(s)}{s^{n-1}}\right) = \int_s^r \frac{2I(\tau)}{\tau} d\tau$$

For $s \leq r \leq \delta$ small enough, we know that the frequency is saturated $|I(r) - I_{x,f}| \leq \varepsilon$, hence

$$\log\left(\frac{H(r)}{r^{n-1}}\right) - \log\left(\frac{H(s)}{s^{n-1}}\right) = (2I_{x,f} + O(\varepsilon))\log\left(\frac{r}{s}\right)\,.$$

By the definition of vanishing order, for small enough $\delta > 0$:

$$\log\left(\frac{H(r)}{r^{n-1}}\right) - \log\left(\frac{H(s)}{s^{n-1}}\right) = (2\kappa_{x,f} + O(\varepsilon))\log\left(\frac{r}{s}\right)\,.$$

This shows that:

$$|I_{x,f} - \kappa_{x,f}| = O(\varepsilon)\,,$$

for all $\varepsilon > 0$ and we conclude.

3.2. **Properties of the vanishing order.** In this section we will show upper-semi continuity of κ_x, and indeed that κ_x is uniformly bounded on B_1, and also study the homogeneity of subsequential blow-up limits.

Lemma 3.6. *Let f be a Dir-stationary multi-valued functions with zero average, then vanishing degree $x \mapsto \kappa_x$ is upper-semi continuous.*

Proof. We can compare close-by points by using the three-sphere inequality. Without loss of generality via a translation we only need to prove upper-semi continuity for the origin, that is we aim to show that for every $\varepsilon > 0$ there is $\delta > 0$ such that $\kappa_0 \geq \kappa_x - \varepsilon$ for any $x \in B_\delta$. This will follow if we can show that for any $x \in B_\delta$ there exists $r_0(x, \varepsilon) > 0$ such that $\forall r \leq r_0(x, \varepsilon)$ we have

$$\int_{B_{2r} \setminus B_r(x)} |f|^2 \geq r^{2\kappa_0 + n + 2\varepsilon}\,.$$

To achieve this, note that for any three radii $r \ll K_1\delta \leq K_2\delta$ with large $2 < K_1 < K_2/2$ (to be chosen later), Corollary 2.1 with $\tau = \kappa_0 + \frac{n}{2} + \varepsilon$ says:

$$\frac{\fint_{B_{K_1\delta} \setminus B_{\frac{K_1\delta}{2}}(x)} |f|^2}{(\log(K_1/K_2)^2 + 1)(K_1\delta)^{2\kappa_0 + 2\varepsilon}} - C\frac{\fint_{B_{K_2\delta} \setminus B_{\frac{K_2\delta}{2}}(x)} |f|^2}{(K_2\delta)^{2\kappa_0 + 2\varepsilon}} \leq C\frac{\fint_{B_{2r} \setminus B_r(x)} |f|^2}{r^{2\kappa_0 + 2\varepsilon}}\,. \tag{3.2}$$

Note that here we take r so small so that $\log(K_2/K_1) \leq \log(K_1\delta/r)$. Now for $x \in B_\delta$ and large enough K_1 we have that $B_{\frac{4}{5}K_1\delta} \setminus B_{\frac{3}{5}K_1\delta}(0) \subset B_{K_1\delta} \setminus B_{\frac{K_1\delta}{2}}(x)$, hence:

$$\fint_{B_{K_1\delta} \setminus B_{\frac{K_1\delta}{2}}(x)} |f|^2 \geq \fint_{B_{\frac{4}{5}K_1\delta} \setminus B_{\frac{3}{5}K_1\delta}(0)} |f|^2 \geq C(K_1\delta)^{2\kappa_0 + \varepsilon}\,, \tag{3.3}$$

for small enough $\delta > 0$. This follows from the vanishing order definition. Similarly, for large enough K_2 and small enough δ we also have that $B_{\frac{4}{5}K_2\delta} \setminus B_{\frac{3}{5}K_2\delta}(x) \subset B_{K_2\delta} \setminus B_{\frac{K_2\delta}{2}}(0)$, hence we see that:

$$\fint_{B_{\frac{4}{5}K_2\delta} \setminus B_{\frac{3}{5}K_2\delta}(x)} |f|^2 \leq \fint_{B_{K_2\delta} \setminus B_{\frac{K_2\delta}{2}}(0)} |f|^2 \leq C(K_2\delta)^{2\kappa_0 - \varepsilon}\,. \tag{3.4}$$

Putting together Eqs. (3.3) and (3.4) with the three sphere inequality Eq. (3.2), we conclude that:

$$C\frac{\fint_{B_{2r} \setminus B_r(x)} |f|^2}{r^{2\kappa_0 + 2\varepsilon}} \geq C\left(\frac{(K_1\delta)^{-\varepsilon}}{\log(K_2/K_1)^2} - (K_2\delta)^{-3\varepsilon}\right)\,.$$

We can take K_2 large enough such that $\frac{(K_1\delta)^{-\varepsilon}}{\log(K_2/K_1)^2} > (K_2\delta)^{-3\varepsilon}$ and we conclude that:

$$C\frac{\fint_{B_{2r} \setminus B_r(x)} |f|^2}{r^{2\kappa_0 + 2\varepsilon}} \geq C(K_1, K_2, \delta, \varepsilon)\,,$$

for all $r > 0$ sufficiently small. Since the right hand side is independent of r, we take a logarithm, and conclude that:

$$\kappa_x \leq \kappa_0 + \varepsilon\,.$$

This is indeed the desired conclusion. We see in fact that for any $x \in B_1$:

$$\kappa_x \geq \limsup_{y \to x} \kappa_y\,.$$

$\square$

The upper semi-continuity of Lemma 3.6 together with strong unique continuation in Lemma 3.3 imply that the vanishing order κ_x is uniformly bounded in x.

In the next proposition, we prove that around any point $x \in B_1$ there exists a sequence of radii such that the solution becomes κ_x homogeneous along the sequence.

Theorem 3.7. *Let $f \in W^{1,2}(B_1; \mathcal{A}_Q)$ be a Dir-stationary multi-valued function. Then around any point x there exists a sequence of radii $r_j \to 0$ and vanishing constants $\varepsilon_j \to 0$ such that:*

$$\int_{B_{2r_j}\setminus B_{r_j}(x)} \sum_{i=1}^{Q} |Df_i.x - \kappa_x f_i|^2 \le \varepsilon_j \int_{B_{2r_j}\setminus B_{r_j}(x)} \sum_{i=1}^{Q} |f_i|^2 . \tag{3.5}$$

Proof. It is enough to prove the result for the origin. The proof is by contradiction. Indeed assume that there exists $\varepsilon_0 > 0$ such that for all small radii $r \le r_0$ we have:

$$\int_{B_{2r}\setminus B_{r}(x)} \sum_{i=1}^{Q} |Df_i.x - \kappa_x f_i|^2 \ge \varepsilon_0 \int_{B_{2r}\setminus B_{r}(x)} \sum_{i=1}^{Q} |f_i|^2 . \tag{3.6}$$

The idea is from [15] and is as follows. Instead of the precise weights used in Theorem 1.1, we use the following

$$\text{inner variations: } xe^{-2\tau\phi(\log(|x|))} \text{ \& outer variation: } ue^{-2\tau\phi(\log(|x|))} .$$

Take a compactly supported function $\chi \in C_c^\infty(B_1 \setminus \{0\})$; The inner variation Eq. (1.2) implies:

$$\begin{aligned}
0 &= \mathcal{I}\left(f, \chi x e^{-2\tau\phi(\log(|x|))}\right) \\
&= \int_{B_1} \chi \frac{n - 2\tau\phi' - 2}{2} \sum_{i=1}^{Q} \left(|Df_i|^2 + 2\tau|\partial_r f_i|^2\right) e^{-2\tau\phi(\log(|x|))} \\
&+ \int_{B_1} \sum_{i=1}^{Q} \left((D\chi.x)\frac{|Df_i|^2}{2} - (D\chi.Df_i)(Df_i.x)\right) e^{-2\tau\phi(\log(|x|))} .
\end{aligned} \tag{3.7}$$

For the outer variation Eq. (1.1), with the same smooth cut-off χ we see that:

$$\begin{aligned}
0 &= \mathcal{O}(f, \chi u e^{-2\tau\phi(\log(|x|))}) \\
&= \int_{B_1} \chi \sum_{i=1}^{Q} \left(|Df_i|^2 - 2\tau\phi' \frac{\partial_r f_i f_i}{|x|}\right) e^{-2\tau\phi(\log(|x|))} \\
&+ \int_{B_1} \sum_{i=1}^{Q} (D\chi.Df_i) f_i e^{-2\tau\phi(\log(|x|))} .
\end{aligned} \tag{3.8}$$

Then we multiply Eq. (3.8) by $\eta = \frac{2\tau - n + 2}{2}$ and add to Eq. (3.8) to get the following inequality:

$$\begin{aligned}
&\int_{B_1} \chi \sum_{i=1}^{Q} \left(|\partial_r f_i|^2 - \eta^2 \frac{|f_i|^2}{|x|^2}\right) e^{-2\tau\phi(\log(|x|))} \\
\le C &\int_{B_1} |D\chi| \sum_{i=1}^{Q} \left(|x||Df_i|^2 + \frac{|f_i|^2}{|x|}\right) e^{-2\tau\phi(\log(|x|))} \\
&+ C\|1 - \phi'\|_\infty \int_{B_1} \chi \sum_{i=1}^{Q} \left(|Df_i|^2 + \frac{|f_i|^2}{|x|^2}\right) e^{-2\tau\phi(\log(|x|))} .
\end{aligned}$$

Performing the same integration by parts in the proof of Theorem 1.1 in Eq. (2.6) we see that:

$$\begin{aligned}&\int_{B_1}\chi\sum_{i=1}^{Q}\left|\partial_r f_i-\frac{f_i}{|x|}\right|^2 e^{-2\tau\phi(\log(|x|))}\\&\le C\int_{B_1}|D\chi|\sum_{i=1}^{Q}\left(|x||Df_i|^2+\frac{|f_i|^2}{|x|}\right)e^{-2\tau\phi(\log(|x|))}\\&+C\left(\|1-\phi'\|_\infty+\|\phi''\|_\infty\right)\int_{B_1}\chi\sum_{i=1}^{Q}\left(|Df_i|^2+\frac{|f_i|^2}{|x|^2}\right)e^{-2\tau\phi(\log(|x|))}.\end{aligned}\tag{3.9}$$

Now we use the contradiction assumption (Eq. (3.6)). The idea is to *bend* slightly the graph of $\phi(t)$ with a convexity of size $\varepsilon_0>0$. With that we gain a three-sphere inequality with $\varepsilon_0>0$ more weight for the left hand side. Using this we gain a contradiction with the fact that the vanishing order is a limit in Lemma 3.4. Now for any two radii $r_1\ll r_2\le r_0(x)$ we introduce ϕ_δ as follows:

$$\begin{cases}\phi_\delta(t)\ge(1-\delta)t & \text{for } \log(r_1)\le t\le\log(2r_1) \text{ or } \log(r_2)\le t\le\log(2r_2)\,,\\ \phi_\delta(t)\le(1+2\delta)t & \text{for } \log(\sqrt{r_1r_2})\le t\le\log(2\sqrt{r_1r_2})\,,\\ |\phi'|+|\phi''|\le C\delta\,. &\end{cases}$$

Moreover we put:

$$\tau=\frac{2\kappa_x+n-2}{2}\Rightarrow\eta=\frac{2\tau-n+2}{2}=\kappa_x$$

Now using Eq. (3.9) and Eq. (3.6) and the Cacciopoli inequality on dyadic annuli between r_1 and r_2 we see that:

$$\int_{B_1}\chi\frac{|f|^2}{|x|^2}e^{-2\tau\phi_\delta(\log(|x|))}\le\frac{C}{\varepsilon_0-C\delta}\int_{B_1}(|D\chi|+|D^2\chi|)\frac{|f|^2}{|x|^2}e^{-2\tau\phi_\delta(\log(|x|))}$$

Now take χ to be the smooth cut-off such that $\chi=1$ on $B_{r_2}\setminus B_{2r_1}(x)$ and it linearly decreases to 0 on $B_{r_1}(x)$ and $(B_{2r_2}(x))^c$. Then we can see that for $\delta=c\varepsilon_0$ for small enough $c>0$, there exists a constant $C(\varepsilon_0,x)$ such that:

$$\frac{\int_{B_{2\sqrt{r_1r_2}}\setminus B_{\sqrt{r_1r_2}}(x)}|f|^2}{\sqrt{r_1r_2}^{2\kappa_x+2c\varepsilon_0+n}}\le C(\varepsilon_0,x)\left[\frac{\int_{B_{2r_1}\setminus B_{r_1}(x)}|f|^2}{r_1^{2\kappa_x-c\varepsilon_0+n}}+\frac{\int_{B_{2r_2}\setminus B_{r_2}(x)}|f|^2}{r_2^{2\kappa_x-c\varepsilon_0+n}}\right].$$

By the definition of vanishing order κ_x, for any $\eta>0$ there exists $r_0(\eta,x)$ such that for all radii $r\le r_0(\eta,x)$ we have:

$$r^{\kappa_x+\eta}\le\left(\fint_{B_{2r}\setminus B_r(x)}|f|^2\right)^{\frac{1}{2}}\le r^{\kappa_x-\eta}.$$

Combining the last two displays, we arrive at:

$$\sqrt{r_1r_2}^{2\eta-2c\varepsilon_0}\le C(\varepsilon_0)\left(r_1^{c\varepsilon_0-2\eta}+r_2^{c\varepsilon_0-2\eta}\right).$$

We can take η small enough so that $2\eta-2c\varepsilon_0<0$ and $c\varepsilon_0-2\eta>0$, hence the right hand side becomes bounded by $C(\varepsilon_0)$. Then taking $r_1\ll r_2$ small enough we reach a contradiction and conclude. □

Remark 3.8. The previous proof actually implies that for every sequence of radii $r_j\to 0$ there exists a subsequence $r_{j_k}\to 0$ and a sequence of constants $\varepsilon_{j_k}\to 0$ for which (3.5) holds.

4. Proof of the dimensional bound: Theorem 1.3

The dimensional bound in Theorem 1.3 follows as in [5, Proof of Theorem 0.11] from the following Lemma.

Lemma 4.1. *Let Ω be connected and $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ be Dir-minimizing. Then, either $f = Q[\![\xi]\!]$ with $\xi\colon \Omega \to \mathbb{R}^n$ harmonic in Ω, or the set*

$$\Sigma_{Q,f} := \{x \in \Omega \,:\, f(x) = Q[\![y]\!]\,,\, y \in \mathbb{R}^n\}$$

(which is relatively closed in Ω since f is continuous) has Hausdorff dimension at most $n-2$ and it is locally finite for $n = 2$.

First we notice that we can assume $\xi, y \equiv 0$ in Lemma 4.1, since we can subtract the average of f by [5, Lemma 3.23]. Next we define the blow-up sequence

$$f_{y,\rho}(x) := \frac{\rho^{\frac{n}{2}} f(y+\rho x)}{\sqrt{\int_{B_\rho(y)} |f|^2}}$$

The proof of Lemma 4.1 will then follow as in the proof of [5, Proposition 3.22] replacing Theorem 3.19 therein with the following

Lemma 4.2. *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^m))$ be Dir-minimizing. Assume $f(0) = Q[\![0]\!]$ and $\|f\|_{L^2(B_\rho))} > 0$ for every $\rho \le 1$. Then, for any sequence $(f_{\rho_k})_k$, with $\rho_k \downarrow 0$, a subsequence, not relabeled, converges locally uniformly to a function $g\colon \mathbb{R}^n \to \mathcal{A}_Q(\mathbb{R}^m)$ with the following properties:*

1. *$\|g\|_{L^2(B_1)} = 1$ and $g|_\Omega$ is Dir-minimizing for any bounded Ω;*
2. *$g(x) = |x|^\alpha g\left(\frac{x}{|x|}\right)$, where $\alpha = k_{0,f} > 0$ is the vanishing order of f at 0.*

Before we prove Lemma 4.2, we need the following doubling estimate to guarantee that the sequence f_ρ is uniformly bounded.

Proposition 4.3 (Doubling estimate)**.** *Let $f \in W^{1,2}(B_1^n, \mathcal{A}_Q)$ be a Dir-stationary multi-function. Then for any point $x \in B_1$ there exists a radius $r_x > 0$ and a constant C_x such that for all radii $r \le r_x$ we have:*

$$\int_{B_{2r}(x)} |f|^2 \le C_x \int_{B_r(x)} |f|^2\,.$$

Proof. It is enough to prove that:

$$\int_{B_{2r}\setminus B_r(x)} |f|^2 \le C_x \int_{B_r(x)} |f|^2\,.$$

The strategy is to use the three sphere inequality for the three radii $\varepsilon \le 2\varepsilon \ll r_3$ for $r_3 > 0$ to be chosen later:

$$C\frac{\int_{B_{2\varepsilon}\setminus B_\varepsilon(x)} |f|^2}{(2\varepsilon)^{2\tau}} \le \frac{\int_{B_\varepsilon\setminus B_{\varepsilon/2}(x)} |f|^2}{\varepsilon^{2\tau}} + \frac{\int_{B_{2r_3}\setminus B_{r_3}(x)} |f|^2}{r_3^{2\tau}}$$

Multiplying and rearranging we see that:

$$\int_{B_{2\varepsilon}\setminus B_\varepsilon(x)} |f|^2 \le C2^{2\tau} \int_{B_\varepsilon\setminus B_{\varepsilon/2}(x)} |f|^2 + C\left(\frac{\varepsilon}{r_3}\right)^{2\tau} \int_{B_{2r_3}\setminus B_{r_3}(x)} |f|^2\,. \tag{4.1}$$

Now by Lemma 3.4, we know that for any $\eta > 0$ there exists a radius $r_0(\eta, x)$ small enough such that for all radii $r \le r_0(\eta, x)$ we have that:

$$r^{2\kappa_x+n+\eta} \le \int_{B_{2r}\setminus B_r(x)} |f|^2 \le r^{2\kappa_x+n-\eta}\,.$$

We take $r_3 \leq r_0(\eta, x)$, and we see that:

$$C\left(\frac{\varepsilon}{r_3}\right)^{2\tau} \int_{B_{2r_3}\setminus B_{r_3}(x)} |f|^2 \leq \left(\frac{\varepsilon}{r_3}\right)^{2\tau} r_3^{2\kappa_x+n-\eta}$$

Take $\tau = \frac{2\kappa_x+n+4\eta}{2}$ and we see that the above display reads:

$$C\left(\frac{\varepsilon}{r_3}\right)^{2\tau} \int_{B_{2r_3}\setminus B_{r_3}(x)} |f|^2 \leq \varepsilon^{2\kappa_x+n+3\eta} \leq \varepsilon^{2\eta} \int_{B_{2\varepsilon}\setminus B_{\varepsilon}(x)} |f|^2 .$$

Taking $\varepsilon > 0$ small enough we see that we can we can reabsorb the second term on the right hand side of Eq. (4.1) and end up with the desired estimate:

$$\int_{B_{2\varepsilon}\setminus B_{\varepsilon}(x)} |f|^2 \leq C(x,n) \int_{B_{\varepsilon}(x)} |f|^2 .$$

□

Proof of Lemma 4.2. We consider any ball B_N of radius N centered at 0. It follows from Proposition 4.3 and the Caccioppoli inequality Eq. (2.9) that $\|f_\rho\|_{W^{1,2}(B_N)}$ is uniformly bounded in ρ. Hence, the functions f_ρ are all Dir-minimizing and [5, Theorem 3.9] implies that they are locally equi-Hölder continuous. Since $f(0) = Q\,[\![0]\!]$, the f_ρ's are also locally uniformly bounded and the Ascoli–Arzelá theorem yields a subsequence (not relabeled) converging uniformly on compact subsets of $\mathbb{R}^n$ to a continuous Q-valued function g. This implies easily the weak convergence (see [5, Definition 2.9]), so we can apply [5, Proposition 3.20] and conclude (1) (note that $\|f_\rho\|_{L^2(B_1)} = 1$ for every ρ).

Next notice that up to choosing a further subsequence g is $k_{0,f}(0)$-homogeneous by Lemma 4.2. Finally, assume by contradiction that $k_{0,f}(0) = 0$. Then, by what shown so far, the blowups converge to a continuous 0-homogeneous function g, with $g(0) = Q\,[\![0]\!]$. This implies that $g \equiv Q\,[\![0]\!]$, a contradiction to $\|g\|_{L^2(B_1)} = 1$. □

5. The 2-dimensional case

For the reader's convenience we give a proof of the 2-dimensional case combining the monotonicity of the Weiss energy with the reasoning above on the vanishing order of a Dir-minimizing multivalued map. We start with the following:

Proposition 5.1 (Weiss' monotonicity formula). *If $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ and $B_r(x) \subset \Omega$, then we define the* Weiss Boundary adjusted energy *by*

$$W_{x,f}(r) = W(f_{x,r}) := \frac{1}{r^{m+2\kappa_{x,f}-2}} \int_{B_r(x)} |Df|^2 - \frac{\kappa_{x,f}}{r^{m+2\kappa_{x,f}-1}} \int_{\partial B_r(x)} |f|^2 .$$

Then the map $r \mapsto W_{x,f}(r)$ is absolutely continuous and

$$\frac{d}{dr} W_{x,f}(r) = \frac{m+2\kappa_{x,f}-2}{r}\left(W(f_{x,r}^{\kappa_{x,f}}) - W(f_{x,r})\right) + \frac{1}{r}\int_{\partial B_1} \sum_{i=1}^{Q} |(Df_{x,r})_i \cdot x - \kappa_{x,f}(f_{x,r})_i|^2 , \tag{5.1}$$

where $f_{x,r}^{\kappa}(y) := |y|^\kappa f_{x,r}(x/|x|)$ is the κ-homogeneous extension of the trace of $f_{x,r}$ in B_1. In particular if f is Dir-minimizing in Ω then

(1) *$\frac{d}{dr} W_{x,f}(r) \geq 0$ and so there exists $W_{x,f}(r) \geq W_{x,f}(0) = \lim_{r\downarrow 0} W_{x,f}(r) = 0$;*
(2) *$\frac{d}{dr} W_{x,f}(r) \equiv 0$ if and only if $f_{x,r}$ is $\kappa_{x,f}$-homogeneous.*

Proof. The proof of the above is standard and can be found for instance in [19, Section 9] (adjusting the constants therein to account for κ). In particular notice $W_{x,f}(0) = 0$ follows from Lemma 4.2 since $W_{x,f}(0) = W_{0,g}(1)$, where g is a $\kappa_{x,f}$ homogeneous function and so $W_{0,g}(1) = 0$. □

Next we have the following standard epiperimetric inequality.

Lemma 5.2 (Epiperimetric Inequality)**.** *There is a positive constant δ, depending only on Q, with the following property. Assume $f \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^m))$, $B_1 \subset \mathbb{R}^2$, is* Dir*-minimizing. Then, for f^κ, the κ homogeneous extension of f, we have*

$$\int_{B_1} \left(|Df^\kappa|^2 - |Df|^2\right) \geq \delta\, W_{0,f}(1)\,. \tag{5.2}$$

Proof. In short, we construct competitors by unwinding the boundary conditions, extending harmonically inside and rewinding again. We use the same notation as in [5, Proposition 5.2]; precisely, fix a radius r and let $f(re^{i\theta}) = g(\theta) = \sum_{j=1}^{J} [\![g_j(\theta)]\!]$ be an irreducible decomposition as in [5, Proposition 1.5]. Then for each g_j we can find $\gamma_j : S^1 \to \mathbb{R}^n$ such that:

$$g_j(\theta) = \sum_{i=1}^{Q_j} \left[\!\!\left[\gamma_i\left(\frac{\theta + 2\pi i}{Q_j}\right)\right]\!\!\right].$$

Now take the Fourier decomposition of γ_j:

$$\gamma_j(\theta) = \frac{a_{j,0}}{2} + \sum_{\ell=1}^{\infty} \left[a_{j,\ell}\sin(\ell\theta) + b_{j,\ell}\cos(\ell\theta)\right],$$

and its harmonic extension:

$$\zeta_j(\rho,\theta) = \frac{a_{j,0}}{2} + \sum_{\ell=1}^{\infty} \rho^\ell \left[a_{j,\ell}\sin(\ell\theta) + b_{j,\ell}\cos(\ell\theta)\right].$$

Calculating the Dirichlet energy as in [5, EQ (5.18)], we obtain:

$$\int_{B_r} |Df|^2 \leq \sum_j \mathrm{Dir}(\zeta_j, B_r) = \pi \sum_j \sum_\ell \ell r^{2\ell}\left(|a_{j,\ell}|^2 + |b_{j,\ell}|^2\right).$$

We also calculate the Dirichlet energy for the κ-homogeneous extension:

$$\int_{B_r} |Df^\kappa|^2 = \sum_j \mathrm{Dir}(\gamma_j^\kappa, B_r) = \pi r^{2\kappa} \sum_j \sum_\ell \left[\frac{\kappa}{2} + \frac{\ell^2}{2\kappa}\right]\left(|a_{j,\ell}|^2 + |b_{j,\ell}|^2\right).$$

Hence we can estimate for $r = 1$:

$$\int_{B_1} |Df^\kappa|^2 - |Df|^2 \geq \pi \sum_j \sum_\ell \left[\frac{\kappa}{2} + \frac{\ell^2}{2\kappa} - \ell\right]\left(|a_{j,\ell}|^2 + |b_{j,\ell}|^2\right). \tag{5.3}$$

Now we calculate the Weiss energy using competitors:

$$W(f_{x,1}) \leq \sum_j \mathrm{Dir}(\zeta_j, B_1) - \kappa \int_{\partial B_1} |f|^2 \leq \pi \sum_j \sum_\ell \left[\ell - \kappa Q_j\right]\left(|a_{j,\ell}|^2 + |b_{j,\ell}|^2\right). \tag{5.4}$$

Since $Q_j \geq 1$, it is enough to show that there exists some δ such that:

$$\frac{\ell^2}{2\kappa} + \frac{\kappa}{2} - \ell \geq \delta(\ell - \kappa)\,. \tag{5.5}$$

It is easy to verify that the following choice for δ satisfies Eq. (5.5)

$$\delta = \frac{\lfloor \kappa \rfloor + 1 - \kappa}{2\kappa}$$

Putting together Eqs. (5.3) and (5.4), we conclude that:

$$\int_{B_1} |Df^\kappa|^2 - |Df|^2 \geq \delta W_{0,f}(1)\,.$$

□

Combining the two propositions above we have the following

Lemma 5.3 (Uniqueness of tangent map)**.** *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^m))$ be a Dir-minimizing Q-valued functions, with $\mathrm{Dir}(f, B_1) > 0$ and $f(0) = Q[\![0]\!]$. Then, the maps $f_{x,r}$ converge locally uniformly to a unique tangent mat g.*

Proof. The proof follows from a standard reasoning, see for instance [19, Lemma 12.14 & Proposition 2.14]. □

The proof of the 2-dimensional case of Theorem 1.3 then follows in the same way as in [5, Subsection 5.3], replacing Theorem 5.3 therein with Lemma 5.3.

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