COUNTEREXAMPLE TO THE TWO-ENDS FURSTENBERG CONJECTURE IN $\mathbb{R}^3$

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Wang and Wu [1] made significant progress on the restriction conjecture by combining refined decoupling estimates with incidence estimates for tubes. They also introduced the *two-ends Furstenberg conjecture* in incidence geometry and showed that it would imply the restriction conjecture. They proved this conjecture in $\mathbb{R}^2$. In this note, we provide a counterexample to the two-ends Furstenberg conjecture in $\mathbb{R}^3$.

1. Two-Ends Furstenberg Conjecture

Let $\mathbb{T}$ denote a collection of $\delta \times \delta \times 1$ tube segments contained inside the ball $B(10) \subset \mathbb{R}^3$. We assume $\mathbb{T}$ is *essentially distinct*, meaning

$$\mathrm{Vol}(T_1 \cap T_2) \leq \frac{1}{2}\mathrm{Vol}(T_1) \qquad \text{for } T_1 \neq T_2 \in \mathbb{T}.$$

We define the direction of a tube, $\theta(T) \in S^2$, to be the direction of the core line. When we say δ-tubes, we really mean $\delta \times \delta \times 1$ tube segments.

For each $T \in \mathbb{T}$, we associate a shading $Y(T) \subset T$. We say $Y(T)$ is $(\varepsilon_1, \varepsilon_2, C)$-*two-ends* if for all $\delta \times \delta \times \delta^{\varepsilon_1}$ tube segments $J \subset T$,

$$\mathrm{Vol}(Y(T) \cap J) \leq C\delta^{\varepsilon_2}\,\mathrm{Vol}(Y(T)).$$

Conjecture 1.1 (Two-Ends Furstenberg Conjecture in $\mathbb{R}^3$ [1, Conjecture 0.9])**.** *Let $\mathbb{T}$ be a collection of δ-tube segments, the directions of which are δ-separated. For each $T \in \mathbb{T}$, let $Y(T)$ be an $(\varepsilon_1, \varepsilon_2)$-two-ends shading, with* $\mathrm{Vol}(Y(T))$ *constant over $T \in \mathbb{T}$. Then for all $\varepsilon > 0$,*

$$\mathrm{Vol}\big(\bigcup_{T\in\mathbb{T}} Y(T)\big) \gtrsim_\varepsilon \delta^\varepsilon \delta^{C\varepsilon_1}\,\#\mathbb{T}\,\mathrm{Vol}(T)\,\big(\frac{\mathrm{Vol}(Y(T))}{\mathrm{Vol}(T)}\big)^2.$$

Implicitly, the constants in this theorem depend on the constant in the $(\varepsilon_1, \varepsilon_2)$-two-ends shading.

Wang and Zahl [2] recently proved the Kakaya conjecture. They showed that if $\mathbb{T}$ is a collection of δ-tubes the directions of which are δ-separated, and $Y(T)$ is a shading of constant density, then

$$\mathrm{Vol}\big(\bigcup_{T\in\mathbb{T}} Y(T)\big) \gtrsim_\varepsilon \delta^\varepsilon \#\mathbb{T}\,\mathrm{Vol}(T)\,\big(\frac{\mathrm{Vol}(Y(T))}{\mathrm{Vol}(T)}\big)^C. \tag{1.1}$$

for some large fixed power C.

Actually, Wang and Zahl proved (1.1) under the superficially weaker hypothesis that the tubes are Katz–Tao Convex–Wolff. The Katz–Tao Convex–Wolff constant of $\mathbb{T}$ is defined as

$$C_{KT-CW}(\mathbb{T}) = \sup_{U \text{ a convex set}} \frac{\#\{T \in \mathbb{T} : T \subset U\}}{\mathrm{Vol}(U)/\mathrm{Vol}(T)}, \qquad \text{where } \mathrm{Vol}(T) = \delta^2.$$

If the tubes of $\mathbb{T}$ have δ-separated directions, $C_{KT-CW}(\mathbb{T}) \lesssim 1$.

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Zakharov [3] used Wang–Zahl's theorem to show that, if $C_{KT-CW}(\mathbb{T}) \lesssim 1$, then after a random projective transformation, $\mathbb{T}$ becomes directionally separated. Thus in Conjecture 1.1, it is equivalent to make the hypothesis $C_{KT-CW}(\mathbb{T}) \lesssim 1$ instead of the hypothesis that $\mathbb{T}$ is directionally separated.

Here is his argument. Suppose $C_{KT-CW}(\mathbb{T}) \lesssim 1$. After a rotation and discarding a constant-sized subfamily, we may assume that all tubes lie within $1/10$ radians of the vertical axis. By Wang–Zahl's theorem,

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrsim \delta^{\varepsilon} \#\mathbb{T}\,\mathrm{Vol}(T).$$

By disintegration,

$$\delta^{\varepsilon} \lessapprox \mathrm{Vol}(\cup\mathbb{T}) = \int_{z_0=-10}^{10} \mathrm{Area}(\cup\mathbb{T} \cap \{z = z_0\})\, dz_0.$$

Hence there exists $z_0 \in [-10, 10]$ such that, letting $H := \{z = z_0\}$,

$$\mathrm{Area}(\cup\mathbb{T} \cap H) \gtrapprox \delta^{\varepsilon} \#\mathbb{T}\,\mathrm{Vol}(T).$$

Each tube intersects H in an ellipse comparable to a δ-ball. Let ℓ_T be the core line of T, and define

$$E_0 := \{\ell_T \cap H : T \in \mathbb{T}\}.$$

Let $|E_0|_\delta$ denote the δ-covering number. Since $\mathrm{Vol}(\cup\mathbb{T} \cap H) \approx |E_0|_\delta$, we get

$$|E_0|_\delta \gtrapprox \delta^{\varepsilon} \#\mathbb{T}.$$

Thus we may choose a δ-separated subset of E_0 of size $\gtrapprox \delta^{\varepsilon}\#\mathbb{T}$. Let $\mathbb{T}' \subset \mathbb{T}$ be the corresponding family of tubes.

Choose a ball B of radius $1/100$, whose distance to H is at least $1/10$, and define

$$\mathbb{T}'' := \{T \in \mathbb{T}' : \mathrm{Vol}(Y(T) \cap B) \gtrapprox \mathrm{Vol}(Y(T))\}.$$

We may choose such a B for which

$$\#\mathbb{T}'' \gtrapprox \#\mathbb{T}' \gtrapprox \delta^{\varepsilon}\#\mathbb{T}.$$

Let $\psi : \mathbb{RP}^3 \to \mathbb{RP}^3$ be a projective transformation with

$$\psi(H) = \text{the plane at infinity}, \qquad \psi(B) = \text{the unit ball}.$$

Since B is separated from H, ψ distorts distances in B by a bounded factor. For each $T \in \mathbb{T}''$:

- $\psi(T \cap B)$ is contained in a $C\delta \times C\delta \times 2$ tube segment, and
- $\mathrm{Vol}(\psi(Y(T) \cap B)) \approx \mathrm{Vol}(Y(T) \cap B)$.

Let $\widetilde{T}$ be a $C\delta \times C\delta \times 2$ tube containing $\psi(T \cap B)$, and set

$$\widetilde{Y}(\widetilde{T}) := \psi(Y(T) \cap B).$$

After discarding constant-sized subsets, we may ensure:

(1) $\mathrm{Vol}(\widetilde{Y}(\widetilde{T}))$ is constant over $\widetilde{T} \in \widetilde{\mathbb{T}}$, and
(2) the tubes in $\widetilde{\mathbb{T}}$ are essentially distinct.

Moreover,

$$\mathrm{Vol}\Big(\bigcup_{T\in\mathbb{T}} Y(T)\Big) \gtrapprox \mathrm{Vol}\Big(\bigcup_{\widetilde{T}\in\widetilde{\mathbb{T}}} \widetilde{Y}(\widetilde{T})\Big).$$

The direction of $\psi(\ell_T)$ corresponds to the intersection of $\psi(\ell_T)$ with the plane at infinity. Distances in the space of directions are distorted only by a constant factor relative to distances in $H \cap B(10)$. Thus for any two distinct $T_1, T_2 \in \mathbb{T}'$,

$$|\theta(\widetilde{T}_1) - \theta(\widetilde{T}_2)| \gtrsim |(\ell_{T_1} \cap H) - (\ell_{T_2} \cap H)| \gtrsim \delta.$$

Passing to a further constant-sized subset ensures the right-hand side is $\geq \delta$. Hence the tubes of $\widetilde{\mathbb{T}}$ are δ-directionally separated.

Applying Conjecture 1.1 to $\widetilde{\mathbb{T}}$ gives

$$\mathrm{Vol}\Big(\bigcup_{T\in\mathbb{T}} Y(T)\Big) \gtrsim \mathrm{Vol}\Big(\bigcup_{\widetilde{T}\in\widetilde{\mathbb{T}}} \widetilde{Y}(\widetilde{T})\Big) \gtrsim \delta^{\varepsilon}\#\widetilde{\mathbb{T}}\,\mathrm{Vol}(\widetilde{T})\Big(\frac{\mathrm{Vol}(\widetilde{Y}(\widetilde{T}))}{\mathrm{Vol}(\widetilde{T})}\Big)^2 \gtrsim \delta^{\varepsilon}\delta^{C\varepsilon_1}\#\mathbb{T}\,\mathrm{Vol}(T)\Big(\frac{\mathrm{Vol}(Y(T))}{\mathrm{Vol}(T)}\Big)^2.$$

We will present an example of consisting of δ^{-2} tubes, estimate $\mathrm{Vol}(\cup Y(T))$, and show the tubes are Katz–Tao. The upshot of the above discussion is that these tubes give a counterexample to Conjecture 1.1.

2. The Example

We identify lines in $\mathbb{R}^3$ with $\mathbb{R}^4$ as follows:

$$(a,b,c,d) \mapsto \{(a,b,0)+t(c,d,1) \,:\, t\in\mathbb{R}\}.$$

Our set of tubes is given by the following subset of $\mathbb{R}^4$,

$$X = \left\{\delta^{1/4}\begin{pmatrix}n_1\\n_2\\n_3\\n_4\end{pmatrix} + \delta^{1/2}\begin{pmatrix}\delta^{\alpha}a_0\\ \delta^{\alpha}a_1\\ \delta^{\alpha}a_2\\ 0\end{pmatrix} + \delta^{1/2}s\begin{pmatrix}b_0\\b_1\\b_2\\1\end{pmatrix}\right\},$$

where

$$\begin{aligned}
& n_j, a_j \in \mathbb{Z}, \\
& b_j \quad \text{are some arbitrary fixed integers} \\
& \delta^{1/4}n_j \in [0,1], && (2.1)\\
& \delta^{\alpha}a_j \in [0,1], \qquad \alpha = \frac{1}{6} && (2.2)\\
& s\in[0,1].
\end{aligned}$$

We identify $X\subset\mathbb{R}^4$ with a set of tubes $\mathbb{T}$ in $\mathbb{R}^3$ by considering the z-axis range $z\in[0,1]$. Each of these tubes point approximately vertically. We take a shading that is a constant subset of the z-axis.

$$Y(T) = \{(x,y,z)\in T \,:\, z\in Z_{\mathrm{LowHeight}} + [0,\delta^{1/2}]\}$$

$$Z_{\mathrm{LowHeight}} = \frac{1}{K}\mathbb{Z}\cap[0,1],$$

where K is an integer. Think of $\frac{1}{K}=\delta^{\varepsilon_1}$ in the definition of $(\varepsilon_1,\varepsilon_2)$-two-ends. This shading consists of just a few $\delta^{1/2}$–segments, but is full inside of each $\delta^{1/2}$-segment. We will verify in Section 4 that this set of tubes is Katz–Tao. We will need $b_1b_2\neq b_0$.

This example has structure at two scales. The set $X\subset\mathbb{R}^4$ is contained in a collection of δ^{-1}-many balls of radius $\delta^{1/2}$ that are arranged in a well-spaced way. By well-spaced, I mean the minimum distance between two $\delta^{-1/2}$-balls is close to the maximum possible (for that number of balls). In $\mathbb{R}^3$, that $\mathbb{T}$ is covered by a collection of δ^{-1}-many $\delta^{1/2}$-tubes that are arranged in a well-spaced way. We let $\mathbb{T}^{\delta^{1/2}}$ denote this collection of $\delta^{1/2}$ tubes. For each $T^{\delta^{1/2}}\in\mathbb{T}^{\delta^{1/2}}$, we let $\mathbb{T}[T^{\delta^{1/2}}]$ denote the δ-tubes inside of it.

In $\mathbb{R}^4$, each $\delta^{1/2}$-ball contains a collection of line segments with fixed angle $\begin{pmatrix}b_0 & b_1 & b_2 & 1\end{pmatrix}$. There are $\delta^{-3\alpha}=\delta^{-1/2}$ of these line segments. If we slice a $\delta^{1/2}$-ball in $\mathbb{R}^4$ with a 3-plane, the line segments intersect that 3-plane in an integer grid with $\delta^{-3\alpha}$ many points.

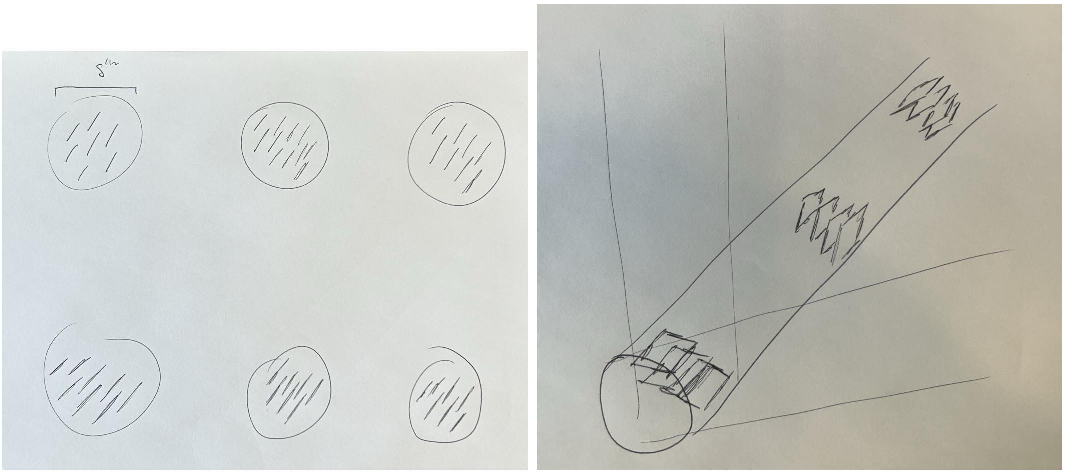

FIGURE 1. The first picture depicts $X \subset \mathbb{R}^4$. It is contained in a collection of $\delta^{1/2}$ balls, and inside each $\delta^{1/2}$ ball, it is a union of parallel line segments. The second picture depicts $\mathbb{T} \subset \mathbb{R}^3$. It is contained in a collection of $\delta^{1/2}$ tubes, and inside each $\delta^{1/2}$ tubes, it is a union of reguli which rotate at the same rate. The δ-tubes in a $\delta^{1/2}$ tube contribute a family of parallel grains to each $\delta^{1/2}$-ball. There are several families of parallel grains in each ball, with different normal vectors.

In tube space, each of these line segments determines a *regulus* inside of a $\delta^{1/2}$-tube. For example, consider the line segment in $\mathbb{R}^4$, $\mathbf{0} + s(1,0,0,1)$ for $s \in [-\delta^{1/2}, \delta^{1/2}]$. This determines the tube set

$$\{(s,0,0) + \mathbb{R}(0,s,1) \,:\, s \in [-\delta^{1/2}, \delta^{1/2}]\}.$$

This is a 1-parameter family of tubes connecting the line segment $\{(s,0,0)\}$ on the $\{z=0\}$ slice with the line segment $\{(0,s,1)\}$ on the $\{z=1\}$ slice. The union of these tubes sweeps out the regulus

$$\cup\{(s,0,0) + \mathbb{R}(0,s,1) \,:\, s \in [-\delta^{1/2}, \delta^{1/2}]\} = \{(x, zx, z) \,:\, x \in [-\delta^{1/2}, \delta^{1/2}], z \in [0,1]\}.$$

When you take a vertical slice of a regulus, you get a line. As you move up along the tube, these lines rotate. The slope of our line segments in $\mathbb{R}^4$ determines how quickly the regulus rotates.

To see this behavior, let $t \in [0,1]$, and let

$$\pi_t(a,b,c,d) = (a+ct, b+dt)$$

be the projection from $\mathbb{R}^4 \to \mathbb{R}^2$ that corresponds to slicing the tube set with the plane $\{z=t\}$. Applying this projection to a line segment in $\mathbb{R}^4$ gives a line segment in $\mathbb{R}^2$, which is why the slice of a regulus is a line. Applying this projection to two parallel line segments in $\mathbb{R}^4$ gives two parallel line segments in $\mathbb{R}^2$.

Thus if we take a vertical slice of $\mathbb{T}[T^{\delta^{1/2}}]$, we get a collection of parallel line segments. Because the reguli are arranged in a nice arithmetic way, there will be many reguli through each of these line segments.

Consider taking a $\delta^{1/2}$-ball B inside of a $\delta^{1/2}$ tube $T^{\delta^{1/2}}$. If we take a regulus inside of $T^{\delta^{1/2}}$ and intersect it with B, we get a $\delta \times \delta^{1/2} \times \delta^{1/2}$ grain. These grains rotate as we move along T. Because the reguli are arranged in an arithmetic way, there are only $\sim (\delta^{1/2})^{-1/3}$ many grains per ball.

Let

$$E = \bigcup_{T \in \mathbb{T}} Y(T).$$

Each $\delta^{1/2}$-ball B active in covering E has several $\delta^{1/2}$-tubes through it, and each of these contribute $\sim (\delta^{1/2})^{-1/3}$ many parallel grains. What is the total number of grains in B?

It turns out that lots of $\delta^{1/2}$ tubes through B contribute the same grains. There are $(\delta^{1/2})^{-1}$ many $\delta^{1/2}$ tubes through B, and these are split into $(\delta^{1/2})^{-1/2}$ many groups of size $(\delta^{1/2})^{-1/2}$ depending on the angle they make to B. All the $\delta^{1/2}$-tubes within each group contribute the same set of parallel grains, and the different groups contribute different grains. So, the total number of $\delta^{1/2}$-grains per ball B is

$$(\delta^{1/2})^{-\frac{1}{3}}(\delta^{1/2})^{-1/2} = (\delta^{1/2})^{-5/6}.$$

By comparison, two-ends Furstenberg predicts the union of these grains to fill out B. There $\ll (\delta^{1/2})^{-1}$ grains, so this does not happen. See Figure 1.

We note that if you take a typical point $p \in E$ and look at the tubes $\mathbb{T}(p)$ through it, these are all contained in a $\delta^{1/2} \times 1 \times 1$ slab. The tubes span that slab.

3. Analyzing the example

We could analyze the example by following the intuitive description above. We could compute the normal vectors of all the grains inside a $\delta^{1/2}$ ball, and find that these normal vectors have low height coordinates, so they overlap a lot.

Instead we will be a bit more rote and compute $\bigcup_{T\in\mathbb{T}} Y(T)$ by taking a vertical slice with the plane $\{z = z_0\}$, where $z_0 \in Y(T)$. Let $t = t_0 + \delta^{1/2} t_1$ where $t_0 \in Z_{\text{LowHeight}}$ and $t_1 \in [0,1]$. We denote by $\pi_t(\mathbb{T})$ the slice of $\mathbb{T}$ with a plane at height t. This corresponds to the projection $\mathbb{R}^4 \to \mathbb{R}^2$ by $(a,b,c,d) \mapsto (a + tc, b + td)$.

We have

$$\begin{aligned}
\pi_t(\mathbb{T}) &= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} + t \begin{pmatrix} c \\ d \end{pmatrix} \,:\, (a,b,c,d) \in \mathbb{T} \right\} \subset \mathbb{R}^2 \\
&= \delta^{1/4} \begin{pmatrix} n_1 + t n_3 \\ n_2 + t n_4 \end{pmatrix} + \delta^{1/2} \begin{pmatrix} \delta^\alpha a_0 + t\delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + \delta^{1/2} s \begin{pmatrix} \delta^\alpha b_0 + \delta^\alpha t b_1 \\ \delta^\alpha b_2 + t \end{pmatrix} \\
&= \delta^{1/4} \begin{pmatrix} n_1 + t_0 n_3 \\ n_2 + t_0 n_4 \end{pmatrix} + \delta^{1/2} \begin{pmatrix} \delta^\alpha a_0 + t_0\delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + \delta^{1/2} t_1 \begin{pmatrix} \delta^{1/4} n_3 \\ \delta^{1/4} n_4 \end{pmatrix} + \delta^{1/2} s \begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}.
\end{aligned}$$

We are interested in the δ-covering number of this subset of $\mathbb{R}^2$.

In the last section, we claimed $E = \cup Y(T)$ consists of a collection of $\delta^{1/2}$-balls, each of which has a collection of $\delta \times \delta^{1/2} \times \delta^{1/2}$ grains inside of it. After slicing, we see a collection of $\delta^{1/2}$-disks, each of which has a collection of $\delta \times \delta^{1/2}$ line segments in it. In the analysis, we will count how many $\delta^{1/2}$-balls we see, and how many line segments we see per $\delta^{1/2}$-ball.

First, $n_1 + t_0 n_3$ and $n_2 + t_0 n_4$ both lie in $\frac{1}{K}\mathbb{Z}$. Thus $\pi_t(\mathbb{T})$ is contained inside the following union of $\delta^{-1/2}K^2$-many $\delta^{1/2}$-balls,

$$\pi_t(\mathbb{T}) \subset \bigcup_{m_1, m_2 \in \frac{\delta^{1/4}}{K}\mathbb{Z}\cap[0,1]} B\big(\delta^{1/4} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, 10\delta^{1/2}\big),$$

where $B(x_0, r)$ is the ball of radius r around x_0.

If we look at a $\delta^{1/2}$ disk in one vertical slice and rescale, the lines in that slice are defined by

$$\begin{pmatrix} \delta^\alpha a_0 + t_0\delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + t_1 \begin{pmatrix} \delta^{1/4} n_3 \\ \delta^{1/4} n_4 \end{pmatrix} + s \begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}, \qquad s \in [0,1].$$

These lines all have a common slope, which is a rational number of height $\lesssim K$.

In order to determine the number of these lines, we take an inner product with the orthogonal vector to the slope, $\begin{pmatrix} -b_1 - t_0 \\ b_0 + t_0 b_2 \end{pmatrix} =: \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, where $q_1, q_2 \in \frac{1}{CK}\mathbb{Z} \cap [-C, C]$. We have

$$\left\langle \begin{pmatrix} \delta^\alpha a_0 + t_0 \delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + t_1 \begin{pmatrix} \delta^{1/4} n_3 \\ \delta^{1/4} n_4 \end{pmatrix} + s \begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle = \delta^\alpha (q_1 a_0 + q_1 t_0 a_2 + q_2 a_1) + t_1 \delta^{1/4} (q_1 n_3 + q_2 n_4).$$

In order to count the total number of lines, we have to count the number of values the right hand side may take as a_0, a_1, a_2, n_3, n_4 vary. As each a_j is an integer and $q_j \in \frac{1}{CK}\mathbb{Z}$,

$$q_1 a_0 + q_1 t_0 a_2 + q_2 a_1 \in \frac{1}{CK}\mathbb{Z},$$

and due to (2.2),

$$\delta^\alpha (q_1 a_0 + q_1 t_0 a_2 + q_2 a_1) \in [-C, C].$$

Thus the first term, $\delta^\alpha (q_1 a_0 + q_1 t_0 a_2 + q_2 a_1)$, takes $\lesssim CK\delta^{-\alpha}$ many values.

As for the second term,

$$q_1 n_3 + q_2 n_4 \in \frac{1}{CK}\mathbb{Z}$$

and

$$\delta^{1/4}(q_1 n_3 + q_2 n_4) \in [-C, C],$$

so the second term, $t_1 \delta^{1/4}(q_1 n_3 + q_2 n_4)$, takes $\lesssim CK\delta^{-1/4}$ many values. Thus there are $\lesssim CK^2 \delta^{-1/4-\alpha}$ many lines inside of our $\delta^{1/2}$ ball.

Because there are $\delta^{-1/2}K^2$ many $\delta^{1/2}$-balls, the total covering number of $\pi_t(\mathbb{T})$ is estimated up to constants by

$$\begin{aligned} |\pi_t(\mathbb{T})|_\delta &\sim |\pi_t(\mathbb{T})|_{\delta^{1/2}} (\#\delta \times \delta^{1/2} \text{ line segments per } \delta^{1/2}\text{-ball}) (\#\delta\text{-balls per } \delta \times \delta^{1/2} \text{ line segment}) \\ &\sim (\delta^{-1/2}K^2)(CK^2\delta^{-1/4-\alpha})\, \delta^{-1/2} \\ &\sim K^4 \delta^{-\frac{5}{4}-\alpha} \\ &\sim K^4 \delta^{-\frac{17}{12}}. \end{aligned}$$

Summing over all $t \in Y(T)$, we find

$$\Big| \bigcup_{T \in \mathbb{T}} Y(T) \Big|_\delta \sim |Y(T)|_\delta \, |\pi_t(\mathbb{T})|_\delta \sim K\delta^{-1/2} K^4 \delta^{-\frac{17}{12}} = K^5 \delta^{-\frac{23}{12}}.$$

On the other hand, the two-ends Furstenberg conjecture (Conjecture 1.1) predicts

$$\Big| \bigcup_{T \in \mathbb{T}} Y(T) \Big|_\delta \gtrsim_\varepsilon \delta^\varepsilon \delta^{C\varepsilon_1} \delta^{-1} |Y(T)|_\delta^2 = \delta^\varepsilon K^C \delta^{-2}.$$

As $\frac{23}{12} < 2$, if $K = \delta^{-\varepsilon_1}$ is chosen sufficiently small, the conjecture does not hold in this example.

4. Verifying that the tube set is Frostman

Our tube set is a union of δ^{-1} many $\delta^{1/2}$-tubes, with δ^{-1} many tubes inside of each $\delta^{1/2}$ tube. We chose $\alpha = \frac{1}{6}$ so that there are δ^{-1} many tubes per $\delta^{1/2}$ tube. Wang–Zahl showed [2, Lemma 4.12]

$$C_{KT-CW}(\mathbb{T}) \lesssim \Big(\sup_{T^{\delta^{1/2}} \in \mathbb{T}^{\delta^{1/2}}} C_{KT-CW}(\mathbb{T}[T^{\delta^{1/2}}] \Big) C_{KT-CW}(\mathbb{T}^{\delta^{1/2}}).$$

Since we have written our tube set in terms of $\mathbb{R}^4$, it is helpful to describe the Katz–Tao condition in terms of $\mathbb{R}^4$. Let U be an $a \times b \times 10$ convex set pointing roughly vertical. Let T be a $\delta \times \delta \times 1$ tube segment in $B(10)$, with

$$T = \{(a,b) + t(c,d) \,:\, t \in [t_0, t_1]\} + B(\delta)$$

and $|(c,d)| \lesssim 1$. When is $T \subset U$?

We need

$$(a,b) + t_0(c,d) \in U \cap \{z = t_0\} + B(\delta)$$

and

$$(a,b) + t_1(c,d) \in U \cap \{z = t_1\} + B(\delta).$$

The convex sets on the right hand side are two translates of a fixed convex set of dimensions close to $a \times b$. The upshot is that for any such U, we can find some convex sets $U_0 \subset U_1 \subset \mathbb{R}^2$ where

$$U_0 \text{ has dimensions } \frac{a}{1000} \times \frac{b}{1000},$$
$$U_1 \text{ has dimensions } 1000a \times 1000b,$$

and translates $\mathbf{v}_0, \mathbf{v}_1 \in B(10) \subset \mathbb{R}^4$ such that, for $\mathbf{x}_T \in \mathbb{R}^4$ the point corresponding to the core line of T,

$$T \subset U \Longrightarrow \mathbf{x}_T \in (U_0 + \mathbf{v}_1) \times (U_0 + \mathbf{v}_2)$$

and

$$\mathbf{x}_T \in (U_0 + \mathbf{v}_1) \times (U_1 + \mathbf{v}_2) \Longrightarrow T \subset U.$$

Our task is to prove the following. Let $U \subset \mathbb{R}^2$ be a convex set with dimensions $a \times b$. Then for any translates $\mathbf{v}_1, \mathbf{v}_2$,

$$|X \cap (U + \mathbf{v}_1) \times (U + \mathbf{v}_2)|_\delta \lesssim \frac{ab}{\delta^2}. \tag{4.1}$$

4.1. $\mathbb{T}^{\delta^{1/2}}$ **is Katz–Tao Convex–Wolff.** The tube collection $\mathbb{T}^{\delta^{1/2}}$ corresponds to

$$X^{\delta^{1/2}} := \left\{ \delta^{1/4} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \,:\, n_j \in \mathbb{Z},\, \delta^{1/4} n_j \in [0,1] \right\}.$$

As each $\delta^{1/4}$-ball only contains one point of $X^{\delta^{1/2}}$, for any convex set W,

$$|X \cap W| \le |W|_{\delta^{1/4}}.$$

If W has dimensions $a \times b \times a \times b$, then

$$|W|_{\delta^{1/4}} \sim \max\{a\delta^{-1/4}, 1\}^2 \max\{b\delta^{-1/4}, 1\} \le ab\delta^{-2}$$

as needed.

4.2. $\mathbb{T}[T^{\delta^{1/2}}]$ **is Katz–Tao Convex–Wolff.** Let us describe the line arrangement inside a $\delta^{1/2}$ tube. Let $\delta' = \delta^{1/2}$, and

$$Y = \left\{ \begin{pmatrix} (\delta')^\alpha a_0 \\ (\delta')^\alpha a_1 \\ (\delta')^\alpha a_2 \\ 0 \end{pmatrix} : a_j \in \mathbb{Z}, (\delta')^\alpha a_j \in [0,1] \right\} \qquad \text{where } \alpha = \frac{1}{3}.$$

For each $\mathbf{y} \in Y$, let

$$\tau_{\mathbf{y}} = \left\{ \mathbf{y} + s \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ 1 \end{pmatrix} : s \in [0,1] \right\}.$$

Then

$$X' = \bigcup_{\mathbf{y} \in \mathbf{Y}} \tau_{\mathbf{y}}$$

describes $\mathbb{T}[T^{\delta^{1/2}}]$.

Let $W = (U + \mathbf{v}_1) \times (U + \mathbf{v}_2)$ where U is an $a \times b$ convex set with $\delta \le a \le b \le 1$.

Because we assume $b_0 b_2 \neq b_1$, the two vectors $\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$ and $\begin{pmatrix} b_2 \\ 1 \end{pmatrix}$ are not parallel. This forces each line segment $\tau_{\mathbf{y}}$ to be transverse to W. To be more specific, let $\tau_{\mathbf{y}}^1$ and $\tau_{\mathbf{y}}^2$ be the projection onto the first two coordinates of $\mathbb{R}^4$ and onto the second two coordinates of $\mathbb{R}^4$, respectively. Then $\tau_{\mathbf{y}}^1$ has slope $\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$ and $\tau_{\mathbf{y}}^2$ has slope $\begin{pmatrix} b_2 \\ 1 \end{pmatrix}$. One of these two slopes is transverse to U. Thus

$$|\tau_{\mathbf{y}} \cap W|_\delta \lesssim \min\left\{ |\tau_{\mathbf{y}}^1 \cap (U + \mathbf{v}_1)|_\delta, |\tau_{\mathbf{y}}^2 \cap (U + \mathbf{v}_2)|_\delta \right\} \lesssim \frac{a}{\delta}$$

where the constant depends on $|b_0 b_2 - b_1|$.

Let B be the smallest b-ball containing W. The number of $\mathbf{y} \in Y$ for which $\tau_{\mathbf{y}}$ intersects B is estimated by

$$|Y \cap B(x_0, b)| \lesssim \max\{b^3 \delta^{-1}, 1\} \lesssim b/\delta.$$

Thus

$$|W \cap X'|_\delta \lesssim (b/\delta)(a/\delta)$$

as desired.

References

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