Higher dimensional fractal uncertainty

by

Alex Cohen

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 2025

Authored by: Alex Cohen
Department of Mathematics
April 18, 2025

Certified by: Larry Guth
Professor of Mathematics, Thesis Supervisor

Accepted by: Davesh Maulik
Professor of Mathematics
Chair, Department Committee on Graduate Theses

Higher dimensional fractal uncertainty

by

Alex Cohen

Submitted to the Department of Mathematics
on April 18, 2025 in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

ABSTRACT

We prove that if a fractal set in $\mathbb{R}^d$ avoids lines in a certain quantitative sense, which we call line porosity, then it has a fractal uncertainty principle. The main ingredient is a new higher dimensional Beurling and Malliavin multiplier theorem, which allows us to construct band-limited functions that decay rapidly on line porous sets.

To prove this theorem, we first explicitly construct certain plurisubharmonic functions on $\mathbb{C}^d$. Then, following Bourgain, we use Hörmander's L^2 theory for the $\bar{\partial}$ equation to construct band-limited functions.

The main theorem has since been applied by Kim and Miller to lower bounds for the mass of eigenfunctions on higher dimensional hyperbolic manifolds.

Thesis supervisor: Larry Guth
Title: Professor of Mathematics

Acknowledgments

I thank my advisor, Larry Guth. He is my role model—in clarity, groundedness, and listening. As his student I learned not just how to do math, but also what my values are as a mathematician.

Semyon Dyatlov has essentially been a second advisor. I learned about the topic of this thesis from him, and it would not have been possible without his guidance and support. I appreciate his willingness to get into the details of any problem—about math, writing, or presentation.

Henry Cohn, my last committee member, showed me a different perspective on math, and I'm grateful I got to be his intern for a summer. Wilhelm Schlag introduced me to analysis in his undergraduate PDEs class, and has supported me every since. Peter Sarnak expanded my view of mathematics and encouraged me to work on hard problems.

I thank my collaborators, Dima Zakharov, Cosmin Pohoata, Dominique Maldague, Felipe Hernandez, and Nitya Mani, who made the work fun; my Baruch REU mentors, Guy Moshkovitz, Frank de Zeeuw, Melvyn Nathanson, and Adam Sheffer, who nurtured my interest in research; and my undegraduate mentors, Stefan Steinerberger and Yair Minsky. I also thank my research group, Alex Ortiz, Shengwen Gan, Sarah Tammen, Yuqiu Fu, Rose Zhang, our weekly lunch was a central line through my PhD.

I am indebted to my academic influences; I wish I could account for all of them. Ruixiang Zhang, Benjamin Jaye, and Tuomas Sahlsten offered insights directly relevant to the topic of this thesis, and Nick Trefethen showed me an applied perspective on band-limited functions and provided helpful feedback on Chapter 2. Malcah Effron provided valuable feedback on the writing.

I am deeply grateful to all of my friends. The high school Avalon crew, Noah, Naomi, Alberto, Axel, Stefan, Alex, and Anna, set me towards academia. Elia, Elena, Jose, David, Hanna, Travis, and all the other department friends made me want to come into the office each day. Finally, I am truly lucky to have Kevin Li as a roommate and as a friend. Not just due to his cooking—but it's surely a benefit.

Meeting Kendra was a gift. She is supportive, silly, and capable of doing pretty much anything. Her presence makes life feel both lighter and more meaningful. I thank my Boston family, Lisa, Amy, and Miles, for all the dinners they provided, and my grandma Ruth, who wanted to know what I was "digging towards" up until the end. My sister Joanna is my closest confidant, and in all of my toughest moments she is the one I relied on. My mom, Holly, showed unbelievable devotion and support, laying the groundwork for every opportunity I've had. My dad, Sandy, never got to see me start graduate school, but the values and support he gave me set me on this path.

Contents

Chapter 1

Introduction

1.1 Main result

A fractal uncertainty principle (FUP) says that a function cannot be simultaneously localized near a fractal set in physical space and near a fractal set in Fourier space. Several such theorems have been proved in different contexts. In 2017, Bourgain and Dyatlov [11] proved an important FUP for porous sets.

Given a parameter $\nu \in (0, 1/3)$, we say a set $\mathbf{X} \subset \mathbb{R}^d$ is *ν-porous on balls* from scales α_0 to α_1 if for every ball B of diameter $R \in (\alpha_0, \alpha_1)$, there exists a point $\mathbf{x} \in \mathsf{B}$ such that the ball $\mathsf{B}_{\nu R}(\mathbf{x})$ (with center $\mathbf{x}$ and radius νR) is disjoint from $\mathbf{X}$. For example, the middle-thirds Cantor set is 1/6-porous on all scales, and an h-neighborhood of this set is porous from scales h to 1.

Theorem 1.1 (Bourgain–Dyatlov [11, Theorem 4]). *Let $\nu > 0$ and suppose that*

- $\mathbf{X} \subset [-1, 1]$ *is ν-porous from scales h to 1, and*
- $\mathbf{Y} \subset [-h^{-1}, h^{-1}]$ *is ν-porous from scales 1 to h^{-1}.*

Then there exist constants $\beta, C > 0$, depending only on ν, such that for all $f \in L^2(\mathbb{R})$

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \quad \Longrightarrow \quad \|f \mathbf{1}_{\mathbf{X}}\|_2 \leq C\, h^{\beta}\, \|f\|_2. \tag{1.1}$$

In this theorem and throughout the thesis, we assume that $h \in (0, 1/100)$ is a small parameter.

Remark. In Bourgain and Dyatlov's paper, the hypothesis is that $\mathbf{X}$ and $\mathbf{Y}$ are Ahlfors–David (AD) regular rather than porous. These two notions are equivalent up to a change in parameters: any AD regular set of dimension less than 1 is porous, and

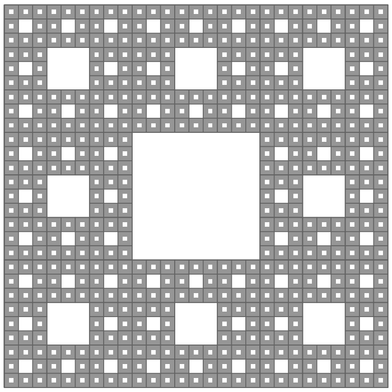

Figure 1.1: The Sierpinski carpet is porous on balls but not on lines

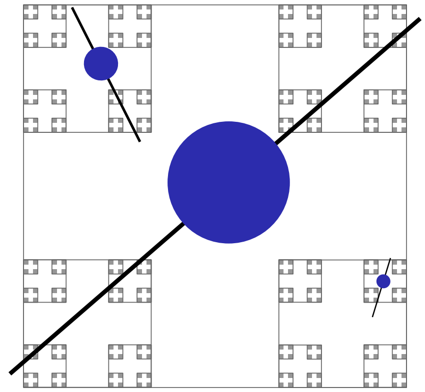

Figure 1.2: The product of two middle-thirds Cantor sets is porous on lines

any porous set is contained in an AD regular set of dimension less than 1. The first statement of FUP using porous sets appeared in [21].

Theorem 1.1 fails in higher dimensions due to the following example. Consider a thin horizontal rectangle $\mathbf{X}$ and a tall vertical rectangle $\mathbf{Y}$,

$$\mathbf{X} = [0,1] \times [0,h] \quad \text{and} \quad \mathbf{Y} = [0,1] \times [0,h^{-1}]. \tag{1.2}$$

Let ψ be a fixed bump function satisfying $\operatorname{supp} \hat{\psi} \subset [0,1] \times [0,1]$, and define the rescaled function $\psi_h(x,y) = \psi(x, h^{-1}y)$. Then

$$\operatorname{supp} \widehat{\psi_h} \subset \mathbf{Y} \quad \text{and} \quad \|\psi_h \mathbf{1}_{\mathbf{X}}\|_2 \geq (\text{const.})\, \|\psi_h\|_2.$$

To rule out this example, we introduce the notion of line porosity. We say a set $\mathbf{X}$ is *ν-porous on lines* from scales α_0 to α_1 if for all line segments τ with length $R \in (\alpha_0, \alpha_1)$, there exists a point $\mathbf{x} \in \tau$ such that $\mathsf{B}_{\nu R}(\mathbf{x})$ is disjoint from $\mathbf{X}$.

Any line porous set is also ball porous, but not the other way around. The rectangle sets $\mathbf{X}$ and $\mathbf{Y}$ in (1.2) are porous on balls, but not on lines. See Figure 1.1 for another set that is porous on balls but not on lines, and Figure 1.2 for an example of a set that is porous on lines. Similar notions have appeared before; for example, Chousionis [12] introduced the notion of *directional porosity* to the study of iterated function systems.

Our main theorem is the following higher-dimensional fractal uncertainty principle, which applies when one set is porous on balls and the other is porous on lines.

Theorem 1.2. *Let $\nu > 0$ and suppose that*

- $\mathbf{X} \subset [-1,1]^d$ *is ν-porous on balls from scales h to 1, and*
- $\mathbf{Y} \subset [-h^{-1}, h^{-1}]^d$ *is ν-porous on lines from scales 1 to h^{-1}.*

Then there exist constants $\beta, C > 0$, depending only on ν and d, such that for all $f \in L^2(\mathbb{R}^d)$

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \quad \Longrightarrow \quad \|f 1_{\mathbf{X}}\|_2 \leq C\, h^{\beta} \|f\|_2. \tag{1.3}$$

See [14] for the journal version of this result.

1.2 Connection to quantum chaos

The fractal uncertainty principle has striking applications to quantum chaos. By applying FUP to fractal sets coming from chaotic dynamical systems, we can control high frequency waves on those systems. Dyatlov–Zahl [23], Dyatlov–Zworski [24], and Bourgain–Dyatlov [11] used Theorem 1.1 to prove a spectral gap for open quantum systems. In what follows we'll focus on applications to compact hyperbolic manifolds—a classic example of a chaotic dynamical system.

To see why hyperbolic manifolds are chaotic, think of geodesic flow lines in the unit tangent bundle. If you place a particle at almost any starting point, the flow line equidistributes as time goes to infinity, a property called *ergodicity*. Also, two nearby flow lines diverge exponentially fast, so the system is highly sensitive to initial conditions.

However, not every flow line on a hyperbolic manifold equidistributes. Some return to themselves, some stay trapped near fractal sets, and there are other, more complicated behaviors as well. This variety of behaviors is one of the hallmarks of chaotic systems.

To study quantum dynamics on a manifold M, we look at the sequence $\{u_j\}_{j=1}^{\infty}$ of L^2 normalized Laplace eigenfunctions. A quantum mechanics version of asking what the flow lines look like is to ask: what do the measures $|u_j|^2 dx$ look like as $j \to \infty$? Actually, we usually study the distribution of u_j in the phase space S^*M, rather than just on the base manifold M, although we'll ignore this point in what follows. In addition to physics, the study of eigenfunctions is motivated by number theory, because eigenfunctions on certain hyperbolic manifolds encode arithmetic information.

One of the first results on this topic was the quantum ergodicity theorem of Shnirelman [45], Zelditch [46], and Colin De Verdére [16], which states that a dense subsequence of eigenfunctions equidistributes. This is a quantum version of the fact that almost every flow line equidistributes.

Rudnick and Sarnak's *quantum unique ergodicity* conjecture hypothesizes that the entire sequence of measures $|u_j|^2 dx$ equidistributes, not just a dense subsequence. In other words, for any open set $U \subset M$,

$$\lim_{j \to \infty} \|u_j\, \mathbf{1}_U\|_2^2 = |U| \qquad \text{where } |U| \text{ is the Lebesgue probability measure.} \tag{1.4}$$

This conjecture gets at a truly quantum phenomenon. Even though some special flow lines do not equidistribute, we expect that every single high-frequency eigenfunction equidistributes. The underlying mechanism is that waves tend to disperse, so we believe they cannot stay trapped near the special flow lines that do not equidistribute.

Lindenstrauss [40] proved the quantum unique ergodicity conjecture in the special setting of Hecke eigenfunctions on arithmetic hyperbolic manifolds. The first major progress about general hyperbolic manifolds was made by Anantharaman [1], who proved that the mass of $|u_j|^2 dx$ cannot be concentrated near too small of a set. Anantharaman and Nonnenmacher [2] significantly strengthened this result by proving a lower bound for the measure-theoretic entropy associated to eigenfunctions. The latest progress was by Dyatlov and Jin [20], who used Theorem 1.1 to prove a uniform lower bound for the L^2 mass of any eigenfunction on a fixed open set.

Theorem 1.3 (Dyatlov & Jin [21])**.** *Let M be a compact hyperbolic surface, and let $U \subset M$ be a nonempty open set. Then for some $c_U > 0$,*

$$\|u_k \, \mathbf{1}_U\|_2 \geq c_U \quad \textit{for all } L^2\textit{-normalized Laplace eigenfunctions } u_k.$$

Proof sketch. We can write

$$M = \Gamma \backslash \mathbb{D},$$

where $\mathbb{D}$ is the Poincaré disk and $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ is a group of isometries. Then u_k lifts to a Γ-invariant eigenfunction $\widetilde{u}_k$ on $\mathbb{D}$, and U lifts to a Γ-invariant open subset $\widetilde{U} \subset \mathbb{D}$.

For $b \in \mathbb{S}^1$ and $z \in \mathbb{D}$, denote by $P_b(z)$ the Poisson kernel. For any $(b,r) \in \mathbb{S}^1 \times \mathbb{R}$, the *hyperbolic plane wave*

$$\psi_b^r(z) := P_b(z)^{\frac{1}{2}+ir}, \quad z \in \mathbb{D}, \tag{1.5}$$

solves the eigenfunction equation

$$-\Delta \psi_b^r = (r^2 + \tfrac{1}{4})\psi_b^r \quad \text{on } \mathbb{D}.$$

If $r > 0$ we call this an outgoing wave and if $r < 0$ it is incoming. Because u_k has eigenvalue λ_k, we take

$$r = \sqrt{\lambda_k - \tfrac{1}{4}}.$$

We can synthesize $\widetilde{u}_k$ in two ways, using either outgoing or incoming waves:

$$\widetilde{u}_k(z) = \int_{\mathbb{S}^1} f(b)\, \psi_b^r(z)\, db,$$

$$\widetilde{u}_k(z) = \int_{\mathbb{S}^1} g(b)\, \psi_b^{-r}(z)\, db, \quad r \sim h^{-1},$$

where f, g are distributions on $\mathbb{S}^1$. These distributions are related by an explicit formula (see, e.g., [9, §4.4])

$$g(b) = c_r \int_{\mathbb{S}^1} e^{-(1+2ir)\log|b-a|} f(a)\, da. \tag{1.6}$$

Now let $\varepsilon > 0$ be small enough that $B_{1-\varepsilon} \subset \mathbb{D}$ covers M. Let γ be a geodesic on $\mathbb{D}$ with endpoints $\gamma_+, \gamma_- \in \mathbb{S}^1$. Define

$$\mathbf{X} = \bigcup \{\gamma_+, \gamma_-\} \quad \text{over all } \gamma \text{ such that } \gamma \cap B_{1-\varepsilon} \neq \emptyset \text{ and } \gamma \cap \widetilde{U} = \emptyset. \tag{1.7}$$

The set $\mathbf{X} \subset \mathbb{S}^1$ represents the geodesics on M that do not intersect U. Using unique ergodicity of the horocycle flow on M, one can show that $\mathbb{X}$ is porous.

Morally speaking, if

$$\|u_k \, 1_U\|_2 = o(1),$$

then f and g are both localized h-close to the set $\mathbb{X}$ where $h = \lambda_k^{-1/2}$. Because f and g are related by the oscillatory integral (1.6), the fractal uncertainty principle applied to the h-neighborhood of $\mathbf{X}$ rules out this scenario.

See Dyatlov's survey [17] for details on the proof (which uses microlocal analysis and does not follow this sketch). □

Dyatlov, Jin, and Nonnenmacher [22] extended Theorem 1.3 to variable curvature. These lower bounds were later applied to control for the Schrödinger equation and exponential decay for the damped wave equation, see [22, 32, 33]. Dyatlov–Jezequel [19] and Athreya–Dyatlov–Miller [5] proved similar results for certain higher-dimensional quantum chaos systems that diverge faster in one direction than others. The applications were limited to these special higher-dimensional systems because Bourgain and Dyatlov's FUP was restricted to subsets of $\mathbb{R}$.

Our main result (Theorem 1.2) has recently been applied to prove mass lower bounds for higher-dimensional eigenfunctions. Kim, Anderson, and Oliver [34] proved a lower bound for the mass of eigenfunctions on quantum cat maps, a model system closely related to hyperbolic manifolds. Kim and Miller [35] proved the following theorem about eigenfunctions on higher-dimensional hyperbolic manifolds. Their actual theorem is more general—we just state a special case.

Theorem 1.4 (Kim–Miller)**.** *Let M be a compact hyperbolic manifold with no immersed totally geodesic submanifolds, and let $U \subset M$ be an open subset. For some $c_U > 0$,*

$$\|u_k \, \mathbf{1}_U\|_2 \geq c_U \quad \textit{for all } L^2\textit{-normalized Laplace eigenfunctions } u_k.$$

The hyperbolic manifold M and open subset U give rise to ball porous sets $\mathbf{X} \subset \mathbb{S}^{d-1}$. If M has no immersed totally geodesic submanifolds, then $\mathbf{X}$ is also porous on lines, so Theorem 1.2 can be applied. In the related setting of cat maps, we really need this extra hypothesis. If the cat map has totally geodesic submanifolds, eigenfunctions might concentrate there.

1.3 Proof sketch of the fractal uncertainty principle

Let $X \subset [-1,1]$ be ν-porous from scales h to 1, and let $Y \subset [-h^{-1}, h^{-1}]$ be ν-porous from scales 1 to h^{-1}. The goal in Theorem 1.1 is to prove

$$\operatorname{supp} \hat{f} \subset Y \Longrightarrow \|f\,\mathbf{1}_X\|_2 \leq h^\beta \|f\|_2. \tag{1.8}$$

It's challenging to use the hypothesis that $\operatorname{supp} \hat{f} \subset Y$. The Lebesgue measure of Y is quite large—roughly $h^{-1+\varepsilon}$, where $\varepsilon \to 0$ as $\nu \to 0$. How can we distinguish a function with Fourier support in Y from a general L^2 function?

Bourgain and Dyatlov developed an innovative strategy for this problem. They brought in tools from complex analysis to control the tail behavior of functions with Fourier support in Y. They proved that if $\operatorname{supp} \hat{f} \subset Y$, then a significant amount of the L^2 mass of f leaks into the holes of X at every scale. They use this information at many scales to prove (1.8).

Two main ingredients are needed to execute this strategy. The first is Beurling and Malliavin's multiplier theorem, which is about constructing functions with compact Fourier support and prescribed decay rates.

Theorem 1.5 (Beurling and Malliavin [8]). *Let $\omega : \mathbb{R} \to \mathbb{R}_{\leq 0}$ be a Lipschitz function satisfying the growth condition*

$$\int_{-\infty}^{\infty} \frac{|\omega(t)|}{1+t^2}\, dt < \infty. \tag{1.9}$$

Then for any $\sigma > 0$, there exists a nonzero $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subset [-\sigma, \sigma]$ and $|f(x)| \leq e^{\omega(x)}$.

The proof of Theorem 1.1 draws on a deep connection between Fourier analysis and complex analysis. Chapter 2 describes this theorem as well as a higher-dimensional version that I proved.

The Beurling and Malliavin theorem lets us construct *damping functions* for Y. These are nonzero functions $\psi \in L^2(\mathbb{R})$ with $\operatorname{supp} \psi \subset [-c_1, c_1]$ and

$$|\hat{\psi}(\xi)| \leq e^{-|\xi|/(\log(2+|\xi|))^\alpha} \qquad \text{for all } \xi \in Y,$$

where $c_1 > 0$ is arbitrary and $\alpha \in (0,1)$ is a parameter. It's important that $\alpha < 1$. If $\alpha > 1$, the Beurling and Malliavin theorem shows that there exists a ψ decaying this fast on all of $\mathbb{R}$. When $\alpha < 1$, it is impossible for a nonzero function to decay this fast on all of $\mathbb{R}$. The construction of damping functions uses porosity, and distinguishes Y from other sets with the same Lebesgue measure.

Damping functions are useful because they let us take the hypothesis that $\operatorname{supp} \hat{f} \subset Y$ and turn it into the knowledge that

$$\widehat{(f * \psi)} = \hat{f}\hat{\psi} \text{ decays rapidly for } \xi \in Y.$$

We exploit this rapid Fourier decay through the following quantitative unique continuation principle. It shows that a function with this decay cannot have too small L^2 mass on certain sets $E \subset \mathbb{R}$.

Theorem 1.6. *Let $E \subset \mathbb{R}$ satisfy*

$$|E \cap [n, n+1]| \geq \lambda \qquad \text{for every } n \in \mathbb{Z}. \tag{1.10}$$

Let $\alpha \in (0,1)$, and suppose

$$\|\hat{g} e^{|\xi|/(\log(2+|\xi|))^{\alpha}}\|_2 \leq A\|g\|_2.$$

Then

$$\|g \, \mathbf{1}_E\|_2 \geq c\|g\|_2 \qquad \text{for some } c = c(\alpha, A, \lambda) > 0. \tag{1.11}$$

We apply Theorem 1.6 to $f * \psi$ to prove the following quantitative unique continuation principle for sets with Fourier support in Y. Actually, we need to apply Theorem 1.6 to modulated copies of f convolved with ψ.

Proposition 1.7. *Let $\nu, \lambda > 0$. Suppose $Y \subset [-h^{-1}, h^{-1}]$ is ν-porous from scales 1 to h^{-1}. Let $E \subset \mathbb{R}$ be a set such that*

$$E \cap [n, n+1] \text{ contains an interval of length } \lambda > 0 \text{ for every } n \in \mathbb{Z}.$$

Then

$$\operatorname{supp} \hat{f} \subset Y \quad \Longrightarrow \quad \|f \, \mathbf{1}_E\|_2 \geq c\|f\|_2 \qquad \text{for some } c = c(\nu, \lambda) > 0.$$

With Proposition 1.7 in hand, we are ready to sketch the proof of Theorem 1.1.

Proof sketch of Theorem 1.1. Let $X \subset [-1,1]$ be ν-porous from scale h to 1, and let $Y \subset [-h^{-1}, h^{-1}]$ be ν-porous from scale 1 to h^{-1}. Let $f \in L^2(\mathbb{R})$ have Fourier support in Y.

Set

$$X_k := X + [-L^{-k}, L^{-k}], \qquad \text{where } L > 10/\nu,$$

and let η_{X_k} be an approximate cutoff to X_k with Fourier support in $[-L^{k+10}, L^{k+10}]$. Because X is porous, every interval of length L^{-k} contains some interval of length $L^{-k-1}/10$ which is disjoint from X_{k+1}. We call the union of all these intervals the *Holes in X at scale L^{-k-1}*. One can use Proposition 1.7 to prove

$$\|f\, \mathbf{1}_{\text{Holes in } X \text{ at scale } L^{-k-1}}\|_2 \geq c(\nu)\|f\, \eta_{X_k}\|_2,$$

which implies

$$\|f\, 1_{\eta_{X_{k+1}}}\|_2 \leq (1 - c(\nu))\|f\, 1_{\eta_{X_k}}\|_2 \qquad \text{as long as } L^{-k} \geq h.$$

Iterating this result yields

$$\|f\, \mathbf{1}_X\|_2 \leq (1 - c(\nu))^{c(\nu)\log h^{-1}}\|f\|_2 \leq h^{\beta(\nu)}\|f\|_2,$$

as desired.

□

We follow the same strategy to prove Theorem 1.2. My contribution was proving a higher-dimensional version of the Beurling and Malliavin multiplier theorem, which is stated in Theorem 2.5. See Chapter 2 for the statement, proof, and exposition. Han and Schlag [27] already proved a higher-dimensional version of the quantitative unique continuation principle, which completes the proof. Jaye and Mitkovskii [31] gave a different proof of the quantitative unique continuation principle, which we present in Chapter 3

Remark. When I read Bourgain and Dyatlov's proof of Theorem 1.1, I was uncomfortable with their use of Beurling and Malliavin's multiplier theorem and quantitative unique continuation. I hadn't encountered those techniques before, so I tried to prove their theorem using more familiar methods such as L^4 bounds, wave packets, the TT^* method, etc, all of which came up short.

Now I understand why. For any two sets $X, Y \subset \mathbb{R}$, one can ask about the operator norm

$$\|1_X \mathcal{F} 1_Y\|_{2\to 2}.$$

There are two easy bounds: the volume bound,

$$\|1_X \mathcal{F} 1_Y\|_{2\to 2} \le |X|^{1/2}|Y|^{1/2},$$

which is proved by interpolating the $L^1 \to L^\infty$ bound for the Fourier transform, and the orthogonality bound

$$\|1_X \mathcal{F} 1_Y\|_{2\to 2} \le 1$$

which comes from ignoring the sets X and Y. Bounding $\|1_X \mathcal{F} 1_Y\|_{2\to 2}$ is related to the *large value problem* in Fourier analysis, see Guth's [26] recent survey about this problem. The familiar methods tend to do well when $|X|^{1/2}|Y|^{1/2}$ is not too large. However, Theorem 1.1 is about the challenging regime

$$|X||Y| \sim h^{-1+\varepsilon} \qquad \text{where } \varepsilon \to 0 \text{ as } \nu \to 0.$$

In this difficult setting, Bourgain and Dyatlov succeed by exploiting subtle features of the tail behavior of functions with Fourier support in Y. For discussion on uncertainty principles in the bulk versus those in the tails, see §2.1.

1.4 Related work

A set $\mathbf{X} \subset \mathbb{R}^d$ is *Ahlfors–David δ-regular* with constant C_{AD} from scales α_0 to α_1 if there is a measure μ supported on $\mathbf{X}$ satisfying the following. For every ball B with diameter $R \in (\alpha_0, \alpha_1)$,

$$\mu(\mathsf{B}) \le C_{\mathrm{AD}}\, R^\delta, \tag{1.12}$$

and if in addition B is centered at a point in $\mathbf{X}$, then

$$\mu(\mathsf{B}) \ge C_{\mathrm{AD}}^{-1}\, R^\delta. \tag{1.13}$$

For $\mathbf{X} \subset [-1,1]^d$ a δ-regular set from scales h to 1 and $\mathbf{Y} \subset [-h^{-1}, h^{-1}]^d$ a δ'-regular set from scales 1 to h^{-1}, there is a trivial bound

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \implies \|f 1_{\mathbf{X}}\|_2 \le C \min(1, h^{(d-(\delta+\delta'))/2})\|f\|_2 \tag{1.14}$$

where C depends only on $\delta, \delta', C_{\mathrm{AD}}, d$. The estimate $\|f 1_{\mathbf{X}}\|_2 \le C h^{(d-(\delta+\delta'))/2}\|f\|_2$ follows from combining $L^1 \to L^\infty$ boundedness of the Fourier transform with a volume bound on the sets $\mathbf{X}$ and $\mathbf{Y}$. An FUP is any improvement over this trivial bound, and the regimes $\delta+\delta' < d$, $\delta+\delta'$ is close to d, and $\delta+\delta' > d$ are quite different.

In the regime $\delta+\delta' < d$, Backus, Leng, and Z. Tao [6] gave a definitive result. They proved an FUP if $\delta+\delta' < d$ and $\mathbf{X}, \mathbf{Y}$ are not orthogonal in a certain sense.

In the regime $\delta + \delta'$ is close to d, Cladek and T. Tao [13] proved an additive energy estimate for fractal sets and used this to prove an FUP when the ambient dimension d is odd and $\mathbf{X}, \mathbf{Y}$ are δ-regular with $d/2 - \varepsilon(d, C_{\mathrm{AD}}) < \delta < d/2 + \varepsilon(d, C_{\mathrm{AD}})$. Shmerkin [44] proved an inverse theorem for additive energy that implies an FUP in this regime in all dimensions, assuming the fractal sets are directionally porous in a certain sense.

The present paper is about the $\delta + \delta' > d$ regime. Han and Schlag [27] proved an FUP when $\mathbf{X}$ is an arbitrary porous set and $\mathbf{Y}$ is a Cartesian product of one dimensional porous sets. A cartesian product of one dimensional porous sets is line porous, so Theorem 1.2 recovers this result. The author [15] proved an FUP when $\mathbf{X}, \mathbf{Y}$ are Cantor sets in $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ which don't contain a pair of orthogonal lines (the ideas in this thesis are unrelated to that work).

We also mention that Dyatlov [18] wrote an expository note giving an alternative point of view on some of the proofs in this thesis.

1.5 Outline

Chapter 2 is about the Beurling and Malliavin multiplier theorem, one of the key ingredients in the proof of FUP.

- §2.1 introduces the Beurling–Malliavin problem in the context of uncertainty principles, and states the higher-dimensional Beurling and Malliavin theorem that I proved.
- §2.2 is about how the Beurling–Malliavin multiplier problem naturally splits into two steps.

  Step 1: Plurisubharmonic Beurling–Malliavin ($\mathcal{PSH}$-BM) is a potential theory problem about constructing plurisubharmonic functions.

  Step 2: Analytic Beurling–Malliavin ($\mathcal{A}$-BM) is a several complex variables problem about constructing entire functions from those plurisubharmonic functions.

  Towards the end of §2.2 we state our solution to each of these steps, Proposition 2.8 and Proposition 2.9, and then use them to prove the higher-dimensional Beurling–Malliavin theorem (Theorem 2.5).

- §§2.3-2.4 together present my solution to $\mathcal{PSH}$-BM.
- §2.5 starts with an exposition of Hörmander's L^2-theory for the $\bar{\partial}$-equation. Then, following Bourgain, we apply this theory to solve $\mathcal{A}$-BM.

Chapter 3 is about quantitative unique continuation. This chapter completes the proof of FUP.

- §3.1 states the quantitative unique continuation principles necessary for the proof of fractal uncertainty. At the end of this section we prove our higher-dimensional FUP (Theorem 1.2) conditional on these results.
- §3.2 proves that functions with rapidly decaying Fourier transform have a quantitative unique continuation principle, following Jaye and Mitkovski's [31] proof using quasi-analytic classes.
- §3.3 uses the results of the prior section to prove that if a set $\mathbf{Y}$ admits damping functions, functions with Fourier support in $\mathbf{Y}$ have a quantitative unique continuation principle. This section also follows Jaye and Mitkovski [31].
- §3.4 uses the results of the prior two sections to prove a fractal uncertainty principle, conditional on the construction of damping functions. The main result of this section was proved by Han and Schlag [27].
- §3.5 uses the higher-dimensional Beurling and Malliavin theorem to construct damping functions for line porous sets, completing the proof of Theorem 1.2.

See Figure 1.3 for a diagram of the proof.

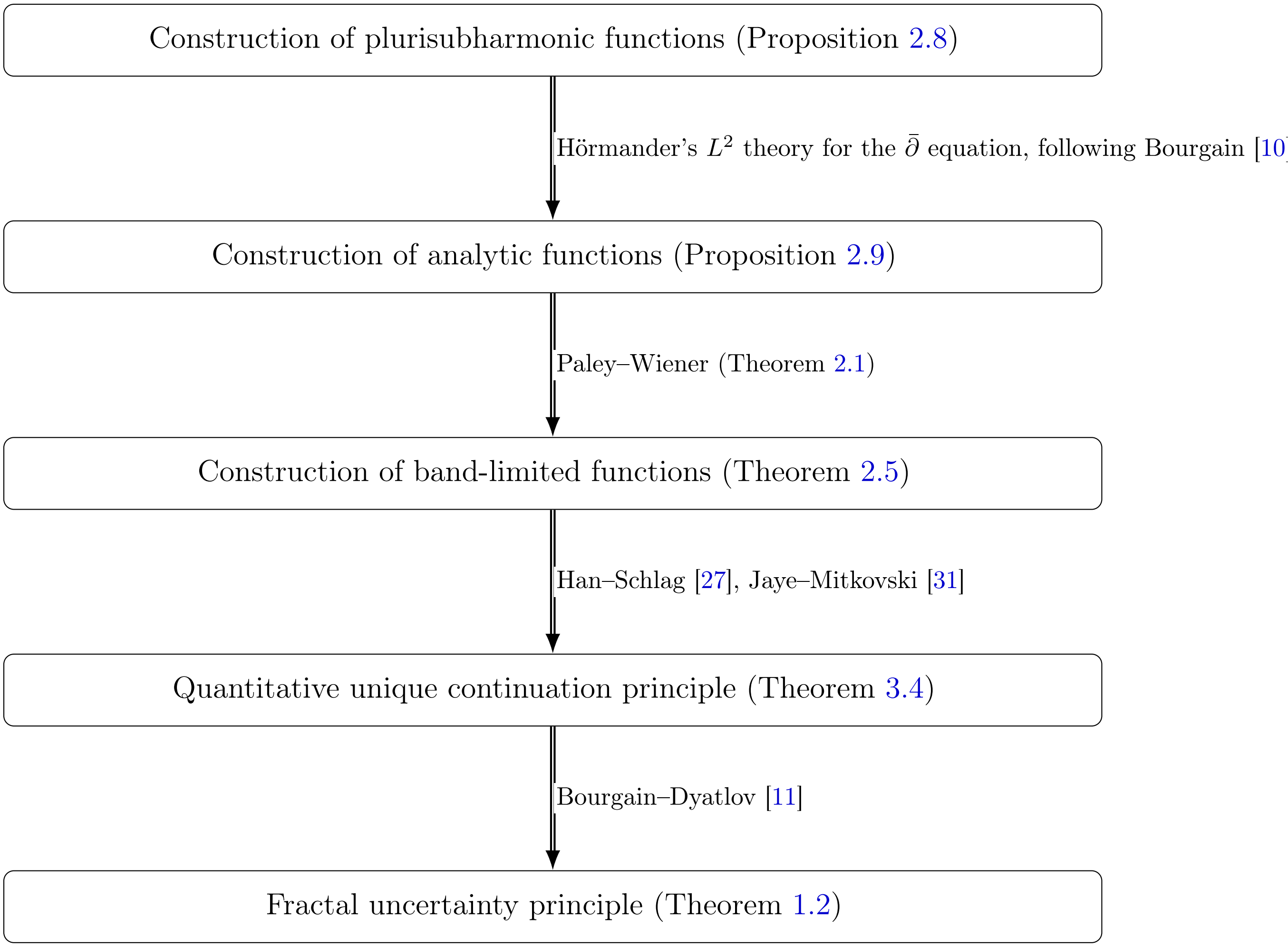

Figure 1.3: Diagram of steps in the proof of Theorem 1.2.

1.6 Notation

For $f \in L^2(\mathbb{R}^d)$, we use the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(\mathbf{x})\, e^{-2\pi i\, \mathbf{x}\cdot\xi}\, d\mathbf{x}$$

and the inverse Fourier transform

$$g^{\vee}(x) = \int_{\mathbb{R}^d} g(\xi)\, e^{2\pi i \mathbf{x}\cdot\xi}\, d\xi.$$

We often denote vectors $\mathbf{z} \in \mathbb{C}^d$ by $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. We use $\hat{\mathbf{y}}$ to denote a unit vector, and if $\mathbf{y} \in \mathbb{R}^d \setminus \{0\}$ we write $\hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$. The ℓ_2 norm on $\mathbb{R}^d$, $\mathbb{C}^d$ is denoted

$|\mathbf{x}|, |\mathbf{z}|$. We let
$$\langle \mathbf{x} \rangle = (1 + |\mathbf{x}|^2)^{1/2}.$$
We denote the Hilbert transform on $L^2(\mathbb{R})$ by $f \mapsto H[f]$. For functions $f \in C_0^1(\mathbb{R})$, this is given by
$$H[f](x) = p.v. \int_{-\infty}^{\infty} \frac{f(x-t)}{t} \frac{dt}{\pi}. \tag{1.15}$$
For $u \in C^2(\mathbb{C}^d)$, $\partial\bar{\partial}u$ is a Hermitian form which can be represented in coordinates as the Hermitian matrix
$$\begin{aligned} \langle (\partial\bar{\partial}u)\hat{\mathbf{e}}_j, \hat{\mathbf{e}}_k \rangle &= \frac{\partial^2 u}{\partial z_j \bar{\partial} z_k} \\ &= \frac{1}{4}(\partial_{x_j}\partial_{x_k} + \partial_{y_j}\partial_{y_k})u + \frac{1}{4}i(\partial_{x_j}\partial_{y_k} - \partial_{x_k}\partial_{y_j})u \end{aligned}$$
where $\hat{\mathbf{e}}_j = (0, \dots, 0, 1, 0, \dots, 0)$.

For functions $f \in C^2(\mathbb{R}^d)$, the quadratic form $D^2 f(\mathbf{x})$ applied to the vector $\mathbf{v}$ is given by
$$\langle (D^2 f(\mathbf{x}))\mathbf{v}, \mathbf{v} \rangle. \tag{1.16}$$
We denote $D^a f = (\partial^\alpha f)_{|\alpha|=a}$ where α ranges over multi indices, and
$$|D^a f(\mathbf{x})| = \sup_{|\alpha|=a} |\partial_\alpha f(\mathbf{x})|. \tag{1.17}$$
We use $A \lesssim B$ to denote that $A \leq C_d B$ where $C_d > 0$ only depends on the ambient dimension. We use $c_d, C_d > 0$ to denote small/large constants depending only on the dimension which may change from line to line.

Chapter 2

The Beurling and Malliavin multiplier theorem

2.1 Introduction

A function is called *band-limited* if it can be synthesized using a finite range of wavelengths,

$$f(x) = \int_{-\sigma}^{\sigma} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi,$$

or equivalently, if its Fourier transform is supported in an interval. This chapter is about Beurling and Malliavin's Theorem 1.5, which constructs band-limited functions with specified decay rate. We restate the theorem here:

Theorem (Beurling and Malliavin [8])**.** *Let $\omega : \mathbb{R} \to \mathbb{R}_{\leq 0}$ be a Lipschitz function satisfying the growth condition*

$$\int_{-\infty}^{\infty} \frac{|\omega(t)|}{1+t^2} \, dt < \infty.$$

Then for any $\sigma > 0$, there exists a nonzero $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subset [-\sigma, \sigma]$ and $|f(x)| \leq e^{\omega(x)}$.

Weights obeying the growth condition (1.9) are called *Poisson integrable*, because the factor $1/(1+t^2)$ comes up in the Poisson kernel.

The story of this theorem involves a connection to complex analysis, and through complex analysis a connection to potential theory—that is, the theory of subharmonic functions. My goal is to explain these connections, and to explain a higher dimensional version of Theorem 1.5 that I proved.

Yet our interest in Beurling and Malliavin's theorem comes not from complex analysis or potential theory, but from uncertainty principles. Uncertainty principles govern the trade-off between localization in physical space and localization in frequency space. There are two broad categories: some control the bulk of the function, while others deal with decay in the tails. The Beurling and Malliavin theorem lands squarely in the tails category, and to put it in context, it is helpful to distinguish these two kinds of uncertainty principles.

2.1.1 Uncertainty principles in the bulk and in the tails

Uncertainty principles in the bulk express the heuristic that

$$\text{If } \operatorname{supp} \hat{f} \subset [-1,1], \text{ then } f \text{ is locally constant at scale } 1.$$

There are several ways to make the notion "locally constant" precise. For example, if η is a smooth bump function which equals one on $[-1,1]$, then $\hat{f} = \hat{f} \cdot \eta$, and inverting the Fourier transform gives $f = f * \eta^\vee$ where $\eta^\vee$ is the inverse Fourier transform. Since $\eta^\vee$ is a Schwarz function it decays faster than any polynomial, so

$$|f(x)| \le C_N \int |f(x-y)|(1+|y|)^{-N}\, dy \qquad \text{for all } N \ge 0 \text{ and } x \in \mathbb{R}. \tag{2.1}$$

Harmonic analysts often pretend that f is constant on every interval of length 1, and then use (2.1) to make the resulting arguments rigorous.

Over the p-adics, this pretend picture—that f is constant on unit intervals—is exactly true. Suppose that $f \in L^2(\mathbb{Q}_p)$ is band-limited in the sense that $\operatorname{supp} \hat{f} \subset B$, where $B \subset \mathbb{Q}_p$ is the p-adic unit ball. The indicator function of the p-adic unit ball is preserved under the Fourier transform, leading to the following convolution equation for f,

$$\operatorname{supp} \hat{f} \subset B \implies \hat{f} = \hat{f} \cdot 1_B \implies f = f * 1_B.$$

If B' is some translate of the unit ball, then for any $x \in B'$

$$f * 1_B(x) = \int_{B'} f(y)\, dy,$$

thus f is constant over any translate of the unit ball.

The locally constant heuristic is not exactly true over the real numbers because, unlike the p-adics, $\eta^\vee$ is not compactly supported. That's why equation (2.1) involves an integral over $\mathbb{R}$ with rapidly decaying weight, rather than an integral over a compact set. This is no accident—it's a consequence of uncertainty in the tails:

If $\hat{f}$ is compactly supported, then f is not compactly supported.

In other words, band-limited functions have tails. A stronger version says that if $\hat{f}$ is compactly supported, f cannot decay exponentially fast, by which we mean there are no constants C and c such that $|f(x)| \le Ce^{-c|x|}$. We will later explore exactly how fast band-limited functions may decay.

When dealing with the locally constant property, tails are an annoying error term—they make equation (2.1) more complicated, and they mess up our heuristic picture that f is constant on intervals of length 1. But in other parts of harmonic analysis, tails are the whole game. This paper is all about tails.

Uncertainty principles in the bulk are robust. They don't care if $\hat{f}$ is truly compactly supported, or if $\hat{f}$ just decays very rapidly. Take the Gaussian function, for example: $f(x) = e^{-\pi x^2}$. Its Fourier transform is also Gaussian, and it decays quickly enough to satisfy the uncertainty principle in the bulk. Tails, on the other hand, are more delicate. A Gaussian just barely fails to have compactly supported Fourier transform, but this slight failure is enough to permit super-exponential decay of the tails.

2.1.2 The decay rate of band-limited functions

We will use complex analysis to prove that a band-limited function cannot be compactly supported. Let $f \in L^2(\mathbb{R})$ have $\operatorname{supp} \hat{f} \subset [-\sigma, \sigma]$, and synthesize f as

$$f(x) = \int_{-\sigma}^{\sigma} \hat{f}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi.$$

By plugging in a complex number we may extend f to an analytic function on the complex plane,

$$\tilde{f}(x+iy) = \int_{-\sigma}^{\sigma} \hat{f}(\xi)\, e^{2\pi i \xi (x+iy)}\, d\xi. \tag{2.2}$$

The integrand is bounded by $|\hat{f}(\xi)|e^{2\pi\sigma|y|}$, with partial derivatives obeying similar bounds, so we may differentiate under the integral sign to prove $\tilde{f}$ is analytic on all of $\mathbb{C}$. Because $\tilde{f}(z)$ is analytic it has at most countably many zeros, and that implies $f(x)$ cannot be compactly supported. If we only knew that $\hat{f}$ decayed exponentially fast, rather than being compactly supported, this argument would still show that f extends to an analytic function in some strip $\{x + iy : |y| \le c\}$, and that would imply f is not compactly supported. Thus band-limited functions cannot decay exponentially quickly.

In order to obtain sharp information about how fast band-limited functions can decay, we need to use the growth of the extension far away from the real axis. If f

is band-limited, the holomorphic extension $\tilde{f}$ grows at most exponentially in the y direction,

$$|\tilde{f}(x+iy)| \le \|\hat{f}\|_{L^1} e^{2\pi\sigma|y|} \le (2\sigma)^{1/2} \|f\|_{L^2}\, e^{2\pi\sigma|y|}.$$

As it turns out, the converse is also true. If $\tilde{f}(x+iy)$ is an analytic function growing at most exponentially in the y-direction, and if its restriction to $\mathbb{R}$ lies in L^2, then its restriction to $\mathbb{R}$ is band-limited. This is known as the Paley-Wiener theorem.

Theorem 2.1 (Paley–Wiener [43, Theorem X]). *A function $f \in L^2(\mathbb{R})$ has Fourier transform supported in $[-\sigma, \sigma]$ if and only f is the restriction to $\mathbb{R}$ of a holomorphic function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that*

$$|\tilde{f}(x+iy)| \le A e^{2\pi\sigma|y|} \qquad \text{for some } A > 0. \tag{2.3}$$

Proof sketch. We already explained that if $f \in L^2(\mathbb{R})$ is band-limited, it extends to a holomorphic function on $\mathbb{C}$ obeying the growth condition (2.3).

For the reverse direction, suppose $\tilde{f}(x+iy)$ is a holomorphic function on $\mathbb{C}$ obeying (2.3), and such that the restriction to $\mathbb{R}$ lies in L^2. For the purpose of this proof sketch we'll also assume

$$\tilde{f}(x+iy) \le A e^{2\pi\sigma|y|}(1+|x|)^{-10}, \tag{2.4}$$

one can remove this hypothesis after the fact. The Fourier transform is given by

$$\widehat{\tilde{f}|_{\mathbb{R}}}(\xi) = \int_{\mathbb{R}} \tilde{f}(x) e^{-2\pi i \xi x}\, dx.$$

The Cauchy integral theorem allows us to shift the contour up or down in the complex plane,

$$\widehat{\tilde{f}|_{\mathbb{R}}}(\xi) = \int_{\mathbb{R}} \tilde{f}(x+iy) e^{-2\pi i \xi (x+iy)}\, dx \qquad \text{for any } y \in \mathbb{R}.$$

To rigorously justify this, apply the Cauchy integral theorem to wider and wider rectangles and use the hypothesis that $\tilde{f}$ decays faster than $|x|^{-10}$ in the x-direction to take a limit. If we put absolute values on the inside and use the growth hypothesis (2.4), we find

$$|\hat{f}(\xi)| \le e^{2\pi\xi y} \int |\tilde{f}(x+iy)|\, dx \le C e^{2\pi(\sigma|y| + \xi y)}.$$

If $\xi > \sigma$ we take $y \to -\infty$ to find $\hat{f}(\xi) = 0$, and if $\xi < -\sigma$ we take $y \to +\infty$ to find $\hat{f}(\xi) = 0$. □

The Paley-Wiener theorem establishes a close connection between band-limited functions and complex analysis. This connection is special to the Euclidean Fourier transform. It doesn't apply to the p-adic Fourier transform, where uncertainty in the tails isn't true.

Our **Main Question** is, what does the magnitude of band-limited functions look like? Or, using the Paley-Wiener theorem, it's equivalent to ask: if an analytic function grows at most exponentially in the y-direction, what can its magnitude on the real axis look like? In order to answer this question we have to understand how the growth of $\tilde{f}$ far from the real axis controls its decay rate on the real axis.

The growth and decay of $\tilde{f}$ is governed by the following fundamental fact of complex analysis:

If $\tilde{f}(z)$ is an analytic function, then $\log|\tilde{f}(z)|$ is a subharmonic function.

Recall that a function $u : \mathbb{C} \to \mathbb{R}$ is called *subharmonic* if it is upper semicontinuous and satisfies the sub-mean value property

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta \qquad \text{for all } z \in \mathbb{C} \text{ and } r > 0.$$

There is an equivalent characterization using the Laplacian: an upper semicontinuous function f is subharmonic if $\Delta f \geq 0$ in the sense of distributions.

The proof that $\log|\tilde{f}|$ is subharmonic is an explicit calculation. In any open ball $U \subset \mathbb{C}$ we may write

$$\tilde{f}(z) = e^{g(z)}(z - \alpha_1)\ldots(z - \alpha_n),$$

where g is analytic and $\alpha_1, \ldots, \alpha_n$ are the roots of f in U. Then

$$\log|\tilde{f}(z)| = \operatorname{Re} g(z) + \log|z - \alpha_1| + \cdots + \log|z - \alpha_n|.$$

$\operatorname{Re} g(z)$ is harmonic, so its Laplacian vanishes. The logarithm function is the fundamental solution to Laplace's equation in two dimensions, so its Laplacian is a delta function:

$$\Delta \log|\tilde{f}(z)| = 2\pi \sum_{1 \leq j \leq n} \delta_{\alpha_j} \qquad \text{where } \delta_{\alpha_j} \text{ is a } \delta\text{-mass centered } \alpha_j.$$

As the Laplacian is a local operator, the same formula extends to all of $\mathbb{C}$

$$\Delta \log|\tilde{f}(z)| = 2\pi \sum_j \delta_{\alpha_j},$$

where the sum may extend over countably many roots. We've learned that if $\tilde{f}$ is an analytic function, then $\Delta \log|\tilde{f}|$ is a sum of delta functions. In particular, $\log|\tilde{f}|$ is subharmonic.

It's often a good idea to just use the fact that $\log|\tilde{f}|$ is subharmonic, and forget the sum of delta functions property. That way we can move away from thinking about analytic functions towards thinking about subharmonic functions. There's a lot to say about subharmonic functions—this is the realm of *potential theory.*

If $\tilde{f}$ is the holomorphic extension of an L^2 function with Fourier transform in $[-\sigma, \sigma]$, then $u(x+iy) = \log|\tilde{f}(x+iy)|$ is a subharmonic function growing at most linearly in the y direction,

$$u(x+iy) \leq A + 2\pi\sigma|y|.$$

Motivated by band-limited functions, we are led to ask: if u is a subharmonic function growing at most linearly in the y-direction, what might u look like on the real axis? Studying this potential theory question tells us most of what we know about the magnitude of band-limited functions. This is the approach we'll take to the Beurling and Malliavin problem. As our first application, we'll use this perspective to prove the following theorem about band-limited functions.

Proposition 2.2. *If $f \in L^2(\mathbb{R})$ is band-limited and nonzero, then*

$$\int_{-\infty}^{\infty} \frac{\log|f(x)|}{1+x^2}\, dx > -\infty. \tag{2.5}$$

Proof. Let $\tilde{f}(x+iy)$ be the holomorphic extension of f, and let $u(x+iy) = \log|\tilde{f}(x+iy)|$. We compare $u(x+iy)$ to the harmonic extension of $\log|f(x)|$ to the upper half plane. This harmonic extension is given by a Poisson integral:

$$v(x+iy) = \int_{-\infty}^{\infty} \log|f(x+t)| \frac{y}{t^2+y^2}\frac{dt}{\pi}, \qquad y > 0$$

is the unique bounded function which is harmonic on the upper half plane $\mathbb{H} = \{x+iy : y > 0\}$ and approaches $\log|f(x)|$ as y goes to zero. To be precise, $v(\bullet + iy) \to \log|f(\bullet)|$ as $y \to 0$ in the L^2 sense.

Consider the auxiliary function

$$w(x+iy) = u(x+iy) - (2\pi\sigma+1)|y| - v(x+iy).$$

As u is subharmonic and the other two terms are harmonic on $\mathbb{H}$, w is subharmonic on $\mathbb{H}$. Because $u(x+iy) \leq A + 2\pi\sigma|y|$, we have $w(x+iy) \to -\infty$ as $y \to \infty$. By the maximum principle for subharmonic functions, w attains its maximum on the real line. But w is zero on the real line, so $w(x+iy) \leq 0$ for all $y \geq 0$.

We evaluate $w(x+i)$ and find

$$0 \geq w(x+i) = u(x+i) - (2\pi\sigma + 1) - \int_{-\infty}^{\infty} \frac{u(x+t)}{1+t^2} \frac{dt}{\pi}.$$

Choose some value of $x \in \mathbb{R}$ where $\tilde{f}(x+i) \neq 0$, so $u(x+i) > -\infty$. Rearranging gives

$$\int_{-\infty}^{\infty} \frac{u(x+t)}{1+t^2} \frac{dt}{\pi} \geq u(x+i) - (2\pi\sigma + 1) > -\infty.$$

The above integral is a shifted version of the integral (2.5), so (2.5) is finite as well. $\square$

The integral in (2.5) is called the *logarithmic integral* of f. The theorem we stated earlier, that band-limited functions cannot decay exponentially quickly, follows from Proposition 2.2. Indeed, if $|f(x)| \leq Ce^{-c|x|}$, then the logarithmic integral would be less than $\int_{-\infty}^{\infty} \frac{\log C - c|x|}{1+x^2}\, dx$, which diverges to negative infinity. On the other hand, Proposition 2.2 permits band-limited functions decaying like $e^{-|x|^{0.9}}$.

The first Beurling and Malliavin multiplier theorem provides a partial converse to Proposition 2.2. It constructs band-limited functions with specified decay rate, provided the logarithmic integral is finite and the logarithmic decay rate is Lipschitz continuous.

In order to understand why the Beurling and Malliavin theorem has a Lipschitz condition, it is helpful to look at a related context where the converse to Proposition 2.2 is sharp. A function $f \in L^2(\mathbb{R})$ has *semi-bounded* Fourier transform if $\operatorname{supp} \hat{f} \subset [0, \infty)$, or equivalently if it can be synthesized as

$$f(x) = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, dx.$$

Just as we extended band-limited functions to analytic functions on $\mathbb{C}$, we may extend semi-bounded functions to analytic functions on the upper half plane

$$\tilde{f}(x+iy) = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i \xi (x+iy)}\, dx, \qquad y \geq 0.$$

One can check that $\tilde{f}$ is analytic on $\mathbb{H}$ and $\tilde{f}(\bullet + iy) \to f(\bullet)$ in the L^2 sense as $y \to 0$. Conversely, if $f \in L^2(\mathbb{R})$ extends to a bounded analytic function on $\mathbb{H}$, then $\operatorname{supp} \hat{f} \subset [0, \infty)$.

Just as it was true for band-limited functions, its also true that semi-bounded functions have finite logarithmic integral. The following Theorem asserts that this is the only constraint on the magnitude of semi-bounded functions.

Theorem 2.3 (Second F. & M. Riesz theorem)**.** *If $f \in L^2(\mathbb{R})$ has semi-bounded Fourier transform, then*

$$\int \frac{\log|f(x)|}{1+x^2}\,dx > -\infty.$$

Conversely, if $g \in L^2(\mathbb{R})$ is nonnegative and has finite logarithmic integral, then there exists a semi-bounded $f \in L^2(\mathbb{R})$ with $|f| = g$.

See Paley and Wiener's book [43, Theorem XII] for a proof.

Proof sketch. The forward direction is just like Proposition 2.2.

For the backward direction, we are given a nonnegative $g \in L^2(\mathbb{R})$ with finite logarithmic integral and must construct a semi-bounded function f with magnitude equal to g.

We construct f as the boundary value of an analytic function on $\mathbb{H}$. Using the hypothesis that the logarithmic integral of g is finite, we can harmonically extend g to the upper half plane by convolving with a Poisson kernel:

$$\begin{aligned} u(x+iy) &= \text{The harmonic extension of } \log|g(x)| \text{ to the upper half plane} \\ &= \int_{-\infty}^{\infty} \log|g(x+t)| \frac{y}{t^2+y^2}\,\frac{dt}{\pi}, \qquad y > 0. \end{aligned}$$

There exists a complementary harmonic function $v(x+iy)$ such that $u+iv$ is analytic on $\mathbb{H}$. Assuming we can make sense of the boundary values $v(x)$, we set $f(x) = e^{u(x)+iv(x)}$, and observe that f is a semi-bounded function with $|f| = g$.

Actually, there is an explicit formula for the boundary values of v on $\mathbb{R}$: this is the *Hilbert transform* of $\log|g|$,

$$v|_{\mathbb{R}} = H[\log|g|](x) = p.v. \int \frac{\log|g(x+t)|}{t}\,\frac{dt}{\pi},$$

and it exists in L^2 because $\log|g|$ does. □

Unlike semi-bounded functions, there's no easy way to determine if there exists a band-limited function with a given magnitude. Theorem 2.3 shows that one needs to impose some regularity condition on the decay rate in addition to logarithmic integrability, and in practice, the Lipschitz condition is easy to verify.

There is an easier version of the Beurling and Malliavin theorem that imposes a harder-to-verify regularity condition on the weight.

Theorem 2.4. *Let $\omega : \mathbb{R} \to \mathbb{R}_{\leq 0}$ satisfy the regularity condition*

$$\|H[\omega']\| < C, \tag{2.6}$$

where $H[\omega']$ is the Hilbert transform of the derivative, and the growth condition

$$\int_{-\infty}^{\infty} \frac{|\omega(t)|}{1+t^2}\, dt < \infty. \tag{2.7}$$

There exists a nonzero $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subset [-2C, 2C]$ and $|f(x)| \leq e^{\omega(x)}$.

If $|\omega'(x)| \leq C$ and $|\omega''(x)| \leq C\langle x\rangle^{-1}$, then (2.6) holds.

The Hilbert transform appears naturally because of its connection to complementary harmonic functions, the same reason it appeared in the proof of Theorem 2.3. While Theorem 2.4 is significantly weaker than Theorem 1.5, it is still useful. For instance, it allows one to construct band-limited functions decaying like $e^{-|x|/(\log(1+|x|)^2)}$, and I don't know any explicit formula for functions like this. It is also strong enough to be used in the proof of Bourgain and Dyatlov's fractal uncertainty principle (Theorem 1.1). We will explain the proof of Theorem 2.4 in §2.2.

2.1.3 Beurling and Malliavin in higher dimensions

I proved a higher dimensional version of the Beurling and Malliavin multiplier theorem. Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ be a weight function, and define the following growth functions that depend on the magnitude of ω on lines through the origin:

$$\begin{aligned} G(\mathbf{x}) &= \int_{1/2}^{2} |\omega(s\mathbf{x})|\, ds, \\ G^*(r) &= \sup_{|\mathbf{x}|=r} G(\mathbf{x}). \end{aligned} \tag{2.8}$$

Also let $\langle \mathbf{x}\rangle = (1+|\mathbf{x}|^2)^{1/2}$.

Theorem 2.5. *Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ be a weight satisfying the three-derivative regularity conditions*

$$|D^a\omega(\mathbf{x})| \leq C_{\mathrm{reg}}\langle \mathbf{x}\rangle^{1-a} \qquad \text{for } 0 \leq a \leq 3, \tag{2.9}$$

and the growth condition

$$\int_0^{\infty} \frac{G^*(r)}{1+r^2}\, dr \leq C_{\mathrm{gr}}. \tag{2.10}$$

Letting $\sigma = C_d \max\{C_{\mathrm{reg}}, C_{\mathrm{gr}}\}$, there exists a nonzero function $f \in L^2(\mathbb{R}^d)$ such that $\operatorname{supp} \hat{f} \subset \mathsf{B}_\sigma$ *and*

$$\begin{aligned} |f(0)| &\geq c(d, C_{\mathrm{reg}}, C_{\mathrm{gr}}) > 0 \\ f(\mathbf{x}) &\leq e^{\omega(\mathbf{x})} && \text{for all } \mathbf{x} \in \mathbb{R}^d, \\ f(\mathbf{x}) &\leq e^{-\frac{\sigma}{C_d}\langle \mathbf{x}\rangle^{1/2}} && \text{for all } \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

See [14, Theorem 1.4] for explicit dependence on the parameters.

The regularity condition (2.9) is a Kohn-Nirenberg symbol condition up to three derivatives. Setting $a = 0$ gives the mild growth condition $|\omega(\mathbf{x})| \le C_{\mathrm{reg}}\langle \mathbf{x} \rangle$, and setting $a = 3$ gives the 3rd derivative condition $|D^3\omega(\mathbf{x})| \le C_{\mathrm{reg}}\langle \mathbf{x} \rangle^{-2}$. Theorem 2.5 is much weaker than the Beurling–Malliavin theorem in one dimension because we require a lot more regularity. Nevertheless, the weights we construct for fractal sets will satisfy (2.9).

Let's discuss the growth condition (2.10). On the one hand, taking $\omega \to G$ smooths out ω and makes it grow less quickly. On the other hand, taking $G \to G^*$ is a maximum and makes it grow more quickly. Morally, G^* is constant on dyadic scales $[2^j, 2^{j+1}]$. In one dimension,

$$\int_0^\infty \frac{G^*(r)}{1+r^2}\, dr \sim \int_{-\infty}^\infty \frac{|\omega(t)|}{1+t^2}\, dt$$

up to constants on both sides, so (2.10) is the same growth condition on $\mathbb{R}$ as in the classical Beurling–Malliavin theorem. The proof of Theorem 2.5 involves estimating different dyadic pieces and then summing them together. We can get a decent estimate for each dyadic piece using only the regularity of ω, and (2.10) is needed to sum these contributions. The growth condition controls the mass of ω on lines through the origin. This makes sense, because the restriction of a band-limited function to any line is also band-limited, thus the restriction of ω to any line must be Poisson integrable. We only need to impose a growth condition on lines through the origin because the regularity hypotheses let us control general lines in terms of their translates that go through the origin.

2.1.4 Outline of the chapter

- In §2.2 we discuss how the Beurling–Malliavin multiplier problem naturally splits into two steps.

  Step 1: Plurisubharmonic Beurling–Malliavin ($\mathcal{PSH}$-BM) is a potential theory problem about constructing plurisubharmonic functions.

  Step 2: Analytic Beurling–Malliavin ($\mathcal{A}$-BM) is a several complex variables problem about constructing entire functions from those plurisubharmonic functions.

  Towards the end of §2.2 we state our solution to each of these steps then use them to prove the higher dimensional Beurling–Malliavin theorem (Theorem 2.5).

- In §2.3 we define an extension operator taking functions on $\mathbb{R}^d$ to functions on $\mathbb{C}^d$ and use this operator to construct plurisubharmonic functions. In §2.4 we show how to take a weight function satisfying the hypotheses of Theorem 2.5 and modify it so the construction in §2.3 is applicable.

  Together, §2.3 and §2.4 complete $\mathcal{PSH}$-BM.

- In §2.5, we give an exposition of Hörmander's L^2-theory for the $\bar{\partial}$-equation. Then, following Bourgain, we apply this theory to solve $\mathcal{A}$-BM.

2.2 Beurling and Malliavin's multiplier problem

2.2.1 Beurling and Malliavin in $\mathbb{R}$

We start by sketching the proof of Theorem 2.4, the easier version of Beurling and Malliavin's multiplier theorem.

Let $\omega : \mathbb{R} \to \mathbb{R}_{\leq 0}$ be a weight function that is Poisson integrable and sufficiently regular. We want to construct a band-limited function decaying like ω. By the Paley-Wiener theorem (Theorem 2.1), it is equivalent to construct a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{cases} |f(x)| \leq e^{\omega(x)} & \text{for } x \in \mathbb{R}, \\ |f(x+iy)| \leq Ae^{2\pi\sigma|y|} & \text{for some } A > 0 \text{ and all } x+iy \in \mathbb{C}, \\ |f(0)| \neq 0. & \end{cases} \tag{2.11}$$

The first equation describes the decay of f; the second ensures the Paley-Wiener criterion is satisfied, so $\operatorname{supp} \hat{f} \subset [-\sigma, \sigma]$; and the third quantifies the non-vanishing of f.

As we discussed in §2.1.2, if $f : \mathbb{C} \to \mathbb{C}$ is analytic then $\log |f|$ is a subharmonic function, and this is the most important piece of information about the magnitude of f. If f satisfies (2.11), then $u = \log |f|$ is a subharmonic function satisfying

$$\begin{cases} u(x) \leq \omega(x) & \text{for } x \in \mathbb{R}, \\ u(x+iy) \leq A + 2\pi\sigma|y| & \text{for some } A > 0 \text{ and all } x+iy \in \mathbb{C}, \\ u(0) > -\infty. & \end{cases} \tag{2.12}$$

Several of the proofs of the Beurling and Malliavin theorem find a converse to this situation. There are two steps: the subharmonic Beurling–Malliavin problem and the analytic Beurling–Malliavin problem.

$\mathcal{SH}$-BM. Find a subharmonic function $u : \mathbb{C} \to \mathbb{R}$ solving (2.12).

$\mathcal{A}$-BM. Find an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $\log |f| \leq u + C$ and $f(0) = 1$.

Each of these steps are approachable problems. As a first attempt towards $\mathcal{SH}$-BM, we'll remove flexibility. We'll try finding a subharmonic function u which is equal to ω on the real line, rather than just being bounded above by ω.

Exact $\mathcal{SH}$-BM. Find a subharmonic function $u : \mathbb{C} \to \mathbb{R}$ such that $u|_{\mathbb{R}} = \omega$ and $u(x + iy) \leq \sigma|y|$.

A natural candidate solution is

$$u = E\omega + C|y|$$

where $E\omega : \mathbb{C} \to \mathbb{R}$ is obtained by separately harmonically extending ω to the upper and lower half planes. There is an explicit formula for $E\omega$ in terms of the Poisson kernel,

$$\begin{aligned} E\omega(x + iy) &= \int_{-\infty}^{\infty} \omega(x + t) \frac{|y|}{t^2 + y^2} \frac{dt}{\pi}, \qquad |y| > 0 \\ &= \int_{-\infty}^{\infty} \frac{\omega(x + ty)}{1 + t^2} \frac{dt}{\pi}. \end{aligned}$$

The function $E\omega$ is harmonic on $\mathbb{C} \setminus \mathbb{R}$, it is symmetric about the real axis, and it extends continuously to $\mathbb{C}$ by taking the value ω on the real line. The Laplacian of $E\omega$, in the sense of distributions, is supported on the real line. The ∂_x^2 part of the Laplacian doesn't contribute, and the ∂_y^2 part contributes twice the normal derivative away from $\mathbb{R}$ times the Lebesgue measure on $\mathbb{R}$. This is a two-dimensional version of the calculation $\frac{d^2}{dx^2}|x| = 2\delta_0$, and can be justified rigorously using integration by parts. The normal derivative of $C|y|$ is just C, so

$$\Delta u = 2\Big(\lim_{y \to 0+} \frac{E\omega(x + iy) - \omega(x)}{y} + C\Big)\delta_{\mathbb{R}}.$$

The operator that takes ω to the normal derivative of $E\omega$ has a name. It is called the Dirichlet-to-Neumann operator of $\mathbb{C} \setminus \mathbb{R}$, because it transforms Dirichlet data for the harmonic extension problem to Neumann data. The Dirichlet-to-Neumann operator is given by the Hilbert transform of the derivative. To see this, let $v(x + iy)$ be the harmonic complement of $E\omega$, so that $E\omega(x + iy) + iv(x + iy)$ is analytic on the upper half plane. The derivative of $E\omega$ in the y-direction is the negative derivative of v in the x-direction,

$$\partial_y E\omega(x+iy) = -\partial_x v(x+iy) \qquad y>0,$$

and taking a limit as $y \to 0$, we find

$$\lim_{y\to 0+} \frac{E\omega(x+iy)-\omega(x)}{y} = -\frac{d}{dx}v(x).$$

The Hilbert transform is precisely the operator mapping $H[\omega](x) = v(x)$. Thus

$$\lim_{y\to 0+} \frac{E\omega(x+iy)-\omega(x)}{y} = H[-\omega'](x),$$

and

$$\Delta u = 2\Big(H[-\omega'] + C\Big)\delta_{\mathbb{R}}. \tag{2.13}$$

Assuming

$$\|H[\omega']\|_\infty < \infty, \tag{2.14}$$

we can take $C = \|H[\omega']\|_\infty$ and then $u = E\omega + C|y|$ will be subharmonic on $\mathbb{C}$. If (2.14) holds, Exact $\mathcal{SH}$-BM is solved in a canonical way.

The main challenge of Theorem 1.5 is solving $\mathcal{SH}$-BM under the weaker condition that ω is just Lipschitz and Poisson integrable. In general the solution will have $u|_{\mathbb{R}} \le \omega$ rather than $u|_{\mathbb{R}} = \omega$. There have been many approaches to this problem over the years. In their original paper Beurling & Malliavin [8] use a variational argument based on the energy method for Dirichlet's problem. Koosis [38] developed an approach based on Perron's method of subsolutions for the Dirichlet problem. Mashreghi, Nazarov, and Havin [41] solve $\mathcal{SH}$-BM by explicitly manipulating a Lipschitz weight ω to a modified weight $\widetilde{\omega} \le \omega$ which satisfies $\|H[\widetilde{\omega}']\| < \infty$.

Stepping back for a moment, $\mathcal{SH}$-BM is a familiar type of problem from potential theory. It is an obstacle problem:

Given an obstacle $v : \mathbb{C} \to \mathbb{R}$, what is the maximal subharmonic function u^* such that

$$u^* \le v?$$

Our obstacle is $v(x+iy) = \omega(x) + 2\pi\sigma|y|$. The pointwise maximum of a family of subharmonic functions is also subharmonic, assuming this pointwise maximum is upper semicontinuous. In the case of Lipschitz obstacles the pointwise maximum is either $-\infty$ or Lipschitz (see Lemma 2.29), so

$$u^*_{\omega,\sigma}(z) = \max\{u(z) : \ u \text{ is subharmonic and } u \le \omega(x) + 2\pi\sigma|y|\} \tag{2.15}$$

is the unique solution to the obstacle problem. If there do not exist any nontrivial subharmonic functions that lie below the obstacle, then $u^*_{\omega,\sigma} = -\infty$. From this perspective, the goal in $\mathcal{SH}$-BM is to prove $u^*_{\omega,\sigma}(0) > -\infty$.

We now turn to the analytic BM problem. We start with a subharmonic function $u : \mathbb{C} \to \mathbb{R}$ and want to construct a nonzero analytic function f with $\log|f| \le u + C$. Beurling and Malliavin's original proof—and many of the proofs since then—solve $\mathcal{A}$-BM by writing f as a product over its roots, as in the Weierstrass product formula:

$$f(z) = \prod_{\alpha_j} (1 - z/\alpha_j).$$

If the roots are real, symmetric around 0, and have bounded density, which we can assume in this setting, then this product converges uniformly in compact sets. The logarithm of the magnitude of f is a sum of logarithmic potentials

$$\log|f| = \sum_{\alpha_j} \log|1 - z/\alpha_j|.$$

Similarly, under some conditions, a subharmonic function u solving (2.12) can be written as a logarithmic potential convolved with the measure $\mu = \Delta u$:

$$u(z) = \int \log|1 - z/\alpha| \, d\mu(\alpha).$$

In order to turn our subharmonic function u into an analytic function, we select a countable sum of delta masses that approximate the measure $\mu = \Delta u$, and make these the roots. For example, if $\|H[-\omega']\|_\infty \le C$, we can take $\mu = (H[-\omega'] + C)\delta_{\mathbb{R}}$ and try to find a discrete approximation to this measure. This approach works to prove Theorem 2.4

Bourgain wrote an unpublished note [10] describing a completely different approach to $\mathcal{A}$-BM, based on the following theorem of Hörmander about the $\bar{\partial}$ equation. Koosis describes a similar approach in his book [38] on the Beurling and Malliavin theorem.

Theorem 2.6 (Hörmander [28, Theorem 2.2.1']). *Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a strictly subharmonic function. Let $\eta \in L^2_{loc}(\mathbb{C})$ satisfy*

$$\int |\eta(z)|^2 \frac{e^{-\varphi(z)}}{\Delta\varphi} \, d\lambda \le C. \tag{2.16}$$

Then there exists $g \in L^2_{loc}(\mathbb{C})$ such that $\partial g/\partial \bar{z} = \eta$ in the sense of distributions and

$$\int |g(z)|^2 e^{-\varphi(z)} \, d\lambda \le 4C.$$

Recall that

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y},$$

and analytic functions are characterized as solutions to $\partial f/\partial \bar{z} = 0$.

Suppose we are given a subharmonic function u solving (2.12), and we would like to construct an analytic function $f : \mathbb{C} \to \mathbb{C}$ solving (2.11). As a first attempt, we could try applying Theorem 2.6 with $\eta = 0$ to find a solution to $\partial f/\partial \bar{z} = 0$ with $\int |f(z)|^2 e^{-u(z)}\, dz < \infty$. This L^2 bound on f is useful: assuming u is Lipschitz, we can combine the L^2 bound with subharmonicity of $|f|^2$ to prove the pointwise bound $\log |f(z)| \leq u(z) + C$, exactly our goal. The problem is, $f(z)$ might equal zero.

In order to construct a nonzero f, we must solve an inhomogeneous $\bar{\partial}$-equation. First set h to be a smooth bump function equal to one on the unit ball and vanishing outside the ball of radius two. Consider the subharmonic weight function

$$\varphi(z) = u(z) + 10 \log |z| + (\text{a strictly subharmonic term}),$$

and use Hörmander's theorem to solve the equation $\bar{\partial} g = \bar{\partial} h$ with respect to this weight. We then set $f = h - g$. The strictly subharmonic term ensures that the denominator in (2.16) remains bounded. The logarithmic term guarantees that $g(0) = 0$, so $f(0) = 1$. As we discussed before, Hörmander's L^2 bound can be used to prove the pointwise bound $\log |f| \leq u + C$, just like we wanted.

Both approaches to $\mathcal{A}$-BM involve some non-canonical choices. In the Weierstrass product approach, the roots can all wiggle a little bit and the resulting analytic function will be just as good of a solution. In Hörmander's theorem, the non-canonical choice is more buried. The proof of Hörmander's theorem uses Hilbert space methods: we work in a weighted L^2 space H, and letting $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$ in this weighted space, we prove an estimate

$$\|f\|_H \leq \|\bar{\partial}^* f\|_H.$$

This estimate is a quantitative version of saying $\bar{\partial}^*$ is injective. As we know from linear algebra, the adjoint being injective corresponds to the operator being surjective. The proof that $\bar{\partial}$ is surjective uses the Hahn-Banach theorem and the Riesz representation theorem. The non-canonical step is the application of the Hahn-Banach theorem to produce a linear extension. I think it is an advantage of Hörmander's approach that the non-canonical step is dealt with in a clean way, and the main work is proving a nice PDE estimate using integration by parts.

2.2.2 Beurling–Malliavin in $\mathbb{R}^d$

Higher dimensional band-limited functions are also characterized by a Paley–Wiener theorem.

Theorem 2.7 (Paley–Wiener). *A function $f \in L^2(\mathbb{R}^d)$ has Fourier support in $\{\xi : |\xi| \le \sigma\}$ if and only if f is the restriction to $\mathbb{R}^d$ of an entire function $\tilde{f} : \mathbb{C}^d \to \mathbb{C}$ such that*

$$|\tilde{f}(\mathbf{x} + i\mathbf{y})| \le A\, e^{2\pi\sigma|\mathbf{y}|} \quad \textit{for some } A > 0. \tag{2.17}$$

In this theorem, $|\mathbf{y}|$ is the ℓ_2-norm of $\mathbf{y}$. The proof is just like the one dimensional case. See [29, Theorem 7.3.1] for a full proof.

Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\le 0}$ be a weight function as in the statement of the higher dimensional Beurling and Malliavin theorem (Theorem 2.5). We would like to construct a band-limited function decaying like ω. By the Paley–Wiener theorem, our goal is to construct an analytic function $f : \mathbb{C}^d \to \mathbb{C}$ such that

$$\begin{cases} |f(\mathbf{x})| \le e^{\omega(\mathbf{x})} & \text{for } x \in \mathbb{R}^d, \\ |f(\mathbf{x} + i\mathbf{y})| \le A e^{2\pi\sigma|\mathbf{y}|} & \text{for some } A > 0 \text{ and all } \mathbf{x} + i\mathbf{y} \in \mathbb{C}^d, \\ |f(0)| \ne 0. \end{cases} \tag{2.18}$$

The key idea in the last section was that

The best way to think about the magnitude of an entire function on $\mathbb{C}$ is to use the fact that its logarithm is subharmonic.

How should we think about the magnitude of an entire function on $\mathbb{C}^d$?

If $f : \mathbb{C}^d \to \mathbb{C}$ is entire, then $\log|f|$ is a *plurisubharmonic* function. A function $u : \mathbb{C}^d \to \mathbb{R}$ is plurisubharmonic if it is upper semicontinuous and its restriction to every complex line is subharmonic. Written explicitly, this means that we have the sub-mean value property

$$u(\mathbf{z}) \le \fint_0^{2\pi} u(\mathbf{z} + e^{i\theta}\mathbf{v})\, d\theta \quad \text{for any } \mathbf{z}, \mathbf{v} \in \mathbb{C}^d. \tag{2.19}$$

Here $\fint_0^{2\pi} = \frac{1}{2\pi}\int_0^{2\pi}$ is a mean value. See the beginning of §2.3.2 for more discussion of the sub-mean value property and equivalence with the definition in terms of a nonnegative Laplacian. A C^2 function u is plurisubharmonic if the Hermitian matrix $\left(\frac{\partial u}{\partial z_j \bar{\partial} z_k}\right)$ is positive semidefinite. It is an key insight from several complex variables that

The best way to think about the magnitude of an entire function on $\mathbb{C}^d$ is to use the fact that its logarithm is plurisubharmonic.

This insight is even more important in several complex variables than it is in one complex variable. In one complex variable the Weierstrass product formula offers an alternative, quite precise way to understand the magnitude of analytic functions, but in higher dimensions we have to rely on plurisubharmonicity.

If $f : \mathbb{C}^d \to \mathbb{C}$ is an analytic function satisfying (2.18), then $u = \log|f|$ is a plurisubharmonic function satisfying

$$\begin{cases} u(\mathbf{x}) \le \omega(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}, \\ u(\mathbf{x}+i\mathbf{y}) \le A + 2\pi\sigma|\mathbf{y}| & \text{for some } A > 0 \text{ and all } x+iy \in \mathbb{C}, \\ u(0) > -\infty. \end{cases} \tag{2.20}$$

Once again, we find a converse to this situation by splitting the Beurling–Malliavin problem into two steps: the plurisubharmonic BM problem and the analytic BM problem.

$\mathcal{PSH}$-BM. Find a plurisubharmonic function $u : \mathbb{C}^d \to \mathbb{R}$ solving (2.20).

$\mathcal{A}$-BM. Find an analytic function $f : \mathbb{C}^d \to \mathbb{C}$ such that $\log|f| \le u + C$ and $f(0) = 1$.

This is the same two steps but with plurisubharmonic in place of subharmonic and vectors $\mathbf{x}, \mathbf{y}$ in place of scalars x, y.

Hörmander's L^2 theorem for the $\bar{\partial}$ equation generalizes naturally to higher dimensions, so we can solve the analytic BM problem using that same approach. (Actually, Hörmander discovered his theorem in several complex variables before it was discovered in one complex variable). Thus our task is to solve the plurisubharmonic BM problem. As a first step, in §2.3 we solve

Exact $\mathcal{PSH}$-BM. Find a plurisubharmonic function $u : \mathbb{C}^d \to \mathbb{R}$ such that $u|_{\mathbb{R}^d} = \omega$ and $u(\mathbf{x}+i\mathbf{y}) \le \sigma|\mathbf{y}|$.

In equation (2.22) we define an extension operator $\omega \to E\omega$ which takes a function on $\mathbb{R}^d$ to a function on $\mathbb{C}^d$. In one dimension E is the Poisson extension operator, and in higher dimensions it is the operator that separately harmonically extends ω to every complex line with real coefficients. Proposition 2.10 says that if ω satisfies two conditions (labeled (i) and (ii)) then $E\omega + C|\mathbf{y}|$ is plurisubharmonic on $\mathbb{C}^d$. Condition (i) says that the Hilbert transform of the derivative is uniformly bounded for every restriction of ω to a line. Condition (ii) is new to higher dimensions, and involves the second derivative of the integral of ω over lines.

It turns out that in higher dimensions, the exact problem really is too restrictive—the weights we care about for fractal sets do not satisfy condition (ii). In §2.4 we show

how to modify a weight ω to a weight $\widetilde{\omega} \leq \omega$ which has similar regularity, but behaves better with respect to integrals over lines. Proposition 2.10 can be applied to $\widetilde{\omega}$, and this solves $\mathcal{PSH}$-BM. Here is the precise statement of our solution to $\mathcal{PSH}$-BM, see §2.4 for the proof.

Proposition 2.8 ($\mathcal{PSH}$-BM)**.** *Suppose that $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ satisfies (2.9) and (2.10), the conditions of Theorem 2.5, with constants C_{reg} and C_{gr}. Then letting $\sigma = C_d \max(C_{\text{reg}}, C_{\text{gr}})$, there exists a plurisubharmonic function $u : \mathbb{C}^d \to \mathbb{R}$ satisfying*

$$u(0) \geq -5C_{\text{reg}},$$
$$u(\mathbf{x} + i\mathbf{y}) \leq \omega(\mathbf{x}) + 2\pi\sigma|\mathbf{y}|.$$

We now turn to the analytic Beurling–Malliavin problem, where we use Hörmander's L^2-theory for the $\bar{\partial}$-equation. Our solution to $\mathcal{A}$-BM is stated using the obstacle problem solution

$$u^*_{\omega,\sigma}(\mathbf{z}) = \sup\{u(\mathbf{z}) \,:\, u \text{ is plurisubharmonic and } u(\mathbf{x} + i\mathbf{y}) \leq \omega(\mathbf{x}) + 2\pi\sigma|\mathbf{y}|\}.$$

In Lemma 2.29, we prove that if $u^*_{\omega,\sigma}$ is finite at any point, it is Lipschitz continuous. Lipschitz continuity guarantees that the supremum defining $u^*_{\omega,\sigma}$ is itself plurisubharmonic. In terms of the obstacle problem, Proposition 2.8 states that if ω satisfies the hypotheses of Theorem 2.5, then $u^*_{\omega,\sigma}(0) \geq -5C_{\text{reg}}$.

Proposition 2.9 ($\mathcal{A}$-BM)**.** *Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ be a Lipschitz weight function, and let $u^*_{\omega,\sigma}$ be the maximal plurisubharmonic function $\leq \omega(\mathbf{x}) + 2\pi\sigma|\mathbf{y}|$. If $u^*_{\omega,\sigma}(0) > -\infty$ then for every $\varepsilon > 0$, there exists an $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat{f} \subset \mathsf{B}_{\sigma+\varepsilon}(0)$ and*

$$\begin{aligned} &|f(0)| \geq c(d,\varepsilon) e^{-2\max\{\|\omega\|_{Lip}, 2\pi\sigma\}} e^{u^*_{\omega,\sigma}(0)}, && \\ &|f(\mathbf{x})| \leq e^{\omega(\mathbf{x})} && \textit{for all } \mathbf{x} \in \mathbb{R}^d, \\ &|f(\mathbf{x})| \leq e^{-\frac{\varepsilon}{C_d}\langle \mathbf{x}\rangle^{1/2}} && \textit{for all } \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

We prove Proposition 2.9 in §2.5.4 following Bourgain's one dimensional argument.

Now we can finish the proof of our higher-dimensional Beurling–Malliavin multiplier theorem by combining $\mathcal{PSH}$-BM and $\mathcal{A}$-BM.

Proof of Theorem 2.5. Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ satisfy the conditions of Theorem 2.5 with constants C_{reg} and C_{gr}, and let $\sigma = 2C_d \max(C_{\text{reg}}, C_{\text{gr}})$ be twice the value of σ in Proposition 2.8. By Proposition 2.8, $u^*_{\omega,\sigma/2}(0) \geq -5C_{\text{reg}}$. By Proposition 2.9, there exists an $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat{f} \subset \mathsf{B}_\sigma$ such that

$$\begin{aligned} &f(0) \geq c(d,\sigma) e^{-2\max\{\|\omega\|_{\text{Lip}}, 2\pi\sigma\}} e^{-5C_{\text{reg}}} \geq c(d, C_{\text{reg}}, C_{\text{gr}}) && \\ &f(\mathbf{x}) \leq e^{\omega(\mathbf{x})} && \text{for all } \mathbf{x} \in \mathbb{R}^d \\ &f(\mathbf{x}) \leq e^{-\frac{\sigma}{C_d}\langle \mathbf{x}\rangle^{1/2}} && \text{for all } \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

□

2.3 Exact plurisubharmonic extension

To solve Exact $\mathcal{SH}$-BM problem in one dimension, we separately harmonically extend ω to each half plane. The harmonic extension is given by integrating against the Poisson kernel,

$$\begin{aligned} E\omega(x+iy) &= \int_{-\infty}^{\infty} \omega(x+t)\frac{|y|}{y^2+t^2}\,\frac{dt}{\pi} \\ &= \int_{-\infty}^{\infty} \frac{\omega(x+ty)}{1+t^2}\,\frac{dt}{\pi} \qquad \text{by change of variables.} \end{aligned} \tag{2.21}$$

In higher dimensions we define an extension operator taking functions on $\mathbb{R}^d$ to functions on $\mathbb{C}^d$ by

$$E\omega(\mathbf{x}+i\mathbf{y}) = \int_{-\infty}^{\infty} \frac{\omega(\mathbf{x}+t\mathbf{y})}{1+t^2}\,\frac{dt}{\pi}. \tag{2.22}$$

If ω is Lipschitz and satisfies the growth condition (2.10) then the integral is finite, see Lemma 2.11. The operator $\omega \to E\omega$ separately harmonically extends ω to every real-linear complex line

$$\ell_{\mathbf{x},\mathbf{y}} = \{\mathbf{x}+z\mathbf{y} \,:\, z\in\mathbb{C}\} \subset \mathbb{C}^d, \quad (\mathbf{x},\mathbf{y}) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}).$$

Equivalently, $E\omega$ is the unique bounded solution to the PDE

$$\begin{cases} \langle (\partial\bar{\partial}E\omega)(\mathbf{x}+i\mathbf{y})\mathbf{y},\mathbf{y}\rangle = 0 & \text{for } \mathbf{x}+i\mathbf{y} \in \mathbb{C}^d \setminus \mathbb{R}^d, \\ E\omega(\mathbf{x}) = \omega(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{2.23}$$

It is not obvious at first that all these separate harmonic extensions combine to give a nice global extension, but equation (2.22) shows that they do.

Given a weight ω,

$$u = E\omega + C|\mathbf{y}| \tag{2.24}$$

will be our candidate plurisubharmonic function. Unlike the one dimensional case $E\omega$ is not plurisubharmonic away from $\mathbb{R}^d$, so adding the term $C|\mathbf{y}|$ will have to both make u satisfy the sub-mean value property (2.19) on complex disks centered at points of $\mathbb{R}^d$ and points off of $\mathbb{R}^d$. Analyzing this equation leads to the following proposition which solves the Exact $\mathcal{PSH}$-BM problem. For $\ell = \{\mathbf{x}+t\hat{\mathbf{y}} \,:\, t\in\mathbb{R}\}$ a line in $\mathbb{R}^d$, let $\omega|_\ell(t) = \omega(\mathbf{x}+t\hat{\mathbf{y}})$ be ω restricted to ℓ (this function just depends on the line itself up to translation and reflection).

Proposition 2.10 (Exact $\mathcal{PSH}$-BM). *Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ be a C^2 and compactly supported function satisfying*

(i) For every line $\ell \subset \mathbb{R}^d$,

$$\|H[\omega|_\ell']\|_\infty \leq C_1. \tag{2.25}$$

(ii) For every $\mathbf{x} \in \mathbb{R}^d$, $\hat{\mathbf{y}}$ a unit vector, and $\hat{\mathbf{v}}$ a unit vector with $\hat{\mathbf{y}} \perp \hat{\mathbf{v}}$,

$$\int_{-\infty}^{\infty} \langle (D^2\omega(\mathbf{x} + t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle \frac{dt}{\pi} \geq -C_2. \tag{2.26}$$

If $C \geq \max(C_1, C_2)$, then

$$u(\mathbf{x} + i\mathbf{y}) = E\omega(\mathbf{x} + i\mathbf{y}) + C|\mathbf{y}|$$

is plurisubharmonic on $\mathbb{C}^d$ and continuous. We have

$$u(\mathbf{x}) \leq u(\mathbf{x} + i\mathbf{y}) \leq u(\mathbf{x}) + 2C|\mathbf{y}|. \tag{2.27}$$

Condition (i) implies that $E\omega + C|\mathbf{y}|$ satisfies the sub-mean value property for complex disks centered at points of $\mathbb{R}^d$. Condition (ii) is new to higher dimensions and it implies that $E\omega + C|\mathbf{y}|$ is plurisubharmonic on $\mathbb{C}^d \setminus \mathbb{R}^d$.

Remarks. 1. It turns out that in $d \geq 2$ condition (i) essentially follows from condition (ii). This observation is due to Semyon Dyatlov, see [18].

2. Proposition 2.10 is strong enough to prove Proposition 2.8 in the special case of radial weights $\omega(\mathbf{x}) = f(|\mathbf{x}|)$.

2.3.1 Basic properties of the extension operator

Let

$$E_R\omega(\mathbf{x} + i\mathbf{y}) = \int_{|t| \leq R} \frac{\omega(\mathbf{x} + t\mathbf{y})}{1 + t^2} \frac{dt}{\pi} \tag{2.28}$$

be the partial integral for $E\omega$.

Lemma 2.11. *Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz with constant C_{Lip} and satisfy the growth condition (2.10) with constant C_{gr}. Then the integral defining $E\omega$ is absolutely convergent and $E_R\omega \to E\omega$ uniformly on compact subsets.*

Proof. First of all,

$$\int \frac{|\omega(\mathbf{x}+t\mathbf{y})|}{1+t^2}\,\frac{dt}{\pi} \le C_{\mathrm{Lip}}|\mathbf{x}| + \int \frac{|\omega(t\mathbf{y})|}{1+t^2}\,\frac{dt}{\pi} < \infty$$

using both the Lipschitz property and the growth condition.

Let $R \ge 1$ and set

$$\mathrm{Err}_R(\mathbf{x}+i\mathbf{y}) = \int_{|t|\ge R} \frac{|\omega(\mathbf{x}+t\mathbf{y})|}{1+t^2}\,\frac{dt}{\pi}.$$

Using that ω is Lipschitz we have

$$\mathrm{Err}_R(\mathbf{x}+i\mathbf{y}) \le \frac{2C_{\mathrm{Lip}}|\mathbf{x}|}{R} + \int_{|t|\ge R} \frac{|\omega(t\mathbf{y})|}{1+t^2}\,\frac{dt}{\pi}.$$

We have

$$\int_{|t|\ge R} \frac{|\omega(t\mathbf{y})|}{1+t^2}\,\frac{dt}{\pi} = \int_{|t|\ge R} \frac{|\omega(t\mathbf{y})|}{1+t^2} 1_{|t\mathbf{y}|\le 1}\,dt + \int_{|t|\ge R} \frac{|\omega(t\mathbf{y})|}{1+t^2} 1_{|t\mathbf{y}|\ge 1}\,dt.$$

The first term is $\lesssim (|\omega(0)| + C_{\mathrm{Lip}})/R$. For the second term we replace R by $\widetilde{R} = \max(R, 1/|\mathbf{y}|)$ and estimate

$$\begin{aligned}
\int_{|t|\ge \widetilde{R}} \frac{|\omega(t\mathbf{y})|}{1+t^2}\,dt &\le \int_{|t|\ge \widetilde{R}} \frac{|\omega(t\mathbf{y})|}{t^2}\,dt \lesssim \int_{|t|\ge \widetilde{R}} \int_{1/2}^{2} \frac{|\omega(t\mathbf{y})|}{(t/r)^2}\,dr dt \\
&\lesssim \int_{|t|\ge \widetilde{R}} \int_{1/2}^{2} \frac{|\omega(rt\mathbf{y})|}{t^2}\,dr dt \lesssim \int_{\widetilde{R}}^{\infty} \frac{G^*(|t\mathbf{y}|)}{t^2}\,dt \\
&\lesssim \int_{\widetilde{R}|\mathbf{y}|}^{\infty} |\mathbf{y}| \frac{G^*(t)}{t^2}\,dt \lesssim |\mathbf{y}| \int_{\widetilde{R}|\mathbf{y}|}^{\infty} \frac{G^*(t)}{1+t^2}\,dt \\
&\lesssim R^{-1/2} \int_{0}^{\infty} \frac{G^*(t)}{1+t^2}\,dt + |\mathbf{y}| \int_{R^{1/2}}^{\infty} \frac{G^*(t)}{1+t^2}\,dt
\end{aligned}$$

where in the last line we split into the two cases $|\mathbf{y}| \le R^{-1/2}$ and $|\mathbf{y}| \ge R^{-1/2}$. Suppose that $|\mathbf{y}|, |\mathbf{x}| \le M$. Then combining our estimates,

$$\mathrm{Err}_R(\mathbf{x}+i\mathbf{y}) \lesssim \frac{C_{\mathrm{Lip}}M}{R} + \frac{|\omega(0)| + C_{\mathrm{Lip}}}{R} + R^{-1/2}C_{\mathrm{gr}} + M \int_{R^{1/2}}^{\infty} \frac{G^*(t)}{1+t^2}\,dt.$$

The right hand side goes to zero as $R \to \infty$ so $\mathrm{Err}_R(\mathbf{x}+i\mathbf{y})$ goes to zero uniformly in compact subsets. It follows that $E_R\omega \to E\omega$ uniformly on compact subsets. □

Next we prove our earlier claim that the Dirichlet-to-Neumann operator of $\mathbb{C} \setminus \mathbb{R}$ is $\omega \to H[-\omega']$.

Lemma 2.12. *Let $\omega \in C_0^1(\mathbb{R})$ and let $u = E\omega$ be the bounded harmonic extension of ω to the upper half plane $\mathbb{H}$. Then*

$$\partial_y u(x + i0) = H[-\omega']. \tag{2.29}$$

Proof. Because u is harmonic on the upper half plane $\mathbb{H}$ and $u(x+iy) \to 0$ as $y \to \infty$, we can write $u = \operatorname{Re} f$ where $f = u + iv$ is analytic on $\mathbb{H}$ and $f(x+iy) \to 0$ as $y \to \infty$. By the Cauchy-Riemann equations,

$$\partial_y u = -\partial_x v. \tag{2.30}$$

For fixed $y > 0$, let $u_y(x) = u(x+iy)$ and $v_y(x) = v(x+iy)$ be functions on $\mathbb{R}$. By the complex analytic characterization of the Hilbert transform,

$$v_y = H[u_y] \quad \text{for all } y > 0. \tag{2.31}$$

Thus

$$\partial_y u(x + i\varepsilon) = -\partial_x v_\varepsilon = H[-u_\varepsilon'](x) \quad \text{for all } \varepsilon > 0. \tag{2.32}$$

We have $u_\varepsilon \to u$ in C^1 as $\varepsilon \to 0$, so taking a limit gives the result. $\square$

Now we establish some basic properties of $\omega \to E\omega$.

Lemma 2.13. *Suppose $\omega \in C_0^2(\mathbb{R}^d)$. Then*

(a) $E\omega$ is C^2 on $\mathbb{C}^d \setminus \mathbb{R}^d$.

(b) For $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^d \setminus \mathbb{R}^d$, let $\ell_{\mathbf{x},\mathbf{y}} = \{\mathbf{x} + t\hat{\mathbf{y}} : t \in \mathbb{R}\}$. We have

$$|E\omega(\mathbf{x} + i\mathbf{y}) - E\omega(\mathbf{x})| \le |\mathbf{y}| \, \|H[\omega|_{\ell_{\mathbf{x},\mathbf{y}}}']\|_\infty. \tag{2.33}$$

(c) Let $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^d \setminus \mathbb{R}^d$. The Hermitian form $\partial\bar{\partial}E\omega(\mathbf{x} + i\mathbf{y})$ has real coefficients. Let $\mathbf{v} \in \mathbb{R}^d$ be given by

$$\mathbf{v} = \mathbf{v}_1 + r\hat{\mathbf{y}}, \quad \mathbf{v}_1 \perp \hat{\mathbf{y}}. \tag{2.34}$$

Then

$$\begin{aligned} \langle(\partial\bar{\partial}E\omega(\mathbf{x} + i\mathbf{y}))\mathbf{v}, \mathbf{v}\rangle &= \langle(\partial\bar{\partial}E\omega(\mathbf{x} + i\mathbf{y}))\mathbf{v}_1, \mathbf{v}_1\rangle \\ &= \frac{|\mathbf{v}_1|^2}{4|\mathbf{y}|} \int_{-\infty}^{\infty} \langle(D^2\omega(\mathbf{x} + t\hat{\mathbf{y}}))\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_1\rangle \, \frac{dt}{\pi}. \end{aligned} \tag{2.35}$$

Proof of (a). Differentiate under the integral sign in (2.22). □

Proof of (b). First we show (2.33) in $d = 1$. Let $\omega \in C_0^2(\mathbb{R})$, and let $u = E\omega$ be the harmonic extension to $\mathbb{H}$. We have

$$u(x + iy) \leq u(x) + y \sup_{z \in \mathbb{H}} \partial_y u(z).$$

The function $\partial_y u$ is harmonic on $\mathbb{H}$, so by the maximum principle

$$\sup_{z \in \mathbb{C}} \partial_y u(z) = \sup_{x \in \mathbb{R}} \partial_y u(x + i0).$$

By Lemma 2.12 we have $\sup_{x \in \mathbb{R}} \partial_y u(x + i0) \leq \|H[\omega']\|_\infty$. Thus $u(x + iy) \leq u(x) + y\|H[\omega']\|_\infty$. The same argument shows $u(x + iy) \geq u(x) - y\|H[\omega']\|_\infty$.

Now let $\mathbf{x} + i\hat{\mathbf{y}} \in \mathbb{C}^d \setminus \mathbb{R}^d$, $\hat{\mathbf{y}}$ a unit vector. Let

$$u(z) = E\omega(\mathbf{x} + z\hat{\mathbf{y}}), \quad z \in \mathbb{C}.$$

Then $u(z)$ harmonically extends $\omega|_\ell(t) = \omega(\mathbf{x} + t\hat{\mathbf{y}})$, so

$$|E\omega(\mathbf{x} + ir\hat{\mathbf{y}}) - E\omega(\mathbf{x})| = |u(ir) - u(0)| \leq r \, \|H[\omega|_\ell']\|_\infty.$$

□

Proof of (c). First of all, $\partial\bar{\partial}E\omega$ has real coefficients because

$$\begin{aligned}
\operatorname{Im} \partial_{z_j} \bar{\partial}_{z_k} E\omega &= \frac{1}{4}(\partial_{x_j}\partial_{y_k} - \partial_{x_k}\partial_{y_j})E\omega \\
&= \frac{1}{4} \int \frac{t(\partial_j\partial_k\omega)(\mathbf{x} + t\mathbf{y}) - t(\partial_k\partial_j\omega)(\mathbf{x} + t\mathbf{y})}{1 + t^2} \, \frac{dt}{\pi} = 0.
\end{aligned}$$

It follows that $\partial\bar{\partial}E\omega = \frac{1}{4}(D_x^2 + D_y^2)E\omega$. For any $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^d \setminus \mathbb{R}^d$ we have

$$\begin{aligned}
\partial\bar{\partial}E\omega(\mathbf{x} + i\mathbf{y}) &= \frac{1}{4}(D_x^2 + D_y^2) \int_{-\infty}^{\infty} \frac{\omega(\mathbf{x} + t\mathbf{y})}{1 + t^2} \, \frac{dt}{\pi} \\
&= \frac{1}{4} \int_{-\infty}^{\infty} D^2\omega(\mathbf{x} + t\mathbf{y}) \, \frac{dt}{\pi}.
\end{aligned}$$

It is nice in this computation that the differentiation on $\mathbf{x}$ and $\mathbf{y}$ combine to give a $1 + t^2$ factor, cancelling the $\frac{1}{1+t^2}$ factor in the Poisson measure. Notice $\partial\bar{\partial}E\omega(\mathbf{x}, \mathbf{y}) = \frac{1}{|\mathbf{y}|}\partial\bar{\partial}E\omega(\mathbf{x}, \hat{\mathbf{y}})$ by change of variables. Also notice that if $\mathbf{v}_1 \perp \hat{\mathbf{y}}$ then

$$\langle (\partial\bar{\partial}E\omega(\mathbf{x} + i\mathbf{y}))\mathbf{v}_1, \mathbf{v}_1 \rangle = \frac{|\mathbf{v}_1|^2}{4|\mathbf{y}|} \int_{-\infty}^{\infty} \langle (D^2\omega(\mathbf{x} + t\hat{\mathbf{y}}))\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_1 \rangle \, \frac{dt}{\pi}$$

and (2.35) holds in this special case. To prove (2.35) in general, we will show that if $\mathbf{v} = \mathbf{v}_1 + r\mathbf{y}$ as in (2.34) then

$$\langle (\partial\bar{\partial} E\omega(\mathbf{x} + i\mathbf{y}))\mathbf{v}, \mathbf{v}\rangle = \langle (\partial\bar{\partial} E\omega(\mathbf{x} + i\mathbf{y}))\mathbf{v}_1, \mathbf{v}_1\rangle. \tag{2.36}$$

Define the X-ray transform by

$$X(f)(\mathbf{x}, \hat{\mathbf{y}}) = \int_{-\infty}^{\infty} f(\mathbf{x} + t\hat{\mathbf{y}})\, dt, \quad (\mathbf{x}, \hat{\mathbf{y}}) \in \mathbb{R}^d \times \mathbb{S}^{d-1}.$$

We have

$$\partial\bar{\partial} E\omega(\mathbf{x} + i\hat{\mathbf{y}}) = \frac{1}{4} X(D^2\omega)(\mathbf{x}, \hat{\mathbf{y}}) = \frac{1}{4} D_x^2 (X\omega)(\mathbf{x}, \hat{\mathbf{y}}).$$

The X-ray transform is constant along lines, meaning $X\omega(\mathbf{x} + a\hat{\mathbf{y}}, \hat{\mathbf{y}}) = X\omega(\mathbf{x}, \hat{\mathbf{y}})$. It follows from this property that

$$D_x^2(X\omega)(\mathbf{x}, \hat{\mathbf{y}})\hat{\mathbf{y}} = 0$$

where $D_x^2(X\omega)$ is viewed as a linear map. Equation (2.36) follows. □

The following lemma isn't used in the proof of Proposition 2.10, but will be used in the application of Proposition 2.10 to Proposition 2.8. Note that this Lemma is not necessary for the application to fractal uncertainty because the weights we construct to prove Theorem 1.2 are compactly supported.

Lemma 2.14. *Let $\omega_j \in C(\mathbb{R}^d)$, $j \geq 1$ be a sequence converging to $\omega \in C(\mathbb{R}^d)$ uniformly on compact subsets. Suppose $\{\omega_j\}$ is uniformly Lipschitz,*

$$|\omega_j(\mathbf{x}_1) - \omega_j(\mathbf{x}_2)| \leq C_{\mathrm{Lip}} |\mathbf{x}_1 - \mathbf{x}_2| \quad \textit{for all } j \geq 1, \textit{ all } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d, \tag{2.37}$$

and satisfies the uniform growth condition

$$\begin{gathered} G^*(r) = \sup_{j\geq 1} \sup_{|\mathbf{y}|=r} \int_{1/2}^{2} |\omega_j(s\mathbf{y})|\, ds, \\ \int_0^\infty \frac{G^*(r)}{1 + r^2}\, dr < \infty. \end{gathered} \tag{2.38}$$

Then $E\omega_j \to E\omega$ uniformly on compact subsets.

Proof. Let

$$G_1^*(r) = \sup_{|\mathbf{y}|=r} \int_{1/2}^{2} |\omega(s\mathbf{y})|\, ds$$

be the growth function of ω. Because this integral is over a compact region, $G_1^*(r) \leq G^*(r)$. Let

$$E_R\omega_j(\mathbf{x}+i\mathbf{y}) = \int_{|t|\leq R} \frac{\omega_j(\mathbf{x}+t\mathbf{y})|}{1+t^2}\,\frac{dt}{\pi}$$

and similarly for ω. Let $\varepsilon > 0$ and $M > 0$ be given. By Lemma 2.11, if $R \geq R_0(\varepsilon, M)$ then for all $|\mathbf{x}|, |\mathbf{y}| \leq M$ we have

$$|E_R\omega_j(\mathbf{x}+i\mathbf{y}) - E\omega_j(\mathbf{x}+i\mathbf{y})| \leq \varepsilon,$$
$$|E_R\omega(\mathbf{x}+i\mathbf{y}) - E\omega(\mathbf{x}+i\mathbf{y})| \leq \varepsilon.$$

If $j \geq j_0(\varepsilon)$ then for all $|\mathbf{x}|, |\mathbf{y}| \leq M$ we have.

$$|E_R\omega_j(\mathbf{x}+i\mathbf{y}) - E_R\omega(\mathbf{x}+i\mathbf{y})| \leq \varepsilon.$$

Combining these we see $|E\omega_j(\mathbf{x}+i\mathbf{y}) - E\omega(\mathbf{x}+i\mathbf{y})| \leq \varepsilon$ in the same region. □

2.3.2 Proof of Proposition 2.10

Let $U \subset \mathbb{C}^d$ be an open set. In §2.2.2 we defined a function $u : U \to \mathbb{R}$ to be plurisubharmonic if it is upper semicontinuous and every restriction to a complex line is subharmonic, meaning the Laplacian is non-negative in the distributional sense. An equivalent condition is that u is upper semicontinuous and satisfies the sub-mean value property

$$u(\mathbf{z}) \leq \fint_0^{2\pi} u(\mathbf{z}+e^{i\theta}\mathbf{v})\,d\theta \quad \text{for all } |\mathbf{v}| < r_0(\mathbf{z}) \tag{2.39}$$

where $r_0(\mathbf{z}) > 0$ may depend arbitrarily on $\mathbf{z}$. For a proof see [29, Theorem 4.1.11].

The upshot is that the proof of Proposition 2.10 can be split into two parts. Let $C > \max(C_1, C_2)$. We show $u = E\omega + C|\mathbf{y}|$ is plurisubharmonic, and it follows from continuity that we can take $C = \max(C_1, C_2)$ as well. First we prove u is plurisubharmonic on $\mathbb{C}^d \setminus \mathbb{R}^d$ using our computation of $\partial\bar{\partial}E\omega$. Then we prove (2.39) holds for all $\mathbf{z} \in \mathbb{R}^d$ using our estimates on $E\omega$ near $\mathbb{R}^d$ (2.33). It is in this step that we use $C > \max(C_1, C_2)$ rather than $C \geq \max(C_1, C_2)$.

Before proving plurisubharmonicity we show (2.27). By (2.33) we have

$$\omega(\mathbf{x}) - C_1|\mathbf{y}| \leq E\omega(\mathbf{x}+i\mathbf{y}) \leq \omega(\mathbf{x}) + C_1|\mathbf{y}|$$

and because $C \geq C_1$, we have

$$\omega(\mathbf{x}) \leq E\omega(\mathbf{x}+i\mathbf{y}) + C|\mathbf{y}| \leq \omega(\mathbf{x}) + 2C|\mathbf{y}|$$

as desired.

Plurisubharmonicity on $\mathbb{C}^d \setminus \mathbb{R}^d$

We start with a Lemma.

Lemma 2.15. *Let $U \subset \mathbb{C}^d$ be an open set. If $v \in C^2(U)$, then v is plurisubharmonic on U if and only if $\partial\bar{\partial}v(\mathbf{z})$ is positive semidefinite for every $\mathbf{z} \in U$.*

See [30, Corollary 4.1.5] for a proof.

By Lemma 2.13(a), $u \in C^2(\mathbb{C}^d \setminus \mathbb{R}^d)$, so it suffices to show $\partial\bar{\partial}u(\mathbf{x} + i\mathbf{y})$ is positive semidefinite for all $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^d \setminus \mathbb{R}^d$ in order to establish (2.39) on $\mathbb{C}^d \setminus \mathbb{R}^d$. For $\mathbf{y} \neq 0$ we have

$$\partial\bar{\partial}|\mathbf{y}| = \frac{1}{4|\mathbf{y}|}(I - \hat{\mathbf{y}}\hat{\mathbf{y}}^t) = \frac{1}{4|\mathbf{y}|}\pi_{\mathbf{y}}^{\perp} \tag{2.40}$$

as a Hermitian form. That is, $\partial\bar{\partial}|\mathbf{y}|$ orthogonally projects away from $\mathbf{y}$ and then scales by $\frac{1}{4|\mathbf{y}|}$. If $\mathbf{v} = \mathbf{v}_1 + r\mathbf{y}$ with $\mathbf{v}_1 \perp \mathbf{y}$, then

$$\langle(\partial\bar{\partial}|\mathbf{y}|)\mathbf{v}, \mathbf{v}\rangle = \frac{|\mathbf{v}_1|^2}{4|\mathbf{y}|}. \tag{2.41}$$

Because $\partial\bar{\partial}E\omega$ is real-linear, the goal is to show that for any $\mathbf{v} \in \mathbb{R}^d$,

$$\langle\partial\bar{\partial}(E\omega + C|\mathbf{y}|)\mathbf{v}, \mathbf{v}\rangle \geq 0. \tag{2.42}$$

Write $\mathbf{v} = \mathbf{v}_1 + r\mathbf{y}$, $\mathbf{v}_1 \perp \mathbf{y}$. Combining (2.35) and (2.41), we have

$$\begin{aligned}\langle(\partial\bar{\partial}E\omega + C|\mathbf{y}|)\mathbf{v}, \mathbf{v}\rangle &= \frac{|\mathbf{v}_1|^2}{4|\mathbf{y}|}\int_{-\infty}^{\infty}\langle(D^2\omega(\mathbf{x} + t\hat{\mathbf{y}}))\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_1\rangle\,\frac{dt}{\pi} + C\frac{|\mathbf{v}_1|^2}{4|\mathbf{y}|}\\ &= \frac{|\mathbf{v}_1|^2}{4|\mathbf{y}|}\left(C + \int_{-\infty}^{\infty}\langle(D^2\omega(\mathbf{x} + t\hat{\mathbf{y}}))\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_1\rangle\,\frac{dt}{\pi}\right)\end{aligned}$$

so if $C \geq C_2$ then (2.42) holds.

Plurisubharmonicity on $\mathbb{R}^d$

This part is analogous to the 1D argument. Let $C \geq C_1 + \varepsilon$. Let $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{v} \in \mathbb{C}^d \setminus \{0\}$. By (2.33),

$$u(\mathbf{x} + e^{i\theta}\mathbf{v}) \geq \omega(\mathbf{x} + \mathrm{Re}(e^{i\theta}\mathbf{v})) + \varepsilon|\mathrm{Im}(e^{i\theta}\mathbf{v})|. \tag{2.43}$$

Because $\omega \in C_0^2(\mathbb{R}^d)$ there is a constant $\lambda > 0$ so that

$$|\omega(\mathbf{x} + \mathbf{h}) - \omega(\mathbf{x}) - \nabla\omega(\mathbf{x}) \cdot \mathbf{h}| \leq \lambda|\mathbf{h}|^2$$

for all $\mathbf{x}, \mathbf{h} \in \mathbb{R}^d$. Integrating, we find

$$\fint_0^{2\pi} \omega(\mathbf{x} + \mathrm{Re}(e^{i\theta}\mathbf{v}))\, d\theta \geq \omega(\mathbf{x}) - \lambda|\mathbf{v}|^2$$

for all $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{v} \in \mathbb{C}^d$. On the other hand,

$$\begin{aligned}\fint_0^{2\pi} |\operatorname{Im}(e^{i\theta}\mathbf{v})|\, d\theta &\geq |\mathbf{v}| \fint_0^{2\pi} |\operatorname{Im}(e^{i\theta}\hat{\mathbf{v}})|^2\, d\theta \\ &= \frac{|\mathbf{v}|}{2} \fint_0^{2\pi} (|\operatorname{Im}(e^{i\theta}\hat{\mathbf{v}})|^2 + |\operatorname{Re}(e^{i\theta}\hat{\mathbf{v}})|^2)\, d\theta = \frac{|\mathbf{v}|}{2},\end{aligned}$$

so

$$\begin{aligned}\fint_0^{2\pi} u(\mathbf{x} + e^{i\theta}\mathbf{v})\, d\theta &\geq \fint_0^{2\pi} \omega(\mathbf{x} + \mathrm{Re}(e^{i\theta}\mathbf{v}))\, d\theta + \varepsilon \fint_0^{2\pi} |\operatorname{Im}(e^{i\theta}\mathbf{v})|\, d\theta \\ &\geq \omega(\mathbf{x}) - \lambda|\mathbf{v}|^2 + \frac{\varepsilon}{2}|\mathbf{v}|.\end{aligned}$$

If $|\mathbf{v}| \leq \frac{\varepsilon}{2\lambda}$ then the sub-mean value property holds.

2.4 Modifying weight functions

In this section we prove Proposition 2.8. Suppose $\omega : \mathbb{R}^d \to \mathbb{R}^d_{\leq 0}$ satisfies the hypotheses of Theorem 2.5. It would be nice if ω also satisfied the hypotheses of Proposition 2.10, because then we could complete the $\mathcal{PSH}$-BM problem. In general condition (i) will be satisfied, but condition (ii) on the integral of the second derivative over lines (2.26) will not. Using regularity of $D^2\omega$ (2.9) we can get a decent estimate for (2.26) on each dyadic scale by putting absolute values inside the integral, but these contributions will not be summable. To fix this issue we modify the weight ω to a new weight $\widetilde{\omega} \leq \omega$ which has a lot of cancellation in (2.26). In our estimates we will not put absolute values inside the integral.

An important observation is that when we zoom out far enough all lines look like they pass through the origin. To be a bit more precise, using the regularity hypothesis (2.9) it suffices to estimate (2.26) for lines through the origin. For a function $f : \mathbb{R}^d \to \mathbb{R}$, let

$$\pi_{\mathbb{S}^{d-1}} f(\hat{\mathbf{v}}) = \int_0^\infty f(t\hat{\mathbf{v}})\, t^{-2} dt$$

be a weighted spherical projection of f. The factor t^{-2} allows us to compare the translational derivative to the rotational derivative.

Lemma 2.16. *Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a C^2 function compactly supported and supported away from the origin. Let $\hat{\mathbf{y}} \perp \hat{\mathbf{v}}$. Then*

$$\int_0^\infty \langle (D^2 f(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt = \pi_{\mathbb{S}^{d-1}} f(\hat{\mathbf{y}}) + \frac{d^2}{d\theta^2}\Big|_{\theta=0} \pi_{\mathbb{S}^{d-1}} f(\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{v}}\sin\theta). \tag{2.44}$$

Proof. We have

$$\frac{d^2}{d\theta^2}\Big|_{\theta=0} f(t\hat{\mathbf{y}}\cos\theta + t\hat{\mathbf{v}}\sin\theta) = t^2\langle (D^2 f(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle - t(\partial_{\hat{\mathbf{y}}} f)(t\hat{\mathbf{y}}).$$

Integrating,

$$\begin{aligned}\frac{d^2}{d\theta^2}\Big|_{\theta=0} \pi_{\mathbb{S}^{d-1}} f(\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{v}}\sin\theta) &= \int_0^\infty \frac{d^2}{d\theta^2}\Big|_{\theta=0} f(t\hat{\mathbf{y}}\cos\theta + t\hat{\mathbf{v}}\sin\theta)\, t^{-2} dt \\ &= \int_0^\infty \langle (D^2 f(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt - \int_0^\infty t^{-1} \frac{d}{dt} f(t\hat{\mathbf{y}})\, dt \\ &= \int_0^\infty \langle (D^2 f(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt - \int_0^\infty f(t\hat{\mathbf{y}})\, t^{-2} dt\end{aligned}$$

using integration by parts in the last step. □

We write the modified weight as a sum of dyadic pieces, $\widetilde{\omega} = \sum_k \widetilde{\omega}_k$. The idea is to design each piece $\widetilde{\omega}_k$ so that $\pi_{\mathbb{S}^{d-1}}\widetilde{\omega}_k \equiv q_k = \text{const}$. Lemma 2.16 then gives

$$\int_0^\infty \langle (D^2 \widetilde{\omega}_k(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt = q_k, \tag{2.45}$$

and as long as $\sum_k |q_k| < \infty$ we obtain a favorable estimate.

We implement this plan with the following two Lemmas. The first Lemma modifies the weight $\omega \to \widetilde{\omega}$.

Lemma 2.17. *Suppose that $\omega : \mathbb{R}^d \to \mathbb{R}_{\le 0}$ satisfies (2.9) and (2.10), the conditions of Theorem 2.5, with constants C_{reg} and C_{gr}. Then there exists a weight $\widetilde{\omega} = \sum_{k\ge 0} \widetilde{\omega}_k$ such that*

- *(i)* $\widetilde{\omega}(\mathbf{x}) \le \omega(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^d$,
- *(ii)* $\widetilde{\omega}(\mathbf{x}) = \omega(\mathbf{x})$ *for* $|\mathbf{x}| \le 2$,
- *(iii)* $\operatorname{supp} \widetilde{\omega}_0 \subset \{\mathbf{x} : |\mathbf{x}| \le 5\}$ *and* $\operatorname{supp} \widetilde{\omega}_k \subset \{\mathbf{x} : 2^{k-1} \le |\mathbf{x}| \le 2^{k+2}\}$ *for* $k \ge 1$,
- *(iv) We have*

$$\|D^a \widetilde{\omega}_k\|_\infty \le C'_{\text{reg}} 2^{(1-a)k} \quad \textit{for } 0 \le a \le 3, \tag{2.46}$$

(v) For $k \geq 5$, $\pi_{\mathbb{S}^{d-1}} \widetilde{\omega}_k = q_k$ is constant over the unit sphere and

$$\sum_k |q_k| \leq C'_{\mathrm{gr}}. \tag{2.47}$$

We may take $C'_{\mathrm{reg}} \leq C_d C_{\mathrm{reg}}$ and $C'_{\mathrm{gr}} \leq C_d C_{\mathrm{gr}}$.

Condition (2.46) is just another way of writing the regularity condition (2.9), and (2.47) is another way of writing the growth condition (2.10).

The second lemma analyzes the modified weight $\widetilde{\omega}$ and shows it is admissible for Proposition 2.10.

Lemma 2.18. *Suppose that $\widetilde{\omega} = \sum_{k \geq 0} \widetilde{\omega}_k$ satisfies the conditions Lemma 2.17(ii)-(v). Let $C = C_d \max(C'_{\mathrm{reg}}, C'_{\mathrm{gr}})$. Then $u = E\widetilde{\omega} + C|\mathbf{y}|$ is continuous and plurisubharmonic on $\mathbb{C}^d$ and satisfies*

$$u(\mathbf{x}) \leq u(\mathbf{x} + i\mathbf{y}) \leq u(\mathbf{x}) + 2C|\mathbf{y}|. \tag{2.48}$$

Combining these two lemmas proves Proposition 2.8 and completes the $\mathcal{PSH}$-BM problem.

2.4.1 Proof of Lemma 2.17: Modifying the weight

Let ω satisfy the conditions of Theorem 2.5 with constants $C_{\mathrm{reg}}, C_{\mathrm{gr}}$. Let

$$1 = \sum_{k=0}^{\infty} \psi_k$$

be a partition of unity of $\mathbb{R}_{\geq 0}$ where $\operatorname{supp} \psi_k \subset A_k$,

$$A_0 = [0, 5],$$
$$A_k = [2^{k-1}, 2^{k+2}] \qquad \text{for } k \geq 1.$$

We may choose $\psi_k(\mathbf{x}) = \psi_1(2^{1-k}\mathbf{x})$ giving a derivative estimate $|D^a \psi_k(\mathbf{x})| \leq C_a 2^{-ak}$ for all $a \geq 0$. Write $\psi_k(\mathbf{x}) = \psi_k(|\mathbf{x}|)$ for $\mathbf{x} \in \mathbb{R}^d$. Let

$$\pi_{\mathbb{S}^{d-1}} \psi_k(\hat{\mathbf{v}}) = p_k, \quad p_k \sim 2^{-k} \text{ up to universal constants.}$$

Write

$$\omega = \sum_{k \geq 0} \omega_k, \quad \omega_k(\mathbf{x}) = \psi_k(\mathbf{x}) \omega(\mathbf{x}).$$

For $k \geq 1$, let

$$q_k = \inf_{\hat{\mathbf{v}} \in \mathbb{S}^{d-1}} \pi_{\mathbb{S}^{d-1}} \omega_k(\hat{\mathbf{v}}).$$

Recall that $\omega \leq 0$, so $|q_k| = \sup_{\hat{\mathbf{v}} \in \mathbb{S}^{d-1}} \pi_{\mathbb{S}^{d-1}} |\omega_k(\hat{\mathbf{v}})|$. Now set

$$g_k(\mathbf{x}) = p_k^{-1} \psi_k(\mathbf{x}) \left(q_k - (\pi_{\mathbb{S}^{d-1}} \omega_k)(\hat{\mathbf{x}})\right), \quad k \geq 1. \tag{2.49}$$

Notice that by the definition of q_k, we have $g_k \leq 0$. We define

$$\widetilde{\omega}_k = \begin{cases} \omega_k & 0 \leq k < 5, \\ \omega_k + g_k & k \geq 5, \end{cases}$$
$$\widetilde{\omega} = \sum_{k \geq 0} \widetilde{\omega}_k.$$

Certainly $\widetilde{\omega} \leq \omega$ because $g_k \leq 0$ for all k. Also, because we only add the modification g_k for $k \geq 5$, we have $\widetilde{\omega}(\mathbf{x}) = \omega(\mathbf{x})$ for $|\mathbf{x}| \leq 2$. By construction,

$$\pi_{\mathbb{S}^{d-1}} \widetilde{\omega}_k = q_k \quad \text{for } k \geq 5. \tag{2.50}$$

We have

$$\begin{aligned} |q_k| &= \sup_{\hat{\mathbf{v}} \in \mathbb{S}^{d-1}} \int_0^\infty |\omega_k(t\hat{\mathbf{v}})| \, t^{-2} dt \lesssim 2^{-2k} \sup_{\hat{\mathbf{v}} \in \mathbb{S}^{d-1}} \int_{2^{k-1}}^{2^{k+2}} |\omega(t\hat{\mathbf{v}})| \, dt \\ &\lesssim 2^{-k} (G^*(2^k) + G^*(2^{k+1})). \end{aligned}$$

Choose $\mathbf{x} \in \mathbb{R}^d$ with $|\mathbf{x}| = r$ so that $G^*(r) = G(\mathbf{x})$. We have

$$G^*(r) = \int_{1/2}^2 |\omega(s\mathbf{x})| \, ds \lesssim \int_{1/2}^2 \int_{1/2}^2 |\omega(st\,\mathbf{x})| \, dsdt \lesssim \int_{1/2}^2 G^*(tr) \, dt$$

leading to the pointwise bound

$$G^*(2^j) \lesssim 2^{-j} \int_{2^{j-1}}^{2^{j+1}} G^*(r) \, dr$$

which gives

$$\sum_k |q_k| \lesssim \sum_k 2^{-k} G(2^k) \lesssim \int_0^\infty \frac{G^*(r)}{1 + r^2} \, dr$$

as needed.

Finally, we must show that $\widetilde{\omega}$ satisfies the regularity condition (2.46). Let $0 \leq a \leq 3$. By the Leibniz rule,

$$\begin{aligned} \|D^a \omega_k\|_\infty &\lesssim \sum_{0 \leq b \leq a} \|D^{a-b} \psi_k\|_\infty \sup_{|\mathbf{x}| \in A_k} |D^b \omega(\mathbf{x})| \\ &\lesssim C_{\text{reg}} \sum_{0 \leq b \leq a} 2^{-(a-b)k} 2^{(1-b)k} \lesssim C_{\text{reg}} 2^{(1-a)k}. \end{aligned}$$

Let $h_k(\mathbf{x}) = \pi_{\mathbb{S}^{d-1}}\omega_k(\hat{\mathbf{x}})$. We have

$$\begin{aligned}
h_k(\mathbf{x}) &= \int_{2^{k-1}}^{2^{k+2}} \omega_k(t\hat{\mathbf{x}})\, t^{-2} dt \\
&= |\mathbf{x}|^{-1} \int_0^\infty \omega_k(s\mathbf{x})\, s^{-2} ds, \\
g_k(\mathbf{x}) &= p_k^{-1}\psi_k(\mathbf{x})(q_k - h_k(\mathbf{x}))
\end{aligned}$$

Thus

$$\begin{aligned}
\|D^a g_k\|_\infty &\lesssim 2^k \sum_{0\le b\le a} \|D^{a-b}\psi_k\|_\infty \sup_{|\mathbf{x}|\in A_k} |D^b(q_k - h_k)(\mathbf{x})| \\
&\lesssim \sum_{0\le b\le a} 2^{-(a-b)k+k} \sup_{|\mathbf{x}|\in A_k} |D^b h_k(\mathbf{x})|.
\end{aligned}$$

Let $|\mathbf{x}| \in A_k$, $0 \le b \le 3$. We have

$$\begin{aligned}
|D^b h_k(\mathbf{x})| &\lesssim \sum_{0\le c\le b} |D^{b-c}|\mathbf{x}|^{-1}| \int_{1/10}^{10} |D^c\omega_k(s\mathbf{x})|\, s^{c-2}\, ds \\
&\lesssim C_{\mathrm{reg}} \sum_{0\le c\le b} 2^{-(1+(b-c))k} 2^{(1-c)k} \lesssim C_{\mathrm{reg}} 2^{-bk}.
\end{aligned}$$

Combining these estimates we obtain that for $0 \le a \le 3$,

$$\begin{aligned}
\|D^a g_k\|_\infty &\lesssim C_{\mathrm{reg}} 2^{(1-a)k} \\
\|D^a \widetilde{\omega}_k\|_\infty &\lesssim C_{\mathrm{reg}} 2^{(1-a)k}
\end{aligned}$$

as needed.

2.4.2 Proof of Lemma 2.18: Analyzing the modified weight

We would like to apply Proposition 2.10 to $\widetilde{\omega}$. Let $\widetilde{\omega} = \sum_{k\ge 0} \widetilde{\omega}_k$ satisfy the conditions Lemma 2.17(ii)-(v). First we prove an estimate on the Hilbert transform of the derivative of $\widetilde{\omega}$ restricted to lines.

Lemma 2.19. *Let $\ell = \{\mathbf{x} + t\hat{\mathbf{y}} : t \in \mathbb{R}\}$ be a line. Let $\widetilde{\omega}_k|_\ell(t) = \widetilde{\omega}_k(\mathbf{x} + t\hat{\mathbf{y}})$ be the restriction of $\widetilde{\omega}_k$ to this line. For all such lines, we have*

$$\sum_{k\ge 0} |H[\widetilde{\omega}_k|_\ell'](0)| \lesssim C'_{\mathrm{reg}} + C'_{\mathrm{gr}}. \tag{2.51}$$

Proof. Let $r = 0$ if $|\mathbf{x}| \le 4$, and otherwise let $r \ge 1$ be such that $|\mathbf{x}| \in [2^{r-1}, 2^r)$.

For any k, r we have the following estimate, although we only use it when $r - 5 \le k \le r + 5$:

$$\begin{aligned} |H[\widetilde{\omega}_k|'_\ell](0)| &= \left| \int_0^\infty \frac{\partial_{\hat{\mathbf{y}}} \widetilde{\omega}_k(\mathbf{x} + t\hat{\mathbf{y}}) - \partial_{\hat{\mathbf{y}}} \widetilde{\omega}_k(\mathbf{x} - t\hat{\mathbf{y}})}{t} \frac{dt}{\pi} \right| \\ &\lesssim 2^k \|D^2 \widetilde{\omega}_k\|_\infty \le C'_{\mathrm{reg}}. \end{aligned}$$

For $k < r - 5$, $\widetilde{\omega}_k$ is supported away from $\mathbf{x}$, and we have

$$\begin{aligned} |H[\widetilde{\omega}_k|'_\ell](0)| &= \left| \int_{-\infty}^\infty \frac{1}{t} \frac{d}{dt} \widetilde{\omega}_k(\mathbf{x} + t\hat{\mathbf{y}}) \frac{dt}{\pi} \right| \\ &= \left| \int_{-\infty}^\infty \frac{\widetilde{\omega}_k(\mathbf{x} + t\hat{\mathbf{y}})}{t^2} \frac{dt}{\pi} \right| && \text{by integration by parts,} \\ &\lesssim 2^{-2r} 2^k \|\widetilde{\omega}_k\|_\infty \le C'_{\mathrm{reg}} 2^{2(k-r)}. \end{aligned}$$

Finally, for $k > r + 5$, $\widetilde{\omega}_k$ is once again supported away from $\mathbf{x}$, and integrating by parts we have

$$\begin{aligned} |H[\widetilde{\omega}_k|'_\ell](0)| &= \left| \int_{-\infty}^\infty \frac{\widetilde{\omega}_k(\mathbf{x} + t\hat{\mathbf{y}})}{t^2} \frac{dt}{\pi} \right| \\ &\lesssim 2^{-2k} \int_{-\infty}^\infty |\widetilde{\omega}_k(\mathbf{x} + t\hat{\mathbf{y}})| \, dt \\ &\lesssim C'_{\mathrm{reg}} 2^{-k} |\mathbf{x}| + 2^{-2k} \int_{-\infty}^\infty |\widetilde{\omega}_k(t\hat{\mathbf{y}})| \, dt && \text{by Lipschitz regularity} \\ &\lesssim C'_{\mathrm{reg}} 2^{-k} |\mathbf{x}| + |q_k|. \end{aligned}$$

Summing these contributions,

$$\begin{aligned} \sum_{k \ge 0} |H[\widetilde{\omega}_k|'_\ell](0)| &\lesssim C'_{\mathrm{reg}} + C'_{\mathrm{reg}} \sum_{k < r-5} 2^{2(k-r)} + C'_{\mathrm{reg}} \sum_{k > r+5} 2^{r-k} + \sum_{k \ge 5} |q_k| \\ &\lesssim C'_{\mathrm{reg}} + C'_{\mathrm{gr}}. \end{aligned}$$

□

Now we prove an estimate on the integral of the second derivative of $\widetilde{\omega}$ over lines.

Lemma 2.20. *Let $\ell = \{\mathbf{x}_0 + t\hat{\mathbf{y}}\}$ be a line, where $\mathbf{x}_0$ is the closest point to the origin. We have*

$$\sum_{k \ge 0} \left| \int_{-\infty}^\infty \langle (D^2 \widetilde{\omega}_k(\mathbf{x}_0 + t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle \, dt \right| \lesssim C'_{\mathrm{reg}} + C'_{\mathrm{gr}} \quad \textit{for all } \hat{\mathbf{v}} \perp \hat{\mathbf{y}}. \tag{2.52}$$

Proof. Let $\hat{\mathbf{v}} \perp \hat{\mathbf{y}}$. Let $r = 0$ if $|\mathbf{x}_0| \le 4$, and otherwise let $r \ge 1$ be so that $|\mathbf{x}_0| \in [2^{r-1}, 2^r)$. For $k < r - 5$ the support of $\widetilde{\omega}_k$ does not intersect ℓ and

$$\int_{-\infty}^{\infty} \langle (D^2\widetilde{\omega}_k(\mathbf{x}_0 + t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt = 0.$$

For $r - 5 < k < r + 5$ we put the absolute values inside the integral and use the second derivative regularity condition,

$$\begin{aligned}\left| \int_{-\infty}^{\infty} \langle (D^2\widetilde{\omega}_k(\mathbf{x}_0 + t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt \right| &\lesssim \int_{-\infty}^{\infty} |D^2\widetilde{\omega}_k(\mathbf{x}_0 + t\hat{\mathbf{y}})| \, dt \\ &\lesssim 2^k \|D^2\widetilde{\omega}_k\|_\infty \le C'_{\mathrm{reg}}.\end{aligned}$$

Next, let $k > r + 5$. We translate the integral to a line through the origin using the third derivative regularity condition,

$$\left| \int_{-\infty}^{\infty} \langle (D^2\widetilde{\omega}_k(\mathbf{x}_0 + t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt \right| \le \left| \int_{-\infty}^{\infty} \langle (D^2\widetilde{\omega}_k(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt \right| + C'_{\mathrm{reg}}|\mathbf{x}_0|2^{-k}.$$

By the hypothesis that $\pi_{S^{d-1}}\omega_k = q_k$ and Lemma 2.16 on the second derivative of spherical projections,

$$\int_{-\infty}^{\infty} \langle (D^2\widetilde{\omega}_k(t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt = 2q_k.$$

Thus

$$\begin{aligned}\sum_k \left| \int_{-\infty}^{\infty} \langle (D^2\widetilde{\omega}_k(\mathbf{x}_0 + t\hat{\mathbf{y}}))\hat{\mathbf{v}}, \hat{\mathbf{v}}\rangle \, dt \right| &\lesssim C'_{\mathrm{reg}} + \sum_{2^k \ge |\mathbf{x}_0|} \left(C'_{\mathrm{reg}}|\mathbf{x}_0|2^{-k} + |q_k| \right) \\ &\lesssim C'_{\mathrm{reg}} + C'_{\mathrm{gr}}.\end{aligned}$$

□

Finally, we finish the proof of Lemma 2.18.

Proof of Lemma 2.18. Let

$$\widetilde{\omega}_{\le k} = \sum_{0 \le j \le k} \widetilde{\omega}_j. \tag{2.53}$$

By (2.51) and (2.52) the compactly supported weights $\widetilde{\omega}_{\le k}$ satisfy the hypotheses of Proposition 2.10 uniformly in k, and there is some $C \lesssim C'_{\mathrm{reg}} + C'_{\mathrm{gr}}$ such that for all $k \ge 1$,

$$u_{\le k} = E\widetilde{\omega}_{\le k} + C|\mathbf{y}|$$

is plurisubharmonic and satisfies

$$u_{\le k}(\mathbf{x}) \le u_{\le k}(\mathbf{x} + i\mathbf{y}) \le u_{\le k}(\mathbf{x}) + 2C|\mathbf{y}|.$$

Notice that the sequence $\{\widetilde{\omega}_k\}_{k=1}^{\infty}$ is uniformly Lipschitz by (2.46), and satisfies the uniform growth condition (2.38) because of (2.47). By Lemma 2.14, $E\widetilde{\omega}_{\le k} \to E\widetilde{\omega}$ uniformly on compact sets. It follows that

$$u = E\widetilde{\omega} + C|\mathbf{y}|$$

is plurisubharmonic and satisfies

$$u(\mathbf{x}) \le u(\mathbf{x} + i\mathbf{y}) \le u(\mathbf{x}) + 2C|\mathbf{y}|.$$

□

2.5 Constructing the analytic function

We crucially use Hörmander's L^2 theory for the $\bar{\partial}$ equation in order to construct analytic functions from plurisubharmonic functions. This section includes an exposition of Hörmander's theorem. We already stated his one dimensional result in the introduction (Theorem 2.6), and his higher-dimensional result is stated in Theorem 2.28. After the exposition we prove Proposition 2.9, which is where we construct analytic functions.

Hörmander's method is related to prior work of Kodaira [36], Andreotti–Vesentini [3], Morrey [42], Kohn [37], and Ash [4]. We are ignorant of this prior work and refer the reader to Hörmander's paper [28] and Berndtsson's survey [7] for more discussion.

Hörmander's theorem uses the following principle from linear algebra.

Theorem 2.21. *Let $T : H_1 \to H_2$ be a linear map between two finite dimensional Hilbert spaces. If*

$$\|u\|_{H_2} \le C\|T^* u\|_{H_1} \qquad \text{for all } u \in H_2, \tag{2.54}$$

then for any $v \in H_2$ there exists $w \in H_1$ with $Tw = v$ and $\|w\|_{H_1} \le C\|v\|_{H_2}$.

Proof. Let $v \in H_2$, and define a linear map on a subspace of H_1 by

$$\ell : \{T^* u,\, u \in H_2\} \to \mathbb{C}$$
$$\ell(T^* u) = \langle v, u\rangle.$$

Because T^* is injective, ℓ is well defined, and by (2.54)

$$|\ell(T^*u)| \le \|v\|_{H_2} \|\|u\|_{H_2} \le C\|v\|_{H_2}\|T^*u\|_{H_1},$$

so ℓ is a bounded linear functional on its domain. By the Hahn-Banach theorem, ℓ can be extended to a linear functional on H_1 with norm bounded by $C\|v\|_{H_2}$. By the Riesz representation theorem there exists $w \in H_1$ with $\|w\|_{H_1} \le C\|v\|_{H_2}$ such that $\langle w, T^*u\rangle = \langle v, u\rangle$ for all $u \in H_2$. This implies $Tw = v$, as $T^{**} = T$. □

Theorem 2.21 lets us solve linear equations by proving estimates. This is very useful. The main ingredient in the proof of Hörmander's theorem is a bound like (2.54) for the adjoint of the $\bar{\partial}$ operator in a weighted Hilbert space.

2.5.1 The $\bar{\partial}$ equation over $\mathbb{C}$.

We would like to solve the inhomogenous equation

$$\frac{\partial g}{\partial \bar{z}} = \eta, \qquad \eta \in C_c^\infty(\mathbb{C}).$$

Let's first try using fundamental solutions. In the sense of distributions $\frac{\partial}{\partial \bar{z}} \frac{1}{z-\tau} = 2\pi\delta_\tau$, so

$$g(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{1}{z-\tau}\eta(\tau)\, d\lambda \qquad (d\lambda \text{ is the Lebesgue measure on } \mathbb{C})$$

solves $\frac{\partial g}{\partial \bar{z}} = \eta$.

This solution is not useful to us. We are interested in constructing nonzero analytic functions with certain decay rates on $\mathbb{C}$. To do so we start with a bump function h, solve the inhomogenous equation $\bar{\partial} g = \bar{\partial} h$, and then set $f = g - h$ to be our analytic function. The method of fundamental solutions will return $g = h$, $f = 0$. In order to construct nonzero analytic functions f, we need to enforce the constraint $g(0) = 0$ so that $f(0) \neq 0$. The Hilbert space method allows us to enforce this constraint.

We consider the weighted Hilbert space $L^2(\varphi)$ with the inner product

$$\langle f, g\rangle_\varphi = \int f(\mathbf{z})\bar{g}(\mathbf{z})e^{-\varphi(z)}\, d\lambda$$

and the norm

$$\|f\|_\varphi^2 = \int |f(z)|^2 e^{-\varphi(z)}\, d\lambda.$$

We want to prove surjectivity of the unbounded linear map $\partial/\partial\bar{z}$ on $L^2(\varphi)$. We compute the formal adjoint of $\partial/\partial\bar{z}$ on $L^2(\varphi)$, which we call δ:

$$\begin{aligned}\langle \partial u/\partial\bar{z}, v\rangle_\varphi &= \int \frac{\partial u}{\partial \bar{z}}\,\bar{v}(z)\,e^{-\varphi(z)}\,d\lambda \\ &= -\int u(z)\Big(e^{\varphi}\frac{\partial}{\partial\bar{z}}e^{-\varphi(z)}\bar{v}(z)\Big)\,e^{-\varphi}d\lambda \\ &= \langle u, \delta v\rangle_\varphi\end{aligned}$$

where

$$\delta = -e^{\varphi}\frac{\partial}{\partial z}e^{-\varphi} = -\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial z}. \tag{2.55}$$

(We use the term "formal adjoint" because we only verified this property for functions in $D_{p,q}$). The commutator between $\partial/\partial\bar{z}$ and δ is positive if φ is strictly subharmonic,

$$[\partial/\partial\bar{z}, \delta] = \Big[\partial/\partial\bar{z}, -\partial/\partial z + \frac{\partial\varphi}{\partial z}\Big] = \frac{\partial^2\varphi}{\partial\bar{z}\partial z} = \frac{1}{4}\Delta\varphi, \tag{2.56}$$

leading to the identity

$$\begin{aligned}\|\delta u\|_\varphi^2 &= \langle \frac{\partial}{\partial\bar{z}}\delta u, u\rangle_\varphi \\ &= \langle \delta\frac{\partial}{\partial\bar{z}}u, u\rangle_\varphi + \langle[\partial/\partial\bar{z}, \delta]u, u\rangle_\varphi \\ &= \|\partial u/\partial\bar{z}\|_\varphi^2 + \frac{1}{4}\int |u(z)|^2\Delta\varphi\, e^{-\varphi}d\lambda.\end{aligned}$$

Subtracting $\|\partial u/\partial\bar{z}\|_\varphi^2$ from both sides gives

Lemma 2.22. *For $u \in C_c^\infty(\mathbb{C})$,*

$$\int |u(z)|^2\Delta\varphi\, e^{-\varphi}d\lambda \leq 4\|\delta u\|_\varphi^2.$$

With Lemma 2.22 in hand, we are ready to prove the $\bar{\partial}$ theorem. We restate Theorem 2.6 for the reader's convenience.

Theorem. *Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a smooth, strictly subharmonic function. Let $\eta \in L^2_{loc}(\mathbb{C})$ satisfy*

$$\int |\eta(z)|^2\frac{e^{-\varphi(z)}}{\Delta\varphi}\,d\lambda \leq C.$$

Then there exists $g \in L^2(\varphi)$ such that $\partial g/\partial\bar{z} = \eta$ in the sense of distributions and

$$\int |g(z)|^2 e^{-\varphi(z)}\,d\lambda \leq 4C.$$

If η is smooth then by elliptic regularity g is smooth as well, and $\partial g/\partial \bar{z} = \eta$ in the classical sense.

Proof. Suppose that

$$\int |\eta(z)|^2 \frac{e^{-\varphi(z)}}{\Delta\varphi}\, d\lambda \leq C.$$

We define the following linear functional on a subspace of $L^2(\varphi)$,

$$\begin{aligned} &\ell : \{\delta v \,:\, v \in C_c^\infty(\mathbb{C})\} \to \mathbb{C} \\ &\ell(\delta v) = \langle \eta, v\rangle_\varphi. \end{aligned}$$

Using Cauchy-Schwarz and Lemma 2.22,

$$\begin{aligned} |\ell(\delta v)| = |\langle \eta, v\rangle_\varphi| &\leq \Big(\int |\eta(z)|^2 \frac{e^{-\varphi(z)}}{\Delta\varphi}\, d\lambda\Big)^{1/2} \Big(\int |v(z)|^2 \Delta\varphi\, e^{-\varphi(z)}\, d\lambda\Big)^{1/2} \\ &\leq 2\sqrt{C}\|\delta v\|_\varphi. \end{aligned}$$

Thus ℓ is well-defined and bounded by $2\sqrt{C}$ on its set of definition, so by the Hahn-Banach theorem ℓ extends to a linear functional on $L^2(\mathbb{C})$, which is also bounded by $2\sqrt{C}$. By the Riesz representation theorem, there is some $g \in L^2(\mathbb{C})$ with $\|g\|_\varphi^2 \leq 4C$ such that for all $v \in C_c^\infty(\mathbb{C})$,

$$\langle g, \delta v\rangle_\varphi = \langle \eta, v\rangle_\varphi. \tag{2.57}$$

This equation almost says that $\bar{\partial} g = \eta$ in the sense of distributions, although not quite because of the φ-weighted inner product. That is just a technicality—its easy to convert to the usual inner product. Given $v \in C_c^\infty(\mathbb{C})$, let $v' = e^\varphi \bar{v}$. Then v' also lies in $C_c^\infty(\mathbb{C})$, and the φ-weighted inner products with v' equal the Lebesgue inner products with v,

$$\begin{aligned} \langle g, \delta v'\rangle_\varphi &= -\int g \overline{\frac{\partial \bar{v}}{\partial z}}\, d\lambda = \int g(z) \frac{\partial v}{\partial \bar{z}}\, d\lambda, \\ \langle \eta, v'\rangle_\varphi &= \int \eta(z) v(z)\, d\lambda. \end{aligned}$$

The left hand sides are equal by (2.57), so the right hand sides are equal as well. The right hand sides being equal is exactly the statement that $\partial g/\partial \bar{z} = \eta$ in the sense of distributions. □

In the sequel we ignore this subtlety around converting between the φ-weighted inner product and the usual inner product.

When I read this proof, it made me uncomfortable that we used the Hahn-Banach theorem to construct linear extensions. There may be several linear extensions of ℓ, and each one might lead to a different solution to the $\bar{\partial}$ equation. Is there a constructive proof that chooses a canonical solution?

I think there is not a canonical solution. There are many solutions to $\bar{\partial} g' = 0$ with $g' \in L^2(\varphi)$, so there are also many solutions to $\bar{\partial} g = \eta$ in $L^2(\varphi)$. It is a feature of Hörmander's method that it deals with non-uniqueness in a clean way—by using the Hahn-Banach theorem.

2.5.2 The $\bar{\partial}$ operator over $\mathbb{C}^d$.

Theorem 2.6 has a higher-dimensional analogue; to state it we need some notation. Let $\Omega_{p,q}$ denote the vector space of (p,q)-forms at a point of $\mathbb{C}^d$. Every element of $\Omega_{p,q}$ can be written in the standard basis

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J}\, dz^I d\bar{z}^J, \qquad \alpha_{I,J} \in \mathbb{C},\ I \text{ and } J \text{ are ordered subsets of } \{1, \dots, d\}.$$

Let $D_{p,q}$ denote the space of smooth, compactly supported (p,q) forms on $\mathbb{C}^d$. An element $u \in D_{p,q}$ can be written as

$$u = \sum_{I,J} u_{I,J}\, dz^I d\bar{z}^J, \qquad u_{I,J} \in C_c^\infty(\mathbb{C}^d).$$

We put an inner product structure on $\Omega_{p,q}$ by asserting this standard basis is orthonormal,

$$\langle \alpha, \beta \rangle_{\Omega_{p,q}} = \sum_{I,J} \alpha_{I,J} \bar{\beta}_{I,J}.$$

We consider the weighted inner product

$$\langle u, v \rangle_\varphi = \sum_{I,J} \int_{\mathbb{C}^d} u_{I,J}(\mathbf{z}) \bar{v}_{I,J}(\mathbf{z})\, e^{-\varphi(\mathbf{z})}\, d\lambda, \qquad d\lambda \text{ is the Lebesgue measure on } \mathbb{C}^d,$$
$$\|u\|_\varphi^2 = \langle u, u \rangle_\varphi.$$

We let

$$L^2_{p,q}(\varphi) = \{u \,:\, u_{I,J} \in L^2_{loc}(\mathbb{C}^d) \text{ and } \|u\|_\varphi < \infty\}$$

be the Hilbert space generated by this inner product.

The $\bar{\partial}$ operator maps $D_{p,q} \to D_{p,q+1}$. For a function $u \in C_c^\infty(\mathbb{C}^d)$,

$$\bar{\partial} u = \sum_j \bar{\partial}_j u \, d\bar{z}_j, \qquad \bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j},$$

for a form $u \in D_{p,q}$,

$$\bar{\partial} u = \sum_{I,J} \bar{\partial} u_{I,J} \wedge dz^I d\bar{z}^J.$$

We have $\bar{\partial}^2 = 0$, so for each p there is a chain complex

$$\cdots \xrightarrow{\bar{\partial}} D_{p,q} \xrightarrow{\bar{\partial}} D_{p,q+1} \xrightarrow{\bar{\partial}} D_{p,q+2} \xrightarrow{\bar{\partial}} \cdots .$$

Given a (p,q)-form η, we are interested in solving the inhomogenous $\bar{\partial}$ equation

$$\bar{\partial} g = \eta \qquad \text{where } g \text{ is a } (p, q-1)\text{-form.}$$

As $\bar{\partial}^2 = 0$, it is a necessary condition that $\bar{\partial}\eta = 0$. This is a new condition in higher dimensions: in one complex dimension, $\bar{\partial}\eta = 0$ for any $\eta \in D_{0,1}$. Hörmander proved the following theorem about the inhomogenous $\bar{\partial}$ equation.

Proposition 2.23. *Let $\varphi : \mathbb{C}^d \to \mathbb{R}$ be a smooth, strictly plurisubharmonic function with $\partial\bar{\partial}\varphi(z) \geq \kappa(z)$, where $\kappa : \mathbb{C}^d \to \mathbb{R}_{>0}$. Let η be a (p,q)-form with L^2_{loc} coefficients satisfy $\bar{\partial}\eta = 0$, and suppose*

$$\int |\eta(\mathbf{z})|^2 \frac{e^{-\varphi(\mathbf{z})}}{\kappa(\mathbf{z})} \, d\lambda \leq C.$$

Then there exists $g \in L^2_{p,q-1}(\varphi)$ such that $\bar{\partial} g = \eta$ in the sense of distributions and $\|g\|^2_{L^2_{p,q-1}(\varphi)} \leq C$.

The first step is to prove a higher-dimensional version of Lemma 2.22 asserting quantitative injectivity of the adjoint. This involves more calculation but no new difficulties. The second step—using this estimate to solve the $\bar{\partial}$ equation—is a bit more subtle than it was in one dimension because of the need to assume $\bar{\partial}\eta = 0$. To solve this step we need to prove an approximation lemma.

We begin by analyzing the adjoint operator. We write $\bar{\partial}$ as

$$\bar{\partial} = \sum_j \bar{\partial}_j d\bar{z}_j. \tag{2.58}$$

In this notation $d\bar{z}_j$ is an operator mapping $D_{p,q} \to D_{p,q+1}$ by wedge product, and $\bar{\partial}_j$ as an operator mapping $D_{p,q+1} \to D_{p,q+1}$ by partial derivatives.

The formal adjoint is the operator $\bar{\partial}^* : D_{p,q+1} \to D_{p,q}$ such that

$$\langle \bar{\partial} u, v \rangle_\varphi = \langle u, \bar{\partial}^* v \rangle_\varphi \qquad \text{for all } u \in D_{p,q} \text{ and } v \in D_{p,q+1}.$$

Notice that $\bar{\partial}^*$ depends on the weight φ. To compute $\bar{\partial}^*$, we first compute the adjoint of $\bar{\partial}_j$ with respect to the weight $e^{-\varphi}$,

$$\int (\bar{\partial}_j u) \bar{v} e^{-\varphi}\, d\lambda = -\int u (e^{\varphi} \bar{\partial}_j e^{-\varphi} \bar{v})\, e^{-\varphi} d\lambda.$$

Just as in (2.55), the adjoint of $\bar{\partial}_j$ is

$$\delta_j u = -\frac{\partial u}{\partial z_j} + \frac{\partial \varphi}{\partial z_j} u \tag{2.59}$$

and $\langle \bar{\partial}_j u, v \rangle_\varphi = \langle u, \delta_j v \rangle_\varphi$. There is a commutator identity generalizing (2.60),

$$[\bar{\partial}_k, \delta_j] = \frac{\partial^2 \varphi}{\partial \bar{z}_k \partial z_j}. \tag{2.60}$$

Using the notation of (2.58), we may write

$$\bar{\partial}^* = \sum_j \delta_j d\bar{z}_j^*. \tag{2.61}$$

Here, $d\bar{z}_j^* : D_{p,q+1} \to D_{p,q}$ is the adjoint of the wedge product operator $d\bar{z}_j$, and $\delta_j : D_{p,q} \to D_{p,q}$ acts coordinate-wise. To be explicit,

$$d\bar{z}_j^*(d\bar{z}^I) = \begin{cases} 0 & j \notin I, \\ \operatorname{sgn}(j \in I) d\bar{z}^{I \setminus \{j\}} & j \in I \end{cases}$$

where $\operatorname{sgn}(j \in I) = 1$ if j occupies an odd index in I, and otherwise equals -1. For example,

$$\bar{\partial}^* \Big(\sum_j u_j d\bar{z}_j \Big) = \sum_j \delta_j u_j.$$

Define the $\bar{\partial}$-Laplacian

$$\begin{aligned} &\Delta_{\bar{\partial}} : L^2_{p,q}(\varphi) \to L^2_{p,q}(\varphi) \\ &\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}. \end{aligned}$$

The $\bar{\partial}$-Laplacian is an important operator in complex geometry. It induces the quadratic form

$$\langle \Delta_{\bar{\partial}} u, u \rangle_\varphi = \| \bar{\partial} u \|^2_\varphi + \| \bar{\partial}^* u \|^2_\varphi.$$

There is a formula, called the Weitzenböck formula, that allows us to compute $\Delta_{\bar\partial}$ explicitly. Just as in the one dimensional case, $\partial\bar\partial\varphi$ appears in the Weitzenböck formula (in one dimension, $\partial\bar\partial\varphi = \frac{1}{4}\Delta\varphi$). The Weitzenböck formula we state is a special case of a more general formula in complex geometry, see [25, p. 97].

Lemma 2.24 (Weitzenböck identity)**.** *We may decompose*

$$\Delta_{\bar\partial} = \Delta' + A$$

where

$$\Delta' = \sum_j \delta_j \bar\partial_j \qquad \textit{and} \tag{2.62}$$

$$A = \sum_{j,k} \frac{\partial^2\varphi}{\partial\bar z_j \partial z_k} d\bar z_j d\bar z_k^*. \tag{2.63}$$

Proof. Using equations (2.58) and (2.61) for ∂ and $\bar\partial$,

$$\Delta_{\bar\partial} = \sum_{j,k} \delta_k \bar\partial_j d\bar z_k^* d\bar z_j + \sum_{j,k} \bar\partial_j \delta_k d\bar z_j d\bar z_k^*.$$

Let $d\bar z^I$ be a basis element of $\Omega_x^{p,q}$. If $j \notin I$ and $k \in I$, then

$$(d\bar z_j d\bar z_k^*)(d\bar z^I) = -d\bar z_k^* d\bar z_j (d\bar z^I).$$

If $j = k \in I$

$$d\bar z_j d\bar z_k^* (d\bar z^I) = d\bar z^I, \qquad d\bar z_k^* d\bar z_j (d\bar z^I) = 0,$$

and conversely if $j = k \notin I$

$$d\bar z_j d\bar z_k^* (d\bar z^I) = 0, \qquad d\bar z_k^* d\bar z_j (d\bar z^I) = d\bar z^I,$$

so

$$d\bar z_k^* d\bar z_j + d\bar z_j d\bar z_k^* = 1_{j=k}$$

as an operator from $\Omega_x^{p,q} \to \Omega_x^{p,q}$. Thus

$$\Delta_{\bar\partial} = \sum_j \delta_j \bar\partial_j + \sum_{j,k} [\bar\partial_j, \delta_k] d\bar z_j d\bar z_k^*.$$

The first summand is Δ', and by the commutator identity (2.60), the second summand is A. $\square$

We can also express the Weitzenböck identity in coordinates. Given two (p, q) forms u and v, we denote by

$$\langle u, v\rangle_{\Omega_{p,q}} = \sum_{I,J} u_{I,J}\bar{v}_{I,J}$$

the function on $\mathbb{C}^d$ which is the pointwise inner product of the forms. For $u = \sum_{I,J} u_{I,J}\, dz^I \wedge d\bar{z}^J$,

$$\langle \Delta' u, u\rangle_{\Omega_{p,q}} = \sum_{I,J}\sum_{k} |\bar{\partial}_k u_{I,J}|^2$$
$$\langle Au, u\rangle_{\Omega_{p,q}} = \sum_{I}\sum_{|J|=q-1}\sum_{j,k} \frac{\partial^2\varphi}{\partial\bar{z}_j\partial z_k}\langle u, d\bar{z}_j d\bar{z}^J\rangle\langle u, d\bar{z}_k d\bar{z}^J\rangle.$$

The Weitzenböck formula quickly yields the following generalization of Lemma 2.22.

Lemma 2.25. *Suppose $\partial\bar{\partial}\varphi(\mathbf{z}) \geq \kappa(\mathbf{z})$ as a quadratic form. For any $u \in D_{p,q}$,*

$$\|\bar{\partial}u\|_{\varphi}^2 + \|\bar{\partial}^* u\|_{\varphi}^2 \geq \int |u(\mathbf{z})|^2\, \kappa(\mathbf{z}) e^{-\varphi(\mathbf{z})}\, d\lambda.$$

Proof. Using that $\partial\bar{\partial}\varphi(\mathbf{z}) \geq \kappa(\mathbf{z})$,

$$\langle Au, u\rangle_{\Omega_{p,q}} \geq q\kappa(\mathbf{z})|u(\mathbf{z})|^2_{\Omega_{p,q}}.$$

Because the operator Δ' in Lemma 2.24 is positive semidefinite,

$$\|\bar{\partial}u\|_{\varphi}^2 + \|\bar{\partial}^* u\|_{\varphi}^2 = \langle \Delta_{\bar{\partial}} u, u\rangle = \langle \Delta' u, u\rangle + \langle Au, u\rangle \geq q\int \kappa(\mathbf{z})|u(\mathbf{z})|^2 e^{-\varphi} d\lambda.$$

□

2.5.3 Solving the $\bar{\partial}$ equation on $\mathbb{C}^d$.

The T^* method in Hilbert space applied to the estimate in Lemma 2.25 naturally leads to the following existence theorem.

Proposition 2.26. *Let $\varphi : \mathbb{C}^d \to \mathbb{C}$ be a smooth, strictly plurisubharmonic function, and let $\kappa : \mathbb{C}^d \to \mathbb{R}_{>0}$ be a lower bound for $\partial\bar{\partial}\varphi$,*

$$\partial\bar{\partial}\varphi(\mathbf{z}) \geq \kappa(\mathbf{z}) \qquad \textit{for all } \mathbf{z} \in \mathbb{C}^d.$$

Let η be a (p,q)-form with L^2_{loc} coefficients satisfying

$$\int |\eta(\mathbf{z})|^2 \frac{e^{-\varphi(\mathbf{z})}}{\kappa(\mathbf{z})}\, d\lambda \leq C. \tag{2.64}$$

Then there exists $g \in L^2_{p,q-1}(\varphi)$ and $h \in L^2_{p,q+1}(\varphi)$ such that

$$\|g\|^2_{L^2_{p,q-1}(\varphi)} + \|h\|^2_{L^2_{p,q+1}(\varphi)} \leq C \tag{2.65}$$

and

$$\bar{\partial} g + \bar{\partial}^* h = \eta \tag{2.66}$$

in the sense of distributions.

Proof. Define the following linear functional on a subspace of $L^2_{p,q-1}(\varphi) \times L^2_{p,q+1}(\varphi)$,

$$\begin{aligned} &\ell \,:\, \{(\bar{\partial}^* u, \bar{\partial} u) \in L^2_{p,q-1} \times L^2_{p,q+1} \,:\, u \in D_{p,q}\} \to \mathbb{C} \\ &\ell(\bar{\partial}^* u, \bar{\partial} u) = \langle \eta, u \rangle_\varphi. \end{aligned}$$

By Lemma 2.25 and assumption (2.64),

$$\begin{aligned} |\ell(\bar{\partial}^* u, \bar{\partial} u)| = |\langle \eta, u\rangle_\varphi| &\leq \Big(\int |\eta(z)|^2 \frac{e^{-\varphi(z)}}{\kappa(z)}\, d\lambda\Big)^{1/2} \Big(\int |u(z)|^2 \kappa(z) e^{-\varphi(z)}\, d\lambda\Big)^{1/2} \\ &\leq \sqrt{C}\sqrt{\|\bar{\partial} u\|^2_\varphi + \|\bar{\partial}^* u\|^2_\varphi}. \end{aligned}$$

Thus ℓ is well defined and bounded by $\sqrt{C}$ on its domain. By the Hahn-Banach theorem ℓ extends to a bounded linear functional on $L^2_{p,q-1}(\varphi) \times L^2_{p,q+1}(\varphi)$, and by the Riesz representation theorem there exist functions $g \in L^2_{p,q-1}(\varphi)$ and $h \in L^2_{p,q+1}(\varphi)$ such that

$$\begin{aligned} \|g\|^2_{L^2_{p,q-1}(\varphi)} + \|h\|^2_{L^2_{p,q+1}(\varphi)} &\leq C \\ \langle g, \bar{\partial}^* u\rangle + \langle h, \bar{\partial} u\rangle &= \langle \eta, u\rangle \end{aligned}$$

for all $u \in D_{p,q}$. This equation means $\bar{\partial} g + \bar{\partial}^* h = \eta$ in the sense of distributions. □

The conclusion of Proposition 2.26 is not quite what we want. Given a (p,q) form η, it produces $g \in L^2_{p,q-1}(\varphi)$ and $h \in L^2_{p,q+1}(\varphi)$ such that

$$\bar{\partial} g + \bar{\partial}^* h = \eta$$

in the sense of distributions. When $\bar{\partial}\eta = 0$, we want to show $\bar{\partial} g = \eta$. In other words, we want to prove $\bar{\partial}^* h = 0$.

Here is a faulty argument that $\bar\partial^* h = 0$. First, consider $\ker\bar\partial$ as a closed subspace of $L^2_{p,q}(\varphi)$,

$$\ker\bar\partial = \{u \in L^2_{p,q}(\varphi) \,:\, \bar\partial u = 0 \text{ in distributions}\}.$$

To check that $\ker\bar\partial$ is a closed subspace, suppose $u_j \to u$ in $L^2_{p,q}(\varphi)$ and $\bar\partial u_j = 0$ in the sense of distributions. For any $v \in D_{p,q+1}$,

$$\langle\bar\partial u, v\rangle = \langle u, \bar\partial^* v\rangle = \lim_{j\to\infty}\langle u_j, \bar\partial^* v\rangle = \lim_{j\to\infty}\langle\bar\partial u_j, v\rangle = 0,$$

so $\bar\partial u = 0$ in distributions as well.

We expect $\bar\partial^* h$ lies in the orthogonal complement, $(\ker\bar\partial)^\perp$. Indeed, for $u \in (\ker\bar\partial)\cap D_{p,q}$

$$\langle u, \bar\partial^* h\rangle = \langle\bar\partial u, h\rangle = 0.$$

Unfortunately, we don't know that $\bar\partial^* h$ lies in $L^2_{p,q}(\varphi)$, and even if we did know that, we would need the above equation to hold for all $u \in \ker\bar\partial$, not just when u is smooth and compactly supported. Yet if it were true that $\bar\partial^* h$ lies in $(\ker\bar\partial)^\perp$, then it would have to equal zero when $\bar\partial\eta = 0$

In order to make this heuristic argument rigorous, we need an approximation lemma. The approximation lemma is important. There are some complex manifolds where the approximation lemma fails, and the $\bar\partial$ theorem also fails—see [7, §6]

Lemma 2.27 (Approximation lemma)**.** *Let $u \in L^2_{p,q}(\varphi)$, and assume $\bar\partial^* u \in L^2_{p,q-1}(\varphi)$ and $\bar\partial u \in L^2_{p,q+1}(\varphi)$ in the sense of distributions. Then there exists a sequence of smooth, compactly supported functions $u^{(j)} \in D_{p,q}$ such that*

$$\|u^{(j)} - u\|_\varphi + \|\bar\partial^*(u^{(j)} - u)\|_\varphi + \|\bar\partial(u^{(j)} - u)\|_\varphi \to 0.$$

Proof. As stated in (2.58) and (2.61), we may write

$$\bar\partial = \sum_j \bar\partial_j \, d\bar z_j, \qquad \bar\partial^* = \sum_j \delta_j \, d\bar z_j^*.$$

Using equation (2.59) for δ_j, we may split $\bar\partial^*$ into two parts: an unweighted adjoint $\bar\partial^*_{\mathbb{C}^d}$, and a multiplication operator involving the derivatives of φ,

$$\begin{aligned}\bar\partial^* &= -\sum_j \partial_j \, d\bar z_j^* + \sum_j \frac{\partial\varphi}{\partial z_j}\, d\bar z_j^* \\ &=: \bar\partial^*_{\mathbb{C}^d} u + Mu.\end{aligned}$$

The first step is to multiply by a smooth cutoff to make u compactly supported. Let η be a fixed smooth bump function which equals one on B_1 and is supported inside B_2. Let $\eta_R = \eta(\bullet/R)$, and let $u_R = \eta_R u$. By the dominated convergence theorem $\|u_R - u\|_\varphi \to 0$ as $R \to \infty$. We have

$$\bar\partial u_R = \sum_j \bar\partial_j \eta_R \, d\bar z_j \wedge u + \eta_R \bar\partial u,$$

in the sense of distributions, so bounding $|\bar\partial \eta_R| \lesssim 1/R$ and using the dominated convergence theorem we find $\|\bar\partial(u_R - u)\|_\varphi \to 0$. Similarly,

$$\bar\partial^* u_R = \sum_j \partial_j \eta_R \, d\bar z_j^*(u) + \eta_R \bar\partial^* u,$$

and the same estimate holds.

The next step is to convolve with a smooth approximation of the identity to make u_R smooth. Let χ be a fixed compactly supported bump function integrating to one, and let $\chi_\varepsilon = \varepsilon^{-d}\chi(\bullet/\varepsilon)$. Let

$$u_{\varepsilon,R} = \chi_\varepsilon * u_R.$$

For any fixed R, $\|u_{\varepsilon,R} - u_R\|_2 \to 0$ as $\varepsilon \to 0$, and because u_R is compactly supported, $\|u_{\varepsilon,R} - u_R\|_{L^2(\varphi)} \to 0$ as well. We can exchange convolution and differentiation to get $\bar\partial u_{\varepsilon,R} = \chi_\varepsilon * \bar\partial u_R$, so $\|\bar\partial(u_{\varepsilon,R} - u_R)\|_\varphi \to 0$ also. Similarly,

$$\bar\partial^* u_{\varepsilon,R} = \chi_\varepsilon * \bar\partial^*_{\mathbb{C}^d} u_R + M u_{\varepsilon,R},$$

and the above converges to $\bar\partial^* u_R = \bar\partial^*_{\mathbb{C}^d} u_R + M u_R$ as $\varepsilon \to 0$. Thus

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} (\|u_{\varepsilon,R} - u\|_\varphi + \|\bar\partial(u_{\varepsilon,R} - u)\|_\varphi + \|\bar\partial^*(u_{\varepsilon,R} - u)\|_\varphi) = 0.$$

We may extract a subsequence along which $R \to \infty$, $\varepsilon \to 0$, and the graph norm above tends to zero, which proves the Lemma. □

We are now ready to prove Proposition 2.23

Proof of Proposition 2.23. By Proposition 2.26, there exists $g \in L^2_{p,q-1}(\varphi)$ and $h \in L^2_{p,q+1}(\varphi)$ such that $\bar\partial g + \bar\partial^* h = \eta$ in distributions and $\|g\|^2_{L^2_{p,q-1}(\varphi)} + \|h\|^2_{L^2_{p,q+1}(\varphi)} \le C$.

We must use the hypothesis $\bar\partial \eta = 0$ to show $\bar\partial^* h = 0$. Let $v \in D_{p,q}$ be a smooth, compactly supported form. Let

$$v = \pi_{\ker \bar\partial} v + \pi_{(\ker \bar\partial)^\perp} v := v_1 + v_2.$$

I claim that in the sense of distributions,

$$\bar\partial v_1 = 0, \qquad \bar\partial^* v_1 = \bar\partial^* v,$$
$$\bar\partial v_2 = \bar\partial v, \qquad \bar\partial^* v_2 = 0.$$

First, $\bar\partial v_1 = 0$ by the definition of $\ker \bar\partial$. Next, for any $w \in D_{p,q-1}$,

$$\langle w, \bar\partial^* v_2\rangle = \langle \bar\partial w, v_2\rangle = 0$$

because $\bar\partial^2 w = 0$ and v_2 lies in the orthogonal complement of $\ker\bar\partial$. Thus $\bar\partial^* v_2 = 0$. The last two identities follow from these two,

$$\langle w, \bar\partial^* v_1\rangle = \langle \bar\partial w, v_1\rangle = \langle \bar\partial w, v - v_2\rangle = \langle \bar\partial w, v\rangle = \langle w, \bar\partial^* v\rangle,$$
$$\langle w, \bar\partial v_2\rangle = \langle \bar\partial^* w, v_2\rangle = \langle \bar\partial^* w, v - v_1\rangle = \langle w, \bar\partial v\rangle.$$

Because $\bar\partial v_j$ and $\bar\partial^* v_j$ lie in $L^2(\varphi)$, we may apply Lemma 2.27 to find subsequences

$$v_1^{(j)} \to v_1, \qquad v_2^{(j)} \to v_2$$

in the graph norm of Lemma 2.27.

We are ready to prove $\bar\partial^* h = 0$. First,

$$\langle \bar\partial^* h, v\rangle = \langle h, \bar\partial(v - v_1^{(j)} + v_2^{(j)})\rangle + \langle h, \bar\partial v_1^{(j)}\rangle + \langle h, \bar\partial v_2^{(j)}\rangle.$$

The first term goes to zero as $j \to \infty$ because $\|\bar\partial(v - v_1^{(j)} - v_2^{(j)})\|_\varphi \to 0$. The second term goes to zero as $j \to \infty$ because $\|\bar\partial v_1^{(j)}\|_\varphi \to 0$ as $j \to \infty$. We further decompose the third term as

$$\langle \bar\partial^* h, v_2^{(j)}\rangle = \langle \eta, v_2^{(j)}\rangle - \langle \bar\partial g, v_2^{(j)}\rangle.$$

As $j \to \infty$, $\langle \eta, v_2^{(j)}\rangle \to \langle \eta, v_2\rangle = 0$ because $\bar\partial\eta = 0$ and v_2 lies in the orthogonal complement of $\ker\bar\partial$. Finally, $\|\bar\partial^* v_2^{(j)}\|_\varphi \to 0$, so all the terms vanish. □

In our application we will want to choose a plurisubharmonic weight φ that equals negative infinity at the origin. For this reason, it's inconvenient to assume φ is smooth. By mollification one can generalize Proposition 2.23 to work with non-smooth weights.

Theorem 2.28 (Hörmander [28, Theorem 2.2.1']). *Let φ be a strictly plurisubharmonic function with $\partial\bar\partial\varphi(\mathbf{z}) \geq \kappa(\mathbf{z})$ in the sense of distributions, where $\kappa : \mathbb{C}^d \to \mathbb{R}_{>0}$. Let $\eta \in L^2_{(p,q)}(\varphi)$ satisfy $\bar\partial\eta = 0$ and*

$$\int |\eta(\mathbf{z})|^2 \frac{e^{-\varphi(\mathbf{z})}}{\kappa(\mathbf{z})}\, d\lambda \leq C.$$

Then there exists $g \in L^2_{p,q-1}(\varphi)$ such that $\bar\partial g = \eta$ and $\|g\|^2_{L^2_{p,q-1}(\varphi)} \leq C$

2.5.4 Application of Hörmander's theorem to the Beurling and Malliavin theorem

In this section we use Theorem 2.28 to prove Proposition 2.9, which is about constructing analytic functions from plurisubharmonic functions. We begin with a few lemmas.

Lemma 2.29. *Let $v : \mathbb{C}^d \to \mathbb{R}$ be Lipschitz with constant C_{Lip}, and let u_* be the maximal plurisubharmonic function $\leq v$. Then either $u_* = -\infty$, or u_* takes finite values everywhere and is Lipschitz with constant C_{Lip}.*

Proof. Because $u_* \leq v$,

$$u_*(\mathbf{x}-\mathbf{z}) \leq v(\mathbf{x}-\mathbf{z}) \leq v(\mathbf{x}) + C_{Lip}|\mathbf{z}|.$$

Thus $u_*(\bullet - \mathbf{z}) - |\mathbf{z}|C_{Lip}$ is a plurisubharmonic function lying below v, and by maximality

$$u_*(\mathbf{x}) \leq u_*(\mathbf{x}-\mathbf{z}) - |\mathbf{z}|C_{Lip} \qquad \text{for all } \mathbf{x}, \mathbf{z} \in \mathbb{C}^d.$$

If u_* takes finite values anywhere, then it takes finite values everywhere and is Lipschitz. □

Lemma 2.30. *For every $\alpha \in (0,1)$, there exists a plurisubharmonic function $u_\alpha : \mathbb{C}^d \to \mathbb{R}$ satisfying*

$$-\langle \mathbf{x} \rangle^\alpha \leq u_\alpha(\mathbf{x}+i\mathbf{y}) \leq -\langle \mathbf{x} \rangle^\alpha + C_{\alpha,d}|\mathbf{y}| \qquad \textit{for all } \mathbf{x}+i\mathbf{y} \in \mathbb{C}^d.$$

Proof. As the function $\langle \mathbf{x} \rangle^\alpha$ satisfies the Kohn-Nirenberg regularity conditions and is radial, we may apply Proposition 2.10 to the function $-\langle \mathbf{x} \rangle^\alpha$. □

Lemma 2.31. *Let f be analytic on an open set $U \subset \mathbb{C}^d$. If $\mathsf{B}_r(\mathbf{z}) \subset U$ then*

$$|f(\mathbf{z})| \leq C_d r^{-d} \|f\|_{L^2(\mathsf{B}_r(\mathbf{z}))}.$$

Proof. Because f is analytic on U, $|f|^2$ is plurisubharmonic and thus subharmonic on U. It follows that

$$|f(\mathbf{z})|^2 \leq \fint_{\mathsf{B}_r(\mathbf{z})} |f(\mathbf{w})|^2 \, d_{\mathrm{Leb}}(\mathbf{w}) \leq C_d r^{-2d} \|f\|^2_{L^2(\mathsf{B}_r(\mathbf{z}))}.$$

□

We restate Proposition 2.9 for the reader's convenience.

Proposition. *Let* $\omega : \mathbb{R}^d \to \mathbb{R}_{\le 0}$ *be a Lipschitz weight function, and let* u_* *be the maximal plurisubharmonic function* $\le \omega(\mathbf{x}) + 2\pi\sigma|\mathbf{y}|$. *If* $u_*(0) > -\infty$ *then for every* $\varepsilon > 0$, *there exists an* $f \in L^2(\mathbb{R}^d)$ *with* $\operatorname{supp} \hat{f} \subset \mathsf{B}_{\sigma+\varepsilon}(0)$, $\|f\|_2 \le 1$, *and*

$$\begin{aligned} |f(0)| &\ge c(d,\varepsilon) e^{-2\max\{\|\omega\|_{Lip}, 2\pi\sigma\}} e^{u^*_{\omega,\sigma}(0)}, \\ |f(\mathbf{x})| &\le e^{\omega(\mathbf{x})} && \text{for all } \mathbf{x} \in \mathbb{R}^d, \\ |f(\mathbf{x})| &\le e^{-\frac{\varepsilon}{C_d}\langle \mathbf{x}\rangle^{1/2}} && \text{for all } \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

Proof. The obstacle $\omega(\mathbf{x}) + 2\pi\sigma|\mathbf{y}|$ is Lipschitz with constant $C_{Lip} \le \max\{\|\omega\|_{Lip}, 2\pi\sigma\}$, so by Lemma 2.29, u_* is Lipschitz with the same constant.

We construct a plurisubharmonic weight by adding three new pieces to u_*. The first piece is

$$\log|\mathbf{z}|_\infty = \max_{1\le j\le d} \log\sqrt{x_j^2 + y_j^2},$$

which we add to provide lower bounds on $|f(0)|$. This term is plurisubharmonic because it is a maximum of plurisubharmonic functions. Next we add $u_{1/2}$, the function from Lemma 2.30 with parameter $\alpha = 1/2$, in order to balance out the prior term when $\mathbf{x}$ is far from the origin. Finally, we add a term proportional to $\langle \mathbf{y}\rangle$ to make the weight strictly plurisubharmonic. Indeed,

$$\partial\bar{\partial}\langle \mathbf{y}\rangle = \frac{1}{4}\langle \mathbf{y}\rangle^{-3}(1 + |\mathbf{y}|^2 - \mathbf{y}\mathbf{y}^t)$$

as a Hermitian matrix. The minimal eigenvector is $\hat{\mathbf{y}}$, and

$$\langle (\partial\bar{\partial}\langle \mathbf{y}\rangle)\hat{\mathbf{y}}, \hat{\mathbf{y}}\rangle = \frac{1}{4}\langle \mathbf{y}\rangle^{-3}$$

so $(\partial\bar{\partial}\langle \mathbf{y}\rangle)(\mathbf{x}+i\mathbf{y}) \ge \frac{1}{4}\langle \mathbf{y}\rangle^{-3}$. Let $\rho \in (0,1)$ be a small parameter to be chosen later, and define

$$\varphi = 2u_* + 20d\log|\mathbf{z}|_\infty + \rho u_{1/2} + \rho(\langle \mathbf{y}\rangle - 1).$$

Because each term is plurisubharmonic, $\partial\bar{\partial}\varphi(\mathbf{x}+i\mathbf{y}) \ge \kappa(\mathbf{x}+i\mathbf{y}) =: \frac{\rho}{4}\langle \mathbf{y}\rangle^{-3}$.

Let h be a smooth bump function on $\mathbb{C}^d$ with $h = 1$ on $\mathsf{B}_{1/2}$ and $\operatorname{supp} h \subset \mathsf{B}_1$. Let $\eta = \bar{\partial}h$, so η is supported on $\mathsf{B}_1 \setminus \mathsf{B}_{1/2}$. On this set,

$$\varphi \ge 2(u_*(0) - C_{Lip}) - C_d,$$

so

$$\int_{\mathbb{C}^d} |\eta(\mathbf{z})|^2 \frac{e^{-\varphi(\mathbf{z})}}{\kappa(\mathbf{z})}\, d_{\mathrm{Leb}}(\mathbf{z}) \le C_{d,\rho} e^{2C_{Lip}} e^{-2u_*(0)}. \tag{2.67}$$

By Theorem 2.28, there exists a smooth g such that $\bar{\partial} g = \eta$ and

$$\int_{\mathbb{C}^d} |g(\mathbf{z})|^2 e^{-\varphi(\mathbf{z})}\, d_{\mathrm{Leb}}(\mathbf{z}) \le C_{d,\rho} e^{2C_{Lip}} e^{-2u_*(0)}.$$

Define $f = h - g$. By construction $\bar{\partial} f = 0$, so f is entire.

First we show $g(0) = 0$. Because g is analytic on $\mathsf{B}_{1/2}$, Lemma 2.31 implies that for any $r \in (0, 1/2)$

$$\begin{aligned} |g(0)| &\le C_d r^{-d} \|g\|_{L^2(\mathsf{B}_r(0))} \\ &\le C_d r^{-d} \|g e^{-\varphi/2}\|_{L^2(\mathbb{C}^d)} \max_{|\mathbf{z}|\le r} e^{\varphi(\mathbf{z})/2}. \end{aligned}$$

When $|\mathbf{z}| \le r$,

$$\varphi \le 20d \log r + 2(u_*(0) + C_{Lip}) + C_d,$$

so $|g(0)| \le C r^{9d}$ for some constant C, and in particular $g(0) = 0$.

Next we estimate $|f(\mathbf{x})|$ far from the origin. To this end, we prove a weighted L^2 bound for f. In order to obtain a bound we must remove the logarithmic term from the weight. We have

$$20d \log |\mathbf{z}|_\infty + \rho u_{1/2}(\mathbf{z}) + \rho(\langle \mathbf{y}\rangle - 1) \le 10d \log |\mathbf{x}| + 10d \log |\mathbf{y}| + \log 2 - \rho \langle \mathbf{x}\rangle^{1/2} + C_d \rho |\mathbf{y}|.$$

Using that $10d \log |\mathbf{x}| - \rho\langle x\rangle^{1/2} \le C_{d,\rho} - \frac{1}{2}\rho\langle \mathbf{x}\rangle^{1/2}$ and $10d \log |\mathbf{y}| \le C_{d,\rho} + \rho|\mathbf{y}|$,

$$20d \log |\mathbf{z}|_\infty + \rho u_{1/2}(\mathbf{z}) + \rho(\langle \mathbf{y}\rangle - 1) \le C_{d,\rho} - \frac{1}{2}\rho\langle \mathbf{x}\rangle^{1/2} + C_d \rho |\mathbf{y}|.$$

We set

$$\widetilde{\varphi} = 2u_* - \frac{1}{2}\rho\langle \mathbf{x}\rangle^{1/2} + C_d\rho|\mathbf{y}|,$$

so that $\widetilde{\varphi} \ge \varphi - C_{d,\rho}$. We estimate the weighted L^2 norm of f,

$$\begin{aligned} \|f e^{-\widetilde{\varphi}/2}\|_{L^2(\mathbb{C}^d)} &\le \|h e^{-\widetilde{\varphi}/2}\|_{L^2(\mathbb{C}^d)} + C_{d,\rho} \|g e^{-\varphi/2}\|_{L^2(\mathbb{C}^d)} \\ &\le C_{d,\rho} e^{C_{Lip}} e^{-u_*(0)} \end{aligned}$$

where we bounded the first term using the Lipschitz property of u_*. For any $z \in \mathbb{C}^d$, Lemma 2.31 implies

$$\begin{aligned} |f(\mathbf{z})| &\le C_d \|f\|_{L^2(\mathsf{B}_1(\mathbf{z}))} \\ &\le C_{d,\rho} e^{C_{Lip}} e^{-u_*(0)} \max_{w \in \mathsf{B}_1(\mathbf{z})} e^{\widetilde{\varphi}(\mathbf{w})/2}. \end{aligned}$$

Now, $\tilde{\varphi}$ is Lipschitz with constant $\leq 2C_{Lip} + C_d\rho$, so

$$|f(\mathbf{z})| \leq C_{d,\rho} e^{2C_{Lip}} e^{-u_*(0)}\, e^{-\frac{1}{2}\rho\langle\mathbf{x}\rangle^{1/2}} e^{u_*(\mathbf{z})+C_d\rho|\mathbf{y}|}.$$

Because $u_*(\mathbf{x}+i\mathbf{y}) \leq \omega(\mathbf{x}) + 2\pi\sigma|\mathbf{y}|$,

$$|f(\mathbf{x}+i\mathbf{y})| \leq C_{d,\rho} e^{2C_{Lip}} e^{-u_*(0)}\, e^{-\frac{1}{2}\rho\langle\mathbf{x}\rangle^{1/2}}\, e^{\omega(\mathbf{x})} e^{(2\pi\sigma+C_d\rho)|\mathbf{y}|}.$$

To make our band-limited function we divide out by the constant factor above and restrict to $\mathbb{R}^d$,

$$\tilde{f} = \frac{1}{C_{d,\rho}} e^{-2C_{Lip}} e^{u_*(0)} f|_{\mathbb{R}^d},$$

and verify the desired properties.

- $\tilde{f}(\mathbf{x}) \leq e^{-\frac{1}{2}\rho\langle\mathbf{x}\rangle^{1/2}}$. In particular, $f \in L^2(\mathbb{R}^d)$.
- $|\tilde{f}(\mathbf{x})| \leq e^{\omega(\mathbf{x})}$.
- Because $g(0) = 0$, $f(0) = 1$, and $\tilde{f}(0) = \frac{1}{C_{d,\rho}} e^{-2C_{Lip}} e^{u_*(0)}$.
- The Paley-Wiener criterion is satisfied, so $\operatorname{supp} \hat{\tilde{f}} \subset \mathsf{B}_{\sigma+C_d\rho/2\pi}$.

Taking $\rho < 2\pi\varepsilon/C_d$ gives the result. □

Chapter 3

Unique continuation and the proof of fractal uncertainty

In this chapter, we use the Beurling and Malliavin multiplier theorem to prove a unique continuation principle for functions with Fourier transform supported in a line-porous set. We then use this unique continuation principle to prove our higher-dimensional fractal uncertainty principle. We follow Jaye and Mitkovski's [31] approach to unique continuation. See §1.3 for a high-level overview of these results and their application to fractal uncertainty.

3.1 Results

Definition 3.1. We say a weight $\mathcal{W} : [0, \infty) \to [0, \infty]$ is a *unique continuation weight* if

1. $\mathcal{W}(0) = 1$,
2. $\mathcal{W}$ is nondecreasing,
3. $\mathcal{W}$ is lower semicontinuous,
4. The mapping $s \mapsto \log \mathcal{W}(e^s)$ is convex on $[0, \infty)$,
5. $\mathcal{W}(t) \geq c_n t^n$ for all $n \geq 0$ and some $c_n > 0$,
6. $\displaystyle\int_0^\infty \frac{\log \mathcal{W}(t)}{1 + t^2}\, dt = \infty.$

For example, a typical choice is

$$\mathcal{W}(t) = e^{t/(\log(2+t))^\alpha},$$

where $\alpha \in (0,1]$. One can verify that $s \mapsto \frac{e^s}{\log(2+e^s)^\alpha}$ is convex on $[0,\infty)$, and if $\alpha \le 1$,

$$\int_0^\infty \frac{\log \mathcal{W}(t)}{1+t^2}\,dt = \int_0^\infty \frac{1}{t(\log(2+t))^\alpha}dt = \infty.$$

For each $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, let

$$\mathsf{Q}_\mathbf{n} = [n_1, n_1+1] \times \cdots \times [n_d, n_d+1]$$

be the integer cube with bottom corner $\mathbf{n}$.

The next result, which is part of Jaye and Mitkovski's [31, Theorem 1.3], is a quantitative unique continuation principle for functions with rapid Fourier decay.

Theorem 3.2. *For any unique continuation weight $\mathcal{W}$ and any set $E \subset \mathbb{R}^d$ satisfying*

$$|E \cap \mathsf{Q}_\mathbf{n}| \ge \lambda \quad \textit{for every integer cube } \mathsf{Q}_\mathbf{n},$$

we have

$$\|\hat{f}\,\mathcal{W}\|_2 \le A\|f\|_2 \quad \implies \quad \|f\|_2 \le C(\mathcal{W}, d, A, \lambda)\|f\,\mathbf{1}_E\|_2.$$

Definition 3.3. Let $\mathbf{Y} \subset \mathbb{R}^d$, let $\mathcal{W}$ be a unique continuation weight, and let $c_1, c_2 \in (0,1]$. We say that $\mathbf{Y}$ admits a $(c_1, c_2, \mathcal{W})$-damping function if there exists a $\psi \in L^2(\mathbb{R}^d)$ satisfying

- $\operatorname{supp}\psi \subset \mathsf{B}_{c_1}$,
- $|\hat{\psi}(\xi)| \ge c_2$ for $\xi \in \mathsf{B}_{c_2}$,
- $|\hat{\psi}(\xi)| \le \langle \xi \rangle^{-d}$ for all $\xi \in \mathbb{R}^d$,
- $|\hat{\psi}(\xi)| \le 1/\mathcal{W}(|\xi|)$ for all $\xi \in \mathbf{Y}$.

The following theorem gives a quantitative unique continuation principle for sets admitting damping functions. It is a variant of Han and Schlag's result [27, Corollary 4.2]. It applies to λ-neighborhoods of the sets appearing in Theorem 3.2, which can be written as a union of λ-cubes, one inside of each integer cube:

$$\mathcal{S} = \bigcup_{\mathbf{n} \in \mathbb{Z}^d} \mathsf{I}_\mathbf{n}, \qquad \mathsf{I}_\mathbf{n} \text{ is a cube of width } \lambda \text{ centered at a point of } \mathsf{Q}_\mathbf{n}. \tag{3.1}$$

Theorem 3.4. *Let $\lambda > 0$. Suppose $\mathbf{Y} \subset \mathbb{R}^d$ is such that every translate $\mathbf{Y} + \eta$, with $\eta \in \mathbb{R}^d$, admits a $(\lambda/4, c_2, \mathcal{W})$-damping function. Let $\mathcal{S}$ be a set of the form (3.1)—that is, a union of λ-cubes, one per integer cube. Then*

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \Longrightarrow \|f\|_2 \le C \|f \, \mathbf{1}_{\mathcal{S}}\|_2$$

where $C = C(c_2, \mathcal{W}, d, \lambda)$.

We can construct damping functions for line-porous sets using our higher-dimensional Beurling–Malliavin theorem (Theorem 2.5).

Proposition 3.5. *Let $\mathbf{Y} \subset [-3h^{-1}, 3h^{-1}]^d$ be ν-porous on lines from scales μ to h^{-1}. For any $c_1 > 0$, $\mathbf{Y}$ admits a $(c_1, c_2, \mathcal{W})$ damping function where $c_2 = c_2(c_1, \nu, d)$ and*

$$\mathcal{W}(t) = \exp\Big(\frac{c_3 t}{(\log(2+t))^{\alpha}}\Big)$$

for some $\alpha(\nu) \in (0, 1)$ and $c_3(c_1, \nu, d) > 0$. As a consequence, if $\mathcal{S}$ is a union of λ-cubes, one inside of each integer cube, then

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \quad \Longrightarrow \quad \|f\|_2 \le C \|f \, \mathbf{1}_{\mathcal{S}}\|_2$$

where $C = C(\lambda, \nu, d)$.

The following theorem, a generalization of Han and Schlag's [27, Theorem 5.1], gives an FUP conditional on the existence of damping functions.

Theorem 3.6. *Suppose*

- *$\mathbf{X} \subset [-1, 1]^d$ is ν-porous on balls from scales h to 1,*
- *$\mathbf{Y} \subset [-h^{-1}, h^{-1}]^d$ satisfies that for all $s \in [h, 1]$ and $\eta \in \mathbb{R}^d$, the set*

$$s\mathbf{Y} + [-4, 4]^d + \eta$$

  admits a $\left(\frac{\nu}{10 d^{1/2}}, c_2, \mathcal{W}\right)$-damping function.

Then

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \Longrightarrow \|f 1_{\mathbf{X}}\|_2 \le C h^{\beta} \|f\|_2.$$

for some constants $\beta, C > 0$ that depend on $c_2, \mathcal{W}, d, \nu$.

Combining Theorem 3.6 with Proposition 3.5 proves Theorem 1.2.

Proof of Theorem 1.2. Let

- $\mathbf{X} \subset [-1,1]^d$ be ν-porous on balls from scales h to 1,
- $\mathbf{Y} \subset [-h^{-1}, h^{-1}]^d$ be ν-porous on lines from scales 1 to h^{-1}.

By Lemma 3.11, for any $s \in (h,1)$ and $\eta \in \mathbb{R}^d$, the set

$$s\mathbf{Y} + [-4,4]^d + \eta$$

is $\nu/2$-porous on lines from scale $10\sqrt{d}/\nu$ to $h^{-1}s$. By Proposition 3.5 with $\mu = 10\sqrt{d}/\nu$, $\mathbf{Y}$ admits a $\mathcal{W}$-damping function where

$$\mathcal{W}(t) = A(\nu,d)\exp\Big(\frac{t}{\log(2+t)^{\alpha(\nu)}}\Big)$$

is a unique continuation weight that depends only on ν and d. By Theorem 3.6, there exists $\beta = \beta(\nu,d) > 0$ and $C = C(\nu,d) > 0$ so that for any $f \in L^2(\mathbb{R}^d)$

$$\operatorname{supp}\hat{f} \subset \mathbf{Y} \implies \|f1_{\mathbf{X}}\|_2 \le \widetilde{C}h^{\beta}\,\|f\|_2. \tag{3.2}$$

□

In the rest of this chapter, we prove the above results in order.

3.2 Unique continuation of functions with rapidly decaying Fourier transform

In this section we prove Theorem 3.2 following [31, Theorem 1.3].

Our starting point is analytic functions. If an analytic function vanishes to infinite order at any point, then it must be identically zero; this property is known as *unique continuation.* A smooth function f on $[0,1]$ is analytic if and only if

$$\|D^n f\|_{L^\infty([0,1])} \le (n!)M^n \qquad \text{for some } M > 0 \text{ and all } n \ge 0.$$

One may then ask: Are there milder derivative growth conditions that still imply unique continuation?

This question was answered decisively by the work of Denjoy and Carleman on quasi-analytic classes. A logarithmically convex sequence $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}_{\ge 0}}$ is called *quasi-analytic* if the following holds for smooth functions on the unit interval:

If $\|D^n f\|_{L^\infty([0,1])} \le M_n$ for all $n \ge 0$ and f vanishes to infinite order at some point, then $f \equiv 0$ on $[0,1]$.

The Denjoy–Carleman theorem characterizes quasi-analytic classes, see [39, Chapter IV] for a proof.

Theorem 3.7 (Denjoy–Carleman)**.** *A logarithmically convex sequence $\mathcal{M}$ generates a quasi-analytic class if and only if*

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty. \tag{3.3}$$

One may prove by induction that if $\{M_n\}$ is quasi-analytic, then the unique continuation property holds in higher dimensions as well:

If $\|D^n f\|_{L^\infty([0,1]^d)} \le M_n$ for all $n \ge 0$ and f vanishes to infinite order at some point, then $f \equiv 0$ on $[0,1]^d$.

The following proposition uses compactness to quantify the unique continuation principle stated above. We follow [31, Remark 2.6].

Proposition 3.8. *For any $t, \gamma > 0$ the following holds. Let $f \in C^\infty([0,1]^d)$ satisfy*

$$\|D^n f\|_{L^\infty([0,1]^d)} \le M_n$$

for all $n \ge 0$, and suppose $\|f\|_{L^\infty([0,1]^d)} \ge t$. Then for any measurable set $E \subset [0,1]^d$ with Lebesgue measure at least γ,

$$\|f\, \mathbf{1}_E\|_{L^1([0,1]^d)} \ge c(\mathcal{M}, d, \gamma, t).$$

Proof. Suppose not. Let f_j be a sequence of smooth functions satisfying

$$\|D^n f_j\|_{L^\infty([0,1]^d)} \le M_n \quad \text{and} \quad \|f_j\|_{L^\infty([0,1]^d)} \ge t,$$

and let $E_j \subset [0,1]^d$ be a sequence of sets with Lebesgue measure at least γ such that

$$\|f_j\, \mathbf{1}_{E_j}\|_{L^1([0,1]^d)} \to 0 \quad \text{as } j \to \infty.$$

Since the derivatives of f_j are uniformly bounded, Arzela-Ascoli implies that after passing to a subsequence, $f_j \to f$ in the L^∞ norm. Working in the sense of distributions, for any multiindex β we have

$$\|D^\beta f\|_{L^\infty([0,1]^d)} = \sup_{\substack{h \in C^\infty([0,1]^d), \\ \|h\|_{L^1([0,1]^d)} \le 1}} \int (D^\beta f)(\mathbf{x})\, h(\mathbf{x})\, d\mathbf{x}.$$

Using integration by parts,

$$\Big|\int (D^\beta f)(\mathbf{x})\, h(\mathbf{x})\, d\mathbf{x}\Big| = \Big|\int f(\mathbf{x})\, D^\beta h(\mathbf{x})\, d\mathbf{x}\Big| = \lim_{j\to\infty}\Big|\int f_j(\mathbf{x})\, D^\beta h(\mathbf{x})\, d\mathbf{x}\Big| \le M_{|\beta|}.$$

Thus, f is a smooth function satisfying $\|D^n f\|_{L^\infty([0,1]^d)} \le M_n$ for all $n \ge 0$.

Define

$$E = \bigcap_{n\ge 0}\Big(\bigcup_{m\ge n} E_m\Big),$$

and observe that the Lebesgue measure of E is at least γ. For any $\mathbf{x} \in E$, there exists a subsequence $\{n_m\}$ with $\mathbf{x} \in E_{n_m}$, thus

$$f(\mathbf{x}) = \lim_{m\to\infty} f_{n_m}(\mathbf{x}) = 0 \qquad \text{for all } \mathbf{x} \in E.$$

For any multiindex β, we can write

$$D^\beta f(\mathbf{x}_0) = \lim_{h\to 0} \frac{\sum_{j=1}^r c_j f(\mathbf{x}_0 + h v_j)}{h^{|\beta|}} \tag{3.4}$$

for some vectors v_j and coefficients c_j. For any $v \in \mathbb{R}^d$,

$$\lim_{h\to 0} |\{\mathbf{x} \in E \,:\, \mathbf{x} + hv \notin E\}| = 0.$$

Thus for a full measure subset $E' \subset E$, there is a subsequence $h_k \to 0$ such that

$$\mathbf{x}, \mathbf{x} + h_k v_1, \ldots, \mathbf{x} + h_k v_r \in E \qquad \text{for all } \mathbf{x} \in E' \text{ and } k \ge 0.$$

For any $\mathbf{x} \in E'$, Eq. (3.4) implies $D^\beta f(\mathbf{x}) = 0$.

Since $\{M_n\}$ is quasi-analytic, this implies $f \equiv 0$. However, by compactness of $[0,1]^d$,

$$\|f\|_{L^\infty([0,1]^d)} \ge \lim_{j\to\infty} \|f_j\|_{L^\infty([0,1]^d)} \ge t,$$

a contradiction. □

Given a unique continuation weight $\mathcal{W}$ as above, we define

$$M_n = \sup_{t\ge 1} \frac{t^n}{\mathcal{W}(t)}. \tag{3.5}$$

For example, if $\mathcal{W}(t) = e^{t/(\log(2+t))^\alpha}$ then the supremum in (3.5) is achieved at $t \sim n(\log n)^\alpha$, so $M_n \sim \left(\frac{n}{e}\right)^n (\log n)^{\alpha n}$, and

$$\frac{M_n}{M_{n+1}} \sim \frac{1}{n(\log n)^\alpha}.$$

Thus $\sum \frac{M_n}{M_{n+1}} = \infty$, and $\{M_n\}$ is a quasi-analytic class.

The following technical Lemma about the sequence $\{M_n\}$ allows us to prove $\{M_n\}$ is quasi-analytic for any unique continuation weight $\mathcal{W}$. This lemma is proved in [31, Page 6 and Proposition 2.2].

Lemma 3.9. *Suppose that $\mathcal{W} : [0, \infty) \to [0, \infty]$ is nondecreasing, lower semicontinuous, the mapping $s \mapsto \log \mathcal{W}(e^s)$ is convex on $[0, \infty)$, and $\mathcal{W}(t) \geq c_n t^n$ for all $n \geq 0$. Then the sequence $\{M_n\}$ defined by (3.5) satisfies:*

- *$\{M_n\}$ is increasing and log–convex; that is,*

$$M_n \leq M_{n+1} \quad \textit{and} \quad M_n^2 \leq M_{n-1}\, M_{n+1}$$

  for all n.

- *The following inequalities hold:*

$$\sum_{n\geq 0} \frac{M_{n-1}}{M_n} \leq \int_1^\infty \frac{\log \mathcal{W}(t)}{t^2}\, dt \leq 1 + \sum_{n \geq 0} \frac{M_{n-1}}{M_n}\,.$$

Proof.

- Monotonicity is clear. For log-convexity, notice that for each $t \geq 1$

$$M_n^2 = \Big(\frac{t^n}{W(t)}\Big)^2 = \Big(\frac{t^{n-1}}{W(t)}\Big)\Big(\frac{t^{n+1}}{W(t)}\Big) \leq M_{n-1} M_{n+1}.$$

- Let

$$t_n := \operatorname{argmax}_{t\geq 1} \frac{t^n}{\mathcal{W}(t)},$$

  if there are several t achieving the maximum, choose the smallest one. The hypothesis that $\mathcal{W}$ grows faster than any polynomial implies that t_n is bounded for all n. For any $t \in [t_n, t_{n+1}]$, write $t = t_n^\alpha t_{n+1}^{1-\alpha}$ for some $\alpha \in [0, 1]$ and use log-convexity of $\mathcal{W}$ to find

$$\begin{aligned} \mathcal{W}(t) = \mathcal{W}(t_n^\alpha t_n^{1-\alpha}) &\leq \mathcal{W}(t_n)^\alpha \mathcal{W}(t_{n+1})^{1-\alpha} \\ &= \Big(\frac{(t_n)^n}{M_n}\Big)^\alpha \Big(\frac{(t_{n+1})^{n+1}}{M_{n+1}}\Big)^{1-\alpha} \\ &\leq \frac{t^{n+1}}{M_n}. \end{aligned} \tag{3.6}$$

  If $t \geq 1$ and $\mathcal{W}(t) < \infty$ then $t \in [t_n, t_{n+1}]$ for some $n \geq 0$. For $r \geq 1$, define

$$\rho(r) := \sup_{n\geq 0} \frac{r^n}{M_n} = \sup_{n\geq 0} \inf_{t\geq 1} (r/t)^n \mathcal{W}(t).$$

Taking $t = r$ shows $\rho(r) \le \mathcal{W}(r)$. By (3.6), if $r \in [t_n, t_{n+1}]$ then

$$\rho(r) \ge \frac{1}{r}\mathcal{W}(r), \tag{3.7}$$

and since this equation does not depend on n, it holds whenever $\mathcal{W}(r) < \infty$. On the other hand, if $\mathcal{W}(r) = \infty$ then $M_n \le r^n$ and $\rho(r') = \infty$ for all $r' > r$. By lower semicontinuity, this implies $\rho(r) = \infty$ whenever $\mathcal{W}(r) = \infty$, thus (3.7) holds for all $r \ge 1$.

The following Legendre transform identity completes the proof:

$$\int_1^\infty \frac{\log \rho(r)}{r^2}\, dr = \sum_{n\ge 1} \frac{M_{n-1}}{M_n}. \tag{3.8}$$

Let $q_n = M_{n+1}/M_n$ for $n \ge 0$, extend q_n to all of $\mathbb{Z}$ by 1. For $r \in [q_{n-1}, q_n]$,

$$\rho(r) = \frac{r^n}{M_n}.$$

Thus

$$\begin{aligned}\int_1^\infty \frac{\log \rho(r)}{r^2}\, dr &= \sum_{n\ge 0} \int_{q_{n-1}}^{q_n} \frac{n \log r - \log M_n}{r^2}\, dr \\ &= \sum_{n\ge 0} \Big[n \Big(\frac{1-\log q_n}{q_n} - \frac{1-\log q_{n-1}}{q_{n-1}} \Big) - \log M_n \Big(\frac{1}{q_{n-1}} - \frac{1}{q_n} \Big) \Big].\end{aligned}$$

We have

$$\sum_{n\ge 0} \log M_n \Big(\frac{1}{q_{n-1}} - \frac{1}{q_n} \Big) = \sum_{n\ge 0} \frac{1}{q_n} \Big(\log \frac{M_{n+1}}{M_n} \Big) = \sum_{n\ge 0} \frac{1}{q_n} \log q_n.$$

On the other hand, by integration by parts

$$\sum_{n\ge 0} n \Big(\frac{1-\log q_n}{q_n} - \frac{1-\log q_{n-1}}{q_{n-1}} \Big) = \sum_{n\ge 0} \frac{1}{q_n} - \sum_{n\ge 0} \frac{1}{q_n} \log q_n.$$

The $\sum \frac{1}{q_n} \log q_n$ terms cancel, and we are left with (3.8)

□

Proof of Theorem 3.2. Suppose $\|\hat{f}(\xi)\mathcal{W}(|\xi|)\|_2 \le A\|f\|_2$. We can estimate the Sobolev norms

$$|f\|_{H^n} = \sum_{|\beta|\le n} \|D^\beta f\|_2$$

using the Fourier transform,

$$\begin{aligned}\|f\|_{H^n} &\lesssim \|f\|_2 + \|(|\xi|^k)\hat{f}\|_2 \\ &\leq \|f\|_2 + M_n\|\hat{f}\mathcal{W}\|_2 \\ &\leq (1+AM_n)\|f\|_2 \leq 2AM_n\|f\|_2.\end{aligned}$$

Let $\mathcal{Q}$ denote the set of all integer cubes. Let

$$\sum \eta_{\mathsf{Q}} = 1 \tag{3.9}$$

be a partition of unity where each η_{Q} is a translate of a fixed bump function, and $\operatorname{supp}\eta_{\mathsf{Q}} \subset 2\cdot\mathsf{Q}$. Decompose f as

$$f = \sum_{\mathsf{Q}\in\mathcal{Q}} f_{\mathsf{Q}}, \qquad f_{\mathsf{Q}} = \eta_{\mathsf{Q}} f.$$

We have

$$\sum_{\mathsf{Q}} \|f_{\mathsf{Q}}\|_{H^n}^2 \lesssim \|f\|_{H^n}^2 \leq 2AM_n\|f\|_2.$$

Let $K > 0$ be a constant to be chosen later, and set

$$\begin{aligned}\mathcal{Q}_{good} &= \{\mathsf{Q}\in\mathcal{Q} \,:\, \|f_{\mathsf{Q}}\|_{H^n} \leq Ke^n M_n\|f_{\mathsf{Q}}\|_2 \text{ for all } n\geq 0\} \\ \mathcal{Q}_{bad} &= \mathcal{Q}\setminus\mathcal{Q}_{good}.\end{aligned}$$

To estimate the bad part, we use the H^n norm of f

$$\begin{aligned}\sum_{\mathsf{Q}\in\mathcal{Q}^{bad}} \|f_{\mathsf{Q}}\|_2^2 &\leq \sum_{n\geq 0} \sum_{\substack{\mathsf{Q}\in\mathcal{Q} \text{ s.t.} \\ \|f_{\mathsf{Q}}\|_{H^n} > Ke^n M_n\|f_{\mathsf{Q}}\|_2}} \|f_{\mathsf{Q}}\|_2^2 \\ &\leq \sum_{n\geq 0} \frac{1}{K^2 e^{2n} M_n^2} \sum_{\substack{\mathsf{Q}\in\mathcal{Q} \text{ s.t.} \\ \|f_{\mathsf{Q}}\|_{H^n} > Ke^n M_n\|f_{\mathsf{Q}}\|_2}} \|f_{\mathsf{Q}}\|_{H^n}^2 \\ &\leq \sum_{n\geq 0} \frac{1}{K^2 e^{2n} M_n^2}\|f\|_{H^n}^2 \leq \Big(\sum_{n\geq 0} \frac{2A}{e^{2n}K^2}\Big)\|f\|_2^2 \\ &\leq \frac{5A^2}{K^2}\|f\|_2^2.\end{aligned}$$

To estimate the good part we use quasi-analyticity. If $\mathsf{Q}\in\mathcal{Q}^{good}$, then we can control the Sobolev norms of f_{Q}. By Sobolev embedding, if β is a multiindex of order n,

$$\begin{aligned}\|D^\beta f_{\mathsf{Q}}\|_\infty &\leq C_d\|f_{\mathsf{Q}}\|_{H^{n+d}} \\ &\leq C_d K e^{n+d} M_{n+d}\|f_{\mathsf{Q}}\|_2.\end{aligned}$$

The sequence $\{e^{n+d}M_{n+d}\}$ is logarithmically convex and forms a quasi-analytic sequence. Let

$$\tilde{f}_{\mathsf{Q}} = \frac{1}{C_d K \|f_{\mathsf{Q}}\|_2} f_{\mathsf{Q}},$$

so $\tilde{f}_{\mathsf{Q}}$ lies in the quasi-analytic class $\{e^{n+d}M_{n+d}\}$. We have $\|f_{\mathsf{Q}}\|_2 \leq 3^d \|f_{\mathsf{Q}}\|_\infty$, so

$$\|\tilde{f}_{\mathsf{Q}}\|_\infty \geq \frac{c_d}{K}.$$

By applying Proposition 3.8 to $\tilde{f}_{\mathsf{Q}}$, we find that

$$\|\tilde{f}_{\mathsf{Q}} \mathbf{1}_{\mathcal{S}\cap\mathsf{Q}}\|_{L^1} \geq c(W, d, \lambda, K),$$

and going back to f_{Q} this gives

$$\|f_{\mathsf{Q}}\|_2 \leq C(W, d, \lambda, K)\|f_{\mathsf{Q}}\mathbf{1}_{\mathcal{S}\cap\mathsf{Q}}\|_2.$$

By summing over all the cubes we find

$$\begin{aligned}
\|f\|_2^2 &\leq C_d \sum_{\mathsf{Q}\in\mathcal{Q}} \|f_{\mathsf{Q}}\|_2^2 \\
&= C_d \sum_{\mathsf{Q}\in\mathcal{Q}_{good}} \|f_{\mathsf{Q}}\|_2^2 + C_d \sum_{\mathsf{Q}\in\mathcal{Q}_{bad}} \|f_{\mathsf{Q}}\|_2^2 \\
&\leq C(\mathcal{W}, d, \lambda, K) \sum_{\mathsf{Q}\in\mathcal{Q}_{good}} \|f\mathbf{1}_{\mathcal{S}\cap\mathsf{Q}}\|_2^2 + C_d \frac{5A^2}{K^2}\|f\|_2^2 \\
&\leq C(\mathcal{W}, d, \lambda, K)\|f\,\mathbf{1}_{\mathcal{S}}\|_2^2 + \frac{5C_dA^2}{K^2}\|f\|_2^2.
\end{aligned}$$

Choose $K > 10C_dA$ and rearrange to obtain

$$\|f\|_2 \leq C(\mathcal{W}, d, \lambda, A)\|f\,\mathbf{1}_{\mathcal{S}}\|_2.$$

□

3.3 Unique continuation for sets admitting damping functions

In this section we prove Theorem 3.4, following [31, Theorem 5.2]

We consider the lattice $\Lambda = \frac{c_2}{\sqrt{d}}\mathbb{Z}^d$. The scaling factor is chosen so that the c_2-neighborhood of Λ covers all of $\mathbb{R}^d$.

For each $\eta \in \Lambda$, let ψ_η be a damping function for $\mathbf{Y}$ centered at η, meaning

- $\operatorname{supp}\psi_\eta \subset \mathsf{B}_{\lambda/4}$
- $|\hat{\psi}_\eta(\xi)| \geq c_2$ for $\xi \in \mathsf{B}_{c_2}(\eta)$
- $|\hat{\psi}_\eta(\xi)| \leq \langle \xi - \eta \rangle^{-d}$ for all $\xi \in \mathbb{R}^d$
- $|\hat{\psi}_\eta(\xi)| \leq 1/\mathcal{W}(|\xi - \eta|)$ for all $\xi \in \mathbf{Y}$.

Let

$$f_\eta = f * \psi_\eta.$$

Our goal is to estimate $\|f\|_2$ in terms of $\|f\,\mathbf{1}_{\mathcal{S}}\|_2$. First, we can control the L^2 norm of f in terms of the $\{f_\eta\}_{\eta\in\Lambda}$,

$$\sum_{\eta\in\Lambda} \|f_\eta\|_2^2 \geq \sum_{\eta\in\Lambda} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\psi_\eta(\xi)|^2 \, d\xi \geq c_2 \|f\|_2^2. \tag{3.10}$$

Say $\eta \in \Lambda$ is *good* if

$$\|\mathcal{W}(|\eta - \xi|)^{1/2} \hat{f}_\eta\|_2 \leq A \|f_\eta\|_2$$

and otherwise η is bad. Here A is a constant to be chosen later, and $\mathcal{W}^{1/2}$ is also a unique continuation weight.

We can control the bad part by

$$\begin{aligned} \sum_{\eta\in\Lambda \text{ is bad}} \|f_\eta\|_2^2 &\leq \frac{1}{A} \sum_{\eta\in\Lambda \text{ is bad}} \|\mathcal{W}(|\eta - \xi|)^{1/2} \hat{\psi}_\eta \hat{f}\|_2^2 \\ &\leq \frac{1}{A} \sum_{\eta\in\Lambda \text{ is bad}} \|\mathcal{W}(|\eta - \xi|)^{-1/2} \hat{f}\|_2^2. \end{aligned}$$

Because $\mathcal{W}$ grows faster than any polynomial, the right hand side is bounded by

$$\sum_{\eta\in\Lambda \text{ is bad}} \|f_\eta\|_2^2 \leq \frac{1}{A} C(\mathcal{W}, c_2) \|f\|_2^2. \tag{3.11}$$

To control the good part we use Theorem 3.2. Let $E \subset \mathcal{S}$ be the union of the $\frac{1}{2}$-dilates of the λ-cubes making up $\mathcal{S}$. For any good $\eta \in \Lambda$,

$$\|f_\eta\|_2 \leq C(\mathcal{W}, d, A, \lambda) \|f_\eta \, \mathbf{1}_E\|_2.$$

We bound the right hand side using that $\operatorname{supp}\psi_\eta \subset \mathsf{B}_{\lambda/4}$,

$$\|f_\eta \, \mathbf{1}_E\|_2 = \|(f * \psi_\eta) \, \mathbf{1}_E\|_2 \leq \|(f \, \mathbf{1}_{\mathcal{S}}) * \psi_\eta\|_2.$$

Thus

$$\begin{aligned}\sum_{\eta\in\Lambda \text{ is good}} \|f_\eta\|_2^2 &\le C \sum_{\eta\in\Lambda \text{ is good}} \|f_\eta\, \mathbf{1}_E\|_2^2 \\ &\le C \sum_{\eta\in\Lambda \text{ is good}} \|\widehat{f\,\mathbf{1}_\mathcal{S}}\,\widehat{\psi_\eta}\|_2^2 \\ &\le C\|f\,\mathbf{1}_\mathcal{S}\|_2^2 \end{aligned} \tag{3.12}$$

using that ψ_η decays faster than $\langle \xi - \eta\rangle^{-d}$. Combining (3.10), (3.11), and (3.12) gives

$$\|f\|_2^2 \le C(c_2, \mathcal{W}, d, A, \lambda)\|f\,\mathbf{1}_\mathcal{S}\|_2^2 + \frac{1}{A}C(\mathcal{W}, c_2)\|f\|_2^2.$$

Taking A large enough that $\frac{1}{A}C(\mathcal{W}, c_2) \le 1/2$, we find

$$\|f\|_2 \le C(c_2, \mathcal{W}, d, \lambda)\|f\,\mathbf{1}_\mathcal{S}\|_2$$

as desired.

3.4 Fractal uncertainty conditional on damping functions

In this section we prove Theorem 3.6.

Fix a nonnegative Schwarz function ϕ on $\mathbb{R}^d$ with

$$\operatorname{supp}\hat{\phi} \subset [-1,1]^d, \qquad \hat{\phi}(0) = 1.$$

Let $T > 0$ be an integer to be chosen later, and set

$$\psi(x) = 2^{Td}\phi(2^T x).$$

Moreover, for any $j \ge 0$ set

$$\psi_j(x) = 2^{jd}\psi(2^j x).$$

Define

$$\Psi_n = \psi_n * \mathbf{1}_{\mathbf{X}+\mathsf{B}_{2^{-n-T/2}}}.$$

There exists a constant $C_{\phi,d}$ depending only on ϕ and d such that for all $n \ge 0$,

$$\Psi_n(\mathbf{x}) \ge 1 - C_{\phi,d}2^{-T} \qquad \text{for all } \mathbf{x} \in \mathbf{X} \tag{3.13}$$

$$\Psi_n(\mathbf{x}) \le C_{\phi,d}2^{-T} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d \text{ with } d(\mathbf{x}, \mathbf{X}) \ge 5\cdot 2^{-n-T/2}. \tag{3.14}$$

Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat{f} \subset \mathbf{Y}$. Set

$$f_m := \Big(\prod_{j=0}^{m-1} \Psi_{jT}\Big) f.$$

When $m = 0$, set $f_0 = f$. We have

$$\begin{aligned}\operatorname{supp} \widehat{f_m} &\subset \mathbf{Y} + \sum_{j=0}^{m-1} [-2^{(j+1)T}, 2^{(j+1)T}]^d \\ &\subset \mathbf{Y} + [-2^{(m+1)T}, 2^{(m+1)T}]^d.\end{aligned}$$

Let m_0 be the largest integer so that $2^{-m_0 T} \geq h$. We have

$$m_0 = \Big\lfloor \frac{\log_2 h^{-1}}{T} \Big\rfloor. \tag{3.15}$$

Claim 3.10. *There exists $\gamma_0 = \gamma_0(\mathcal{W}, \nu, d) \in (0,1)$ such that as long as $T \geq T_0(\nu, d)$ above,*

$$\|f_{m+1}\|_2 \leq (1 - \gamma_0) \|f_m\|_2 \qquad \textit{for } 0 \leq m \leq m_0.$$

Proof. Let $\mathcal{D}_{mT}$ denote the half-open dyadic cubes with side length 2^{-mT}.

Using the hypothesis that $\mathbf{X}$ is ν-porous from scale h to 1, for each dyadic cube $\mathsf{Q} \in \mathcal{D}_{mT}$, there is some $\mathbf{x}_{\mathsf{Q}} \in \mathsf{Q}$ with

$$\operatorname{dist}(\mathbf{x}_{\mathsf{Q}}, \mathbf{X}) \geq 2^{-mT} \nu.$$

Let

$$\begin{aligned}\mathsf{I}_{\mathsf{Q}} &= \mathbf{x}_{\mathsf{Q}} + [-\lambda/2, \lambda/2]^d \qquad \text{where } \lambda = d^{-1/2} 2^{-mT} \nu, \\ \mathcal{S} &= \bigcup_{\mathsf{Q} \in \mathcal{D}_{mT}} \mathsf{I}_{\mathsf{Q}}.\end{aligned}$$

For this choice of λ, $\operatorname{dist}(\mathcal{S}, \mathbf{X}) \geq 2^{-mT}\nu/2$. By (3.14), if T is sufficiently large in terms of ν

$$\Psi_{mT}(\mathbf{x}) \leq \frac{1}{2} \qquad \text{for } \mathbf{x} \in \mathcal{S}.$$

Thus

$$\begin{aligned}\|f_{m+1}\|_2^2 = \|\Psi_{mT} f_m\|_2^2 &\leq \|\Psi_{mT}\|_\infty^2 \|f_m \mathbf{1}_{\mathcal{S}^c}\|_2^2 + \|\Psi_{mT} \mathbf{1}_{\mathcal{S}}\|_\infty^2 \|f_m \mathbf{1}_{\mathcal{S}}\|_2^2 \\ &\leq (\|f_m\|_2^2 - \|f_m \mathbf{1}_{\mathcal{S}}\|_2^2) + \frac{1}{2} \|f_m \mathbf{1}_{\mathcal{S}}\|_2^2 \\ &\leq \|f_m\|_2^2 - \frac{1}{2} \|f_m \mathbf{1}_{\mathcal{S}}\|_2^2.\end{aligned}$$

Let $f_m^{\text{resc}} = f_m(2^{mT}x)$, and let $\mathcal{S}^{\text{resc}} = 2^{mT}\mathcal{S}$. We have

$$\begin{aligned}\operatorname{supp} \widehat{f_m^{\text{resc}}} &\subset 2^{-mT}\mathbf{Y} + \ell_m 2^{-mT}[-1,1]^d \\ &\subset 2^{-mT}\mathbf{Y} + [-2,2]^d.\end{aligned}$$

Apply Theorem 3.4 to f_m^{resc} and $\mathcal{S}^{\text{resc}}$ to find

$$\|f_m^{\text{resc}}\, \mathbf{1}_{\mathcal{S}^{\text{resc}}}\|_2 \geq c(\mathcal{W}, d, \nu)\|f_m^{\text{resc}}\|_2.$$

Rescaling back gives

$$\|f_m\, \mathbf{1}_{\mathcal{S}}\|_2 \geq c(\mathcal{W}, d, \nu)\|f_m\|_2,$$

which implies

$$\|f_{m+1}\|_2^2 \leq \Big(1 - \frac{1}{2}c(\mathcal{W}, d, \nu)\Big)\|f_m\|_2^2$$

as desired. □

Iterating Claim 3.10 several times gives

$$\|f_{m_0}\|_2 \leq (1-\gamma_0)^{m_0}\|f\|_2 \tag{3.16}$$

By (3.13),

$$\begin{aligned}\|f_{m_0}\|_2 &= \|\Big(\prod_{j=0}^{m_0-1} \Psi_{jT}\Big) f\|_2 \\ &\geq (1 - C_{\phi,d}2^{-T})^{m_0}\|f\, \mathbf{1}_{\mathbf{X}}\|_2.\end{aligned}$$

Comparing the above with (3.16) gives

$$\|f\, \mathbf{1}_{\mathbf{X}}\|_2 \leq \Big(\frac{1-\gamma_0}{1 - C_{\phi,d}2^{-T}}\Big)^{m_0}\|f\|_2.$$

If we choose T large enough that the denominator is $\geq 1 - \gamma_0/2$ and use that $m_0 \geq \frac{1}{T}\log_2 h^{-1} - 1$, we find

$$\|f\, \mathbf{1}_{\mathbf{X}}\|_2 \leq Ch^{\beta}\|f\|_2$$

where β and C depend on $\mathcal{W}, d, \nu$.

3.5 Construction of damping functions for line porous sets

3.5.1 Properties of line porous sets

Lemma 3.11. *Let $\mathbf{X} \subset \mathbb{R}^d$ be ν-porous on lines from scales α_0 to α_1.*

(a) Let $r \in (\alpha_0, \alpha_1)$ and let $\nu' < \nu$. Then $\mathbf{X} + \mathsf{B}_r$ is ν'-porous on lines from scales $r/(\nu - \nu')$ to α_1.

(b) For any $s > 0$, the dilate $s \cdot \mathbf{X}$ is ν-porous on lines from scales $s\,\alpha_0$ to $s\,\alpha_1$.

(c) Let $\ell \subset \mathbb{R}^d$ be a line. Let $\mathbf{X}|_\ell = \mathbf{X} \cap \ell$, and view $\mathbf{X}|_\ell$ as a subset of $\mathbb{R}$. Then $\mathbf{X}|_\ell$ is ν-porous from scales α_0 to α_1.

Proof.

(a) Let τ be a line segment of length R with $r/(\nu - \nu') < R < \alpha_1$. Let $\mathbf{x} \in \tau$ be such that $\mathsf{B}_{\nu R}(\mathbf{x}) \cap \mathbf{X} = \emptyset$. Then $\mathsf{B}_{\nu' R}(\mathbf{x}) \cap (\mathbf{X} + \mathsf{B}_{(\nu-\nu')R}) = \emptyset$ as well. By the choice of R, $(\nu - \nu')R > r$ as needed.

(b) Let τ be a segment of length R with $s\,\alpha_0 < R < s\,\alpha_1$. There is some $\mathbf{x} \in s^{-1} \cdot \tau$ such that $\mathsf{B}_{s^{-1}\nu R}(\mathbf{x}) \cap \mathbf{X} = \emptyset$. Then $\mathsf{B}_{\nu R}(s\mathbf{x}) \cap (s \cdot \mathbf{X}) = \emptyset$.

(c) Let $\tau \subset \ell$ be a segment of length R. There is some $\mathbf{x} \in \tau$ such that $\mathsf{B}_{\nu R}(\mathbf{x}) \cap \mathbf{X} = \emptyset$. Then $(\mathsf{B}_{\nu R}(\mathbf{x}) \cap \ell) \cap \mathbf{X}|_\ell = \emptyset$.

□

Lemma 3.12. *Let $\mathbf{X} \subset \mathbb{R}^d$ be ν-porous on balls from scales α_0 to α_1. Then there is some $C, \gamma > 0$ depending only on ν and d such that for any ball B of radius $R \in (\alpha_0, \alpha_1)$,*

$$|\mathbf{X} \cap \mathsf{B}| \le C R^d \left(\frac{\alpha_0}{R}\right)^\gamma.$$

Proof. Let

$$V(R) = \max_{\mathsf{Q} \text{ is a cube of side length } R} |\mathbf{X} \cap \mathsf{Q}|.$$

Given a cube Q^R with side length R, we may split up Q^R into a union of K^d-many cubes with side length R/K,

$$\mathsf{Q}^R = \bigcup_j \mathsf{Q}_j^{R/K}.$$

By porosity, as long as $R \in (\alpha_0, \alpha_1)$, there is some $\mathbf{x} \in \mathsf{Q}^R$ such that $\mathsf{B}_{\nu R}(\mathbf{x}) \cap \mathbf{X} = \emptyset$. If $K > \nu\sqrt{d}$, then any cube $\mathsf{Q}_j^{R/K}$ containing $\mathbf{x}$ must be disjoint from $\mathbf{X}$. Thus

$$V(R) \leq (K^d - 1)V(R/K) \qquad \text{for all } R \in (\alpha_0, \alpha_1).$$

Iterating this inequality yields

$$V(R) \leq (K^d - 1)^m V(R/K^m).$$

Choose m to be the largest integer such that $R/K^m > \alpha_0$ to obtain the result. □

Remark. We can take

$$\gamma \geq c\frac{L^{-d}}{\log L} = c_d \frac{\nu^d}{|\log \nu|}. \tag{3.17}$$

By combining Lemma 3.12 and Lemma 3.11(c), we find that line porous sets have small intersections with lines.

Corollary 3.13. *Let $\mathbf{Y} \subset \mathbb{R}^d$ be ν-porous on lines from scales α_0 to α_1. Let τ be a line segment of length $\alpha_0 < R < \alpha_1$. Then there is some $C, \gamma > 0$ depending only on ν such that*

$$|\tau \cap \mathbf{Y}| \leq CR\left(\frac{\alpha_0}{R}\right)^\gamma.$$

Here $|\bullet|$ is the one-dimensional Lebesgue measure on τ.

Proof. Let τ lie on the line ℓ. By Lemma 3.11(c), $\mathbf{Y}|_\ell$ is ν-porous. By Lemma 3.12 in $d = 1$ we obtain the result. □

Remark. We can take $\gamma \geq c\frac{\nu}{|\log \nu|}$.

3.5.2 Proof of Proposition 3.5

We now construct weight functions adapted to line porous sets, and use the higher-dimensional Beurling and Malliavin theorem (Theorem 2.5) to prove Proposition 3.5. Let $\mathbf{Y}$ be ν-porous on lines from scales μ to h^{-1}. Let $\alpha \in (0, 1)$ be the damping function parameter to be chosen later.

Consider the sequence of dyadic annuli

$$A_k = \{\mathbf{x} \in \mathbb{R}^d : 2^k \leq |\mathbf{x}| \leq 2^{k+1}\}, \quad \text{for } k \geq 1.$$

Define

$$W_k = \frac{2^k}{k^s} \tag{3.18}$$

where $s \in (0,1)$ is a parameter to be chosen later (we will eventually choose $s = 0.2$).

Next, we cover each annulus A_k with a family of finitely overlapping cubes. Specifically, let $\mathcal{Q}_k = \{\mathsf{Q}\}$ be a collection of finitely overlapping cubes, each with side length W_k, satisfying

$$A_k \subset \bigcup_{\mathsf{Q} \in \mathcal{Q}_k} \frac{1}{2}\mathsf{Q},$$

where $\frac{1}{2}\mathsf{Q}$ denotes the cube sharing the same center as Q but having half the side length. Additionally, we impose the condition that the union of these cubes lies within a slightly thicker annulus, explicitly:

$$\bigcup_{\mathsf{Q} \in \mathcal{Q}_k} \mathsf{Q} \subset \{\mathbf{x} \in \mathbb{R}^d : 2^{k-1} \leq |\mathbf{x}| \leq 2^{k+2}\}.$$

For each cube $\mathsf{Q} \in \mathcal{Q}_k$, let η_{Q} be a bump function supported on Q and taking the value 1 on $\frac{1}{2}\mathsf{Q}$. We construct η_{Q} by scaling and translating a fixed bump function of width 1, which leads to the derivative estimates

$$\|D^a \eta_{\mathsf{Q}}\|_\infty \lesssim_{d,a} W_k^{-a} \qquad \text{for all integers } a \geq 0. \tag{3.19}$$

The resulting family of bump functions covers A_k in the sense that

$$\sum_{\mathsf{Q} \in \mathcal{Q}_k} \eta_{\mathsf{Q}}(\mathbf{x}) \in [1, C], \quad \text{for all } \mathbf{x} \in A_k$$

for some universal constant C. Let $\mathcal{S}_{\mathbf{Y},k}$ be the cubes in this family that intersect $\mathbf{Y} \cap A_k$, and let $\mathbf{Y}_k$ be the union of these:

$$\begin{aligned} \mathcal{S}_{\mathbf{Y},k} &= \{\mathsf{Q} \in \mathcal{Q}_k : \mathsf{Q} \cap (\mathbf{Y} \cap A_k) \text{ is nonempty}\}, \\ \mathbf{Y}_k &= \bigcup_{\mathsf{Q} \in \mathcal{S}_{\mathbf{Y},k}} \mathsf{Q}. \end{aligned}$$

Set

$$\omega_k = -\frac{2^k}{k^\alpha} \sum_{\mathsf{Q} \in \mathcal{S}_{\mathbf{Y},k}} \eta_{\mathsf{Q}}. \tag{3.20}$$

Notice that $\operatorname{supp} \omega_k \subset \{\mathbf{x} \in \mathbb{R}^d : 2^{k-1} \leq |\mathbf{x}| \leq 2^{k+2}\}$, and that

$$\omega_k(\mathbf{x}) \leq -\frac{2^k}{k^\alpha} \quad \text{for } \mathbf{x} \in \mathbf{Y} \cap A_k. \tag{3.21}$$

The difference from Bourgain and Dyatlov's construction is that they take $\alpha = s$, and we allow for α to be much closer to 1. Let $k_0 \geq 2$ be the smallest integer such that $W_{k_0} > \mu$ (this choice will be clear when we discuss the growth condition). Set

$$\omega = 20 \sum_{k \geq k_0} \omega_k, \tag{3.22}$$

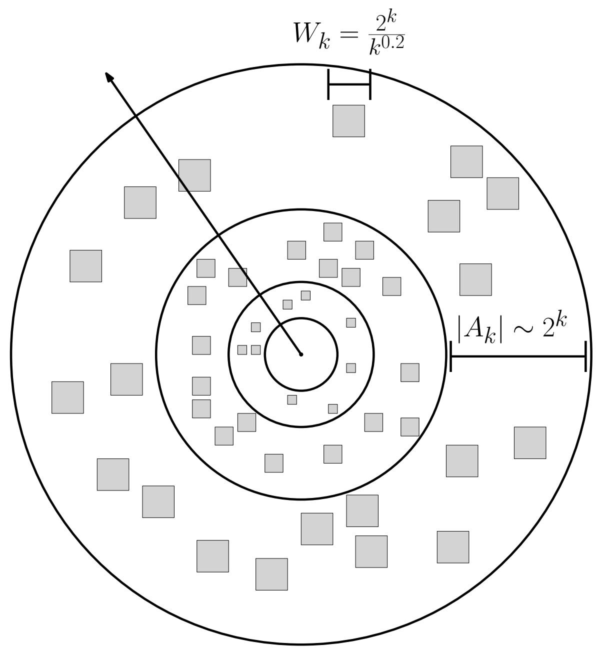

Figure 3.1: Within each dyadic annulus the weight is a sum of bump functions on boxes.

we add the factor of 20 in order to make (3.23) below true. Notice that $\omega(\mathbf{x}) = 0$ for $|\mathbf{x}| \leq 2$. See Figure 3.1 for an image representing the weight. By (3.21),

$$\omega(\mathbf{x}) \leq -\frac{|\mathbf{x}|}{(\log(2+|\mathbf{x}|))^\alpha} \quad \text{for } |\mathbf{x}| > 2^{k_0} \text{ and } \mathbf{x} \in \mathbf{Y}, \tag{3.23}$$

so

$$\omega(\mathbf{x}) \leq -\frac{1}{20}\frac{|\mathbf{x}|}{(\log(2+|\mathbf{x}|))^\alpha} + C(\mu) \quad \text{for all } \mathbf{x} \in \mathbf{Y}. \tag{3.24}$$

Now we establish some regularity. For any $a \geq 0$, $k \geq 1$, we have

$$|D^a \omega_k| \lesssim_{a,d} W_k^{-a} 2^k k^{-\alpha} \sum_{\mathsf{Q} \in \mathcal{S}_{\mathbf{Y},k}} 1_{\mathsf{Q}} \lesssim 2^{(1-a)k} k^{as-\alpha} 1_{\mathbf{Y}_k} \tag{3.25}$$

where we use (3.19) for the first inequality and finite overlapping of the cubes in $\mathcal{Q}$ for the second inequality. As long as $3s < \alpha$, ω satisfies the regularity condition (2.9) with a constant C_{reg} that depends only on the dimension.

Next we discuss the growth condition of Theorem 2.5. We have

$$\mathbf{Y}_k \subset (\mathbf{Y} \cap A_k) + \mathsf{B}_{2W_k\sqrt{d}}. \tag{3.26}$$

Because $\mathbf{Y} \subset [-3h^{-1}, 3h^{-1}]^d$, $\mathbf{Y}_k$ is empty if $2^k > 3h^{-1}\sqrt{d}$ (this is the only place we use that $\mathbf{Y} \subset [-3h^{-1}, 3h^{-1}]^d$). Increasing k_0 if necessary by a value that only depends on d, we may assume $2W_k\sqrt{d} < h^{-1}$. If $k \geq k_0$ then $\mu < 2W_k\sqrt{d} < h^{-1}$ and by Lemma 3.11(a), $\mathbf{Y}_k$ is $\nu/2$-porous on lines from scales $4W_k\sqrt{d}/\nu$ to h^{-1} (this is a vacuous statement if $4W_k\sqrt{d}/\nu > h^{-1}$).

Let ℓ be any line. If $4W_k\sqrt{d}/\nu > 2^k/\sqrt{d}$ then $k^s < 4d/\nu$ and

$$|\mathbf{Y}_k \cap \ell| \lesssim 2^k \lesssim_{\nu,d} 2^k k^{-s}.$$

Here $|\bullet|$ is the one-dimensional Lebesgue measure on ℓ. Otherwise, we can split up $Y_k \cap \ell = \bigcup_j Y_k \cap \tau_j$ where each τ_j is a line segment on ℓ of length $2^k/\sqrt{d}$, and there are $\lesssim \sqrt{d}$-many line segments in the union. Applying Corollary 3.13 to each line segment and summing,

$$|\mathbf{Y}_k \cap \ell| \lesssim_{\nu,d} 2^k\, k^{-s\gamma} \qquad \text{for all lines } \ell \tag{3.27}$$

for some $\gamma = \gamma(\nu) > 0$. Thus if $\ell = \{t\hat{\mathbf{y}} : t \in \mathbb{R}\}$ is a line through the origin, we see

$$2^{-k} \int_0^\infty |\omega_k(t\hat{\mathbf{y}})|\, dt \lesssim k^{-\alpha}|\mathbf{Y}_k \cap \ell| \lesssim 2^k\, k^{-(\alpha+s\gamma)}. \tag{3.28}$$

Let $G^*(r)$ be the growth function defined in (2.8). Let $r \in [2^k, 2^{k+1})$. We have the pointwise bound

$$\begin{aligned} G^*(r) &\lesssim \sup_{\hat{\mathbf{y}} \in \mathbb{S}^{d-1}} 2^{-k} \int_{2^{k-1}}^{2^{k+2}} |\omega(t\hat{\mathbf{y}})|\, dt \\ &\lesssim \sup_{\hat{\mathbf{y}} \in \mathbb{S}^{d-1}} 2^{-k} \sum_{k-3 \leq j \leq k+3} \int_0^\infty |\omega_j(t\hat{\mathbf{y}})|\, dt \\ &\lesssim \frac{r}{(\log(2+r))^{\alpha+s\gamma}}. \end{aligned} \tag{3.29}$$

As long as $\alpha + s\gamma > 1$, the growth condition (2.10) is satisfied with a constant that depends on $\alpha + s\gamma$, ν, and d. We may choose $s = 0.2$ universally and $\alpha > 1 - 0.1\gamma(\nu)$. Then $-\alpha + 3s < -0.3$ and $\alpha + s\gamma > 1 + 0.1\gamma$.

The weight ω satisfies (2.9) and (2.10) with constants C_{reg} and C_{gr} that depend only on ν and d. In order to construct a function with Fourier support in B_{c_1}, we apply Theorem 2.5 to the rescaled weight $c_3\omega$. If c_3 is chosen small enough in terms of c_1, d, ν, we get a function $f \in L^2(\mathbb{R}^d)$ satisfying

$$\begin{aligned} &\operatorname{supp} \hat{f} \subset \mathsf{B}_{c_1}, \\ &|f(0)| \geq c(c_1, d, \nu) \\ &|f(\mathbf{x})| \leq C(\mu) \exp\Big(-c_3 \frac{|\mathbf{x}|}{(\log(2+|\mathbf{x}|))^{\alpha}}\Big) && \text{for } \mathbf{x} \in \mathbf{Y}, \\ &|f(\mathbf{x})| \leq \exp\Big(-\frac{c_1}{C_d}\langle \mathbf{x}\rangle^{1/2}\Big) && \text{for } \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

The last bound implies $f(\mathbf{x})$ decays faster than any polynomial, so after multiplying through by a constant depending only on c_1 and d we may assume

$$|f(\mathbf{x})| \leq \langle \mathbf{x} \rangle^{-d}$$

and we may also assume the third equation holds with no constant $C(\mu)$. As f is band-limited to a ball of radius ≤ 1, the first derivative of f is bounded by $C\|\hat{f}\|_1 \leq C\|f\|_2 \leq C(d, c_1)$. Thus

$$|f(\mathbf{x})| \geq c_2 \qquad \text{for all } \mathbf{x} \in \mathsf{B}_{c_2}$$

for some $c_2 = c_2(c_1, d, \nu)$. Finally, setting $\psi = f^\vee$ gives the desired damping function.

Remark. We may take

$$\alpha = 1 - 0.1\gamma(\nu) = 1 - c\frac{\nu}{|\log \nu|} \tag{3.30}$$

for some absolute $c > 0$.

References

[1] Nalini Anantharaman, *Entropy and the localization of eigenfunctions*, Ann. of Math. (2) **168** (2008), no. 2, 435–475. MR2434883

[2] Nalini Anantharaman and Stéphane Nonnenmacher, *Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold*, 2007, pp. 2465–2523. Festival Yves Colin de Verdière. MR2394549

[3] Aldo Andreotti and Edoardo Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 81–130. MR175148

[4] M. E. Ash, *The basic estimate of the $\bar{\partial}$-Neumann problem in the non-Kählerian case*, Amer. J. Math. **86** (1964), 247–254. MR168916

[5] Jayadev Athreya, Semyon Dyatlov, and Nicholas Miller, *Semiclassical measures for complex hyperbolic quotients* (2024), Preprint, available at arXiv:2402.06477.

[6] Aidan Backus, James Leng, and Zhongkai Tao, *The fractal uncertainty principle via dolgopyat's method in higher dimensions* (2023), Preprint, available at arXiv:2302.11708.

[7] Bo Berndtsson, *An introduction to things $\overline{\partial}$*, Analytic and algebraic geometry, 2010, pp. 7–76. MR2743815

[8] Arne Beurling and Paul Malliavin, *On Fourier transforms of measures with compact support*, Acta Math. **107** (1962), 291–309. MR147848

[9] David Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*, Second, Progress in Mathematics, vol. 318, Birkhäuser/Springer, [Cham], 2016. MR3497464

[10] Jean Bourgain, *Beurling–Malliavin from Hörmander*, 2017. Private communication.

[11] Jean Bourgain and Semyon Dyatlov, *Spectral gaps without the pressure condition*, Ann. of Math. (2) **187** (2018), no. 3, 825–867. MR3779959

[12] Vasilis Chousionis, *Directed porosity on conformal iterated function systems and weak convergence of singular integrals*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 1, 215–232. MR2489023

[13] Laura Cladek and Terence Tao, *Additive energy of regular measures in one and higher dimensions, and the fractal uncertainty principle*, Ars Inven. Anal. (2021), Paper No. 1, 38. MR4462469

[14] Alex Cohen, *Fractal uncertainty in higher dimensions*, Annals of Mathematics (2024). To appear.

[15] ———, *Fractal uncertainty for discrete two-dimensional Cantor sets*, Anal. PDE **18** (2025), no. 3, 743–772. MR4874193

[16] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Bony-Sjöstrand-Meyer seminar, 1984–1985, 1985, pp. Exp. No. 13, 8. MR819779

[17] Semyon Dyatlov, *Control of eigenfunctions on hyperbolic surfaces: an application of fractal uncertainty principle*, Journées équations aux dérivées partielles (2017), 1–14.

[18] ———, *Fractal uncertainty in higher dimensions: notes on Cohen's paper* (2023), Preprint, available at arXiv:2305.17857.

[19] Semyon Dyatlov and Malo Jézéquel, *Semiclassical measures for higher-dimensional quantum cat maps*, Ann. Henri Poincaré **25** (2024), no. 2, 1545–1605. MR4703425

[20] Semyon Dyatlov and Long Jin, *Dolgopyat's method and the fractal uncertainty principle*, Anal. PDE **11** (2018), no. 6, 1457–1485. MR3803716

[21] ———, *Semiclassical measures on hyperbolic surfaces have full support*, Acta Math. **220** (2018), no. 2, 297–339. MR3849286

[22] Semyon Dyatlov, Long Jin, and Stéphane Nonnenmacher, *Control of eigenfunctions on surfaces of variable curvature*, J. Amer. Math. Soc. **35** (2022), no. 2, 361–465. MR4374954

[23] Semyon Dyatlov and Joshua Zahl, *Spectral gaps, additive energy, and a fractal uncertainty principle*, Geom. Funct. Anal. **26** (2016), no. 4, 1011–1094. MR3558305

[24] Semyon Dyatlov and Maciej Zworski, *Fractal uncertainty for transfer operators*, Int. Math. Res. Not. IMRN **3** (2020), 781–812. MR4073195

[25] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original. MR1288523

[26] Larry Guth, *Large value estimates in number theory, harmonic analysis, and computer science* (2025), Preprint, available at arXiv:2503.07410.

[27] Rui Han and Wilhelm Schlag, *A higher-dimensional Bourgain-Dyatlov fractal uncertainty principle*, Anal. PDE **13** (2020), no. 3, 813–863. MR4085124

[28] Lars Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), 89–152. MR179443

[29] Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. MR1996773

[30] Lars Hörmander, *Notions of convexity*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. Reprint of the 1994 edition [of MR1301332]. MR2311920

[31] Benjamin Jaye and Mishko Mitkovski, *Quantitative uniqueness properties for L^2 functions with fast decaying, or sparsely supported, Fourier transform*, Int. Math. Res. Not. IMRN **16** (2022), 12148–12179. MR4465998

[32] Long Jin, *Control for Schrödinger equation on hyperbolic surfaces*, Math. Res. Lett. **25** (2018), no. 6, 1865–1877. MR3934848

[33] ———, *Damped wave equations on compact hyperbolic surfaces*, Comm. Math. Phys. **373** (2020), no. 3, 771–794. MR4061399

[34] Elena Kim, *Characterizing the support of semiclassical measures for higher-dimensional cat maps* (2024), Preprint, available at arXiv:2410.13449. with an appendix by Theresa C. Anderson and Robert J. Lemke Oliver.

[35] Elena Kim and Nicholas Miller, *Semiclassical measures on hyperbolic manifolds* (2025), Preprint, available at arXiv:2503.01528.

[36] K. Kodaira, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1268–1273. MR66693

[37] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds. I*, Ann. of Math. (2) **78** (1963), 112–148. MR153030

[38] Paul Koosis, *Leçons sur le théorème de Beurling et Malliavin*, Université de Montréal, Les Publications CRM, Montreal, QC, 1996. MR1430571

[39] ______, *The logarithmic integral. I*, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, 1998. Corrected reprint of the 1988 original. MR1670244

[40] Elon Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2) **163** (2006), no. 1, 165–219. MR2195133

[41] J. Mashregi, F. L. Nazarov, and V. P. Havin, *The Beurling-Malliavin multiplier theorem: the seventh proof*, St. Petersburg Math. J. **17** (2006), no. 5, 699–744. MR2241422

[42] Charles B. Morrey Jr., *The analytic embedding of abstract real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 159–201. MR99060

[43] Raymond E. A. C. Paley and Norbert Wiener, *Fourier transforms in the complex domain*, American Mathematical Society Colloquium Publications, vol. 19, American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original. MR1451142

[44] Pablo Shmerkin, *Inverse theorems for discretized sums and L^q norms of convolutions in* $\mathbb{R}^d$ (2024), Preprint, available at arXiv:2308.09846.

[45] Alexander Shnirelman, *Ergodic properties of eigenfunctions*, Uspehi Mat. Nauk **29** (1974), no. 6(180), 181–182. MR402834

[46] Steven Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55** (1987), no. 4, 919–941. MR916129