DISCRETE AND CONTINUOUS PROJECTIONS IN $\mathbb{R}^2$

ALEX COHEN

ABSTRACT. These are lecture notes for a colloquium about projection theory given at Stony Brook University on April 2, 2026. These notes cover more than the lecture.

1. Discrete projection theory

Theorem 1.1 (Szemerédi–Trotter [17]). *Let $\mathcal{L}$ be a finite set of lines in $\mathbb{R}^2$. For $r \geq 2$, let $\mathcal{P}_r$ be the set of points with $\geq r$ lines through them. These are called the r-rich points. We have*

$$|\mathcal{P}_r| \lesssim \frac{|\mathcal{L}|^2}{r^3} + \frac{|\mathcal{L}|}{r} \qquad \text{for all } r \geq 2.$$

Here, $A \lesssim B$ means $A \leq CB$ for some absolute constant C.

This theorem has many applications. We mention just a few.

- Szemerédi–Trotter-Spencer [16] adapted the proof of Theorem 1.1 to apply to circles. They used this to prove better bounds for the unit distance problem: If $\mathcal{P}$ is a finite set in the plane, then the number of pairs of points in $\mathcal{P}$ at distance 1 is bounded by $C|\mathcal{P}|^{4/3}$.
- Elekes [3] used Theorem 1.1 to prove the following. For any finite set $A \subset \mathbb{R}$,

  $$\max\{|A+A|, |A \cdot A|\} \gtrsim \frac{|A|^{4/3}}{\log |A|}.$$

  Here, $A + A = \{a + a' : a, a' \in A\}$ is the sumset of A, and similarly, $A \cdot A$ is the product set. For a long time, this was the best bound for the sum-product problem. The current best bounds are bit better.
- Herr–Kwak [8] used the Szemeredi–Trotter theorem to prove sharp Strichartz estimates for the Schrödinger equation on $\mathbb{T}^2$. They used this to prove global well posedness for the cubic nonlinear Schrödinger equation on $\mathbb{T}^2$ in any H^s space with $s > 0$, under minimal assumptions on the L^2 norm of the initial data.

Theorem 1.1 has been generalized in several directions. Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl [2] showed the theorem extends to pseudolines—that is, a family of plane curves with the property that any two curves meet in at most one point. Székely [18] gave a simple proof using the crossing number inequality, which is a lower bound for the number of edge intersections in a planar graph. His proof works for pseudolines. Pach–Sharir [14] extended Székely's method to collections of curves that may intersect more than once, although the bound depends on the number of intersection points. These results show that Theorem 1.1 has a lot to do with the topology of Euclidean space.

Theorem 1.1 has an immediate Corollary about projections. For $\theta \in \mathbb{R}$, define the linear projection $\pi_\theta(x, y) = x + \theta y$.

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Corollary 1.2. *Let $\mathcal{P} \subset \mathbb{R}^2$ and $\Theta \subset \mathbb{R}$ be finite sets. Then*

$$\sup_{\theta \in \Theta} |\pi_\theta \mathcal{P}| \gtrsim \min\{|\mathcal{P}|,\, |\mathcal{P}|^{1/2}\, |\Theta|^{1/2}\}.$$

Proof. We take $\mathcal{L}$ to be the fibers of $\mathcal{P}$ over all directions in Θ,

$$\mathcal{L}_\theta = \{\pi_\theta^{-1}(x) \,:\, x \in \pi_\theta \mathcal{P}\}, \qquad \mathcal{L} = \bigcup_{\theta \in \Theta} \mathcal{L}_\theta.$$

The number of lines is bounded by $|\mathcal{L}| \leq (\sup_\theta |\pi_\theta \mathcal{P}|)\, |\Theta|$, and every point in $\mathcal{P}$ is $|\Theta|$-rich with lines. By Theorem 1.1,

$$|\mathcal{P}| \lesssim \frac{(\sup_{\theta \in \Theta} |\pi_\theta \mathcal{P}|)^2 |\Theta|^2}{|\Theta|^3} + \frac{(\sup_{\theta \in \Theta} |\pi_\theta \mathcal{P}|) |\Theta|}{|\Theta|},$$

and rearranging gives the result. □

Example 1.3. The following example is due to Elekes [4]. It is closely related to an example of Erdős [5].

Let $N_1, N_2 \geq 1$ be integers. Let

$$A = \frac{1}{N_1 N_2} \mathbb{Z} \cap [0,1), \qquad B = \frac{1}{N_1} \mathbb{Z} \cap [0,1), \qquad \mathcal{P} = A \times B$$

and

$$\Theta = \frac{1}{N_2} \mathbb{Z} \cap [0,1).$$

Then, $\pi_\theta(\mathcal{P}) = A + \theta B$. But $A + \theta B \subset \frac{1}{N_1 N_2} \mathbb{Z} \cap [0,2)$, so $|\pi_\theta(\mathcal{P})| \leq 2 N_1 N_2$ for all $\theta \in \Theta$. This agrees with the bound in Corollary 1.2.

There are two easy projection bounds that Corollary 1.2 should be compared to. I call the first one Cauchy–Schwarz(Lines) because it uses the fact that two distinct lines intersect in one point. I call the second Cauchy–Schwarz(Points), because it uses that two distinct points span one line.

Lemma 1.4 (Cauchy–Schwarz of Lines)**.** *If $|\Theta| \geq 2$, then $\sup_\theta |\pi_\theta \mathcal{P}| \geq |\mathcal{P}|^{1/2}$.*

Proof. If θ_1 and θ_2 are distinct directions, then $|\mathcal{P}| \leq |\pi_{\theta_1} \mathcal{P}|\, |\pi_{\theta_2} \mathcal{P}|$, because the location of a point is determined by its two projections. So, one of these two projections has size $\geq |\mathcal{P}|^{1/2}$. □

Lemma 1.5 (Cauchy–Schwarz of Points)**.** *For any finite $\mathcal{P} \subset \mathbb{R}^2$ and $\Theta \subset \mathbb{R}$, we have $\sup_{\theta \in \Theta} |\pi_\theta \mathcal{P}| \geq \frac{1}{2} \min\{|\Theta|, |\mathcal{P}|\}$.*

Proof. Let $\mathcal{L}$ be the line family from the proof of Corollary 1.2. An incidence is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ with $p \in \ell$. The number of incidences is $|\mathcal{P}|\, |\Theta|$, because each point has exactly $|\Theta|$ lines through it. The number of incidences is also equal to $\sum_{\ell \in \mathcal{L}} |\ell \cap \mathcal{P}|$. Applying Cauchy–Schwarz to this sum yields

$$|\mathcal{P}|\, |\Theta| \leq \big(\sum_{\ell \in \mathcal{L}} |\ell \cap \mathcal{P}|^2\big)^{1/2} |\mathcal{L}|^{1/2}.$$

Using that $|\mathcal{L}| \leq (\sup_{\theta \in \Theta} |\pi_\theta \mathcal{P}|) |\Theta|$, we can rearrange to get

$$\sup_{\theta \in \Theta} |\pi_\theta \mathcal{P}| \geq \frac{|\mathcal{P}|^2 |\Theta|}{\sum_{\ell \in \mathcal{L}} |\ell \cap \mathcal{P}|^2}.$$

We recognize the denominator as counting triples,

$$\sum_{\ell \in \mathcal{L}} |\ell \cap \mathcal{P}|^2 = \#\{(p_1, p_2, \ell) \in \mathcal{P} \times \mathcal{P} \times \mathcal{L} \,:\, p_1, p_2 \in \ell\}.$$

There are at most $|\mathcal{P}||\Theta|$ triples with $p_1 = p_2$. Since two distinct points span one line, there are at most $|\mathcal{P}|^2$ triples with $p_1 \neq p_2$. Thus, the denominator is bouded by $|\mathcal{P}||\Theta| + |\mathcal{P}|^2$, giving the result. $\square$

It is helpful to organize our easy estimates and our sharp estimate on a log-log plot.

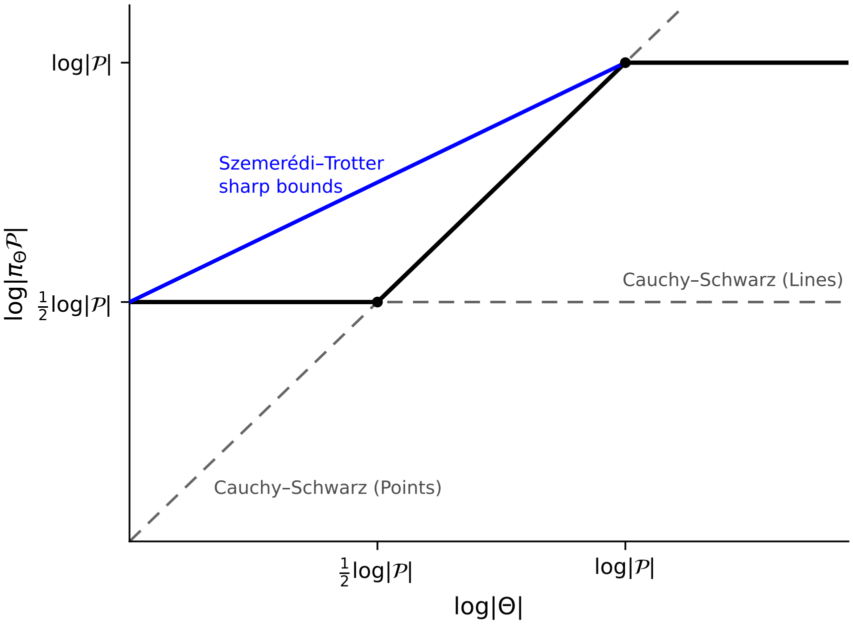

FIGURE 1. Log-log plot of projection bounds as $|\mathcal{P}|$ is fixed and $|\Theta|$ varies.

1.1. **Special case.** Let N be a perfect square. Split up the unit square into a grid of N squares with side length $N^{-1/2}$, and place one point in each square. A set of points $\mathcal{P}$ constructed this way is called *well spaced*. For example, $\mathcal{P} = N^{-1/2}\mathbb{Z} \cap [0,1)^2$ is a well-spaced set of points. In fact, Corollary 1.2 is sharp for this set of points if we choose Θ to be rationals of low height, so this is an important special case.

Let $w \in [N^{-1/2}, 1]$ be an intermediate scale. We can split up $[0,1]^2$ into a grid $\mathcal{Q}$ of $w \times w$ squares. Each square $Q \in \mathcal{Q}$ contains $w^2 N$ points of $\mathcal{P}$.

We make a hypothesis that for each $Q \in \mathcal{Q}$, and each $\theta \in \Theta$, $|\pi_\theta(\mathcal{P} \cap Q)|$ is roughly the same. After some technical considerations, this hypothesis can be justified by pigeonholing.

Using this hypothesis, we can estimate $|\pi_\theta \mathcal{P}|$ in two steps. Let $|\pi_\theta \mathcal{P}|_w$ be the w-covering number, that is, the minimal number of w-intervals needed to cover $\pi_\theta \mathcal{P}$. Because every $w \times w$ square is active in covering $\mathcal{P}$, it is easy to check $|\pi_\theta \mathcal{P}|_w \sim w^{-1}$ up to constants.

Next, for each $x \in \pi_\theta \mathcal{P}$, choose some point $p \in \mathcal{P}$ mapping to x. Let $Q \in \mathcal{Q}$ be the $w \times w$ grid square containing p. By just considering the points in Q, we can lower bound

$$|\pi_\theta \mathcal{P} \cap [x - 2w, x + 2w]| \geq |\pi_\theta(\mathcal{P} \cap Q)|. \tag{1.1}$$

Putting these two pieces together, we find

$$|\pi_\theta \mathcal{P}| \gtrsim |\pi_\theta \mathcal{P}|_w \inf_{x \in \pi_\theta \mathcal{P}} |\pi_\theta \mathcal{P} \cap [x - 2w, x + 2w]| \gtrsim w^{-1} |\pi_\theta(\mathcal{P} \cap Q)|. \tag{1.2}$$

Our pigeonholing hypothesis tells us $|\pi_\theta(\mathcal{P}\cap Q)|$ is roughly constant as θ and Q vary.

Assume $|\Theta| \le |\mathcal{P}|$. Choose $w = \frac{|\Theta|^{1/2}}{|\mathcal{P}|^{1/2}} \in [N^{-1/2}, 1]$. For this choice, $|\mathcal{P}\cap Q| = w^2|\mathcal{P}| = |\Theta|$, so Cauchy–Schwarz(Points) implies $|\pi_\theta(\mathcal{P}\cap Q)| \gtrsim |\Theta|$. By (1.2), $|\pi_\theta\mathcal{P}| \gtrsim |\mathcal{P}|^{1/2}|\Theta|^{1/2}$ as desired.

Recall Figure 1: our goal was a bound at the midpoint of the blue line. By zooming in, we reduced this to a sharp projection estimate at the right endpoint, which follows from Cauchy–Schwarz(Points).

I like to think of the Szemerédi–Trotter theorem as interpolating the blue line from its two easy endpoints. This interpolation crucially uses the topology of $\mathbb{R}^2$. Over finite fields, one can study incidences between points and lines, and the two easy bounds are true with the same proof, but the Szemeredi–Trotter theorem fails.

Szemeredi–Trotter's proof of Theorem 1.1 follows a similar strategy to this special case. They carefully chop $\mathbb{R}^2$ into axis-aligned squares, called cells, and apply the Cauchy–Schwarz(Points) bound inside each square. The tricky part is to simultaneously control the number of points in each square, and the number of lines entering each square. Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl [2] introduced a simpler cell decomposition method, using the lines of $\mathcal{L}$ to chop up $\mathbb{R}^2$. See also Tao's proof via the same method [19]. Yet another cell decomposition proof, due to Katz and reported by Tao in [20], uses the polynomial method to partition $\mathbb{R}^2$.

2. CONTINUOUS PROJECTION THEORY

The following Theorem is a continuum version of Corollary 1.2. We use $\dim_H$ to denote the Hausdorff dimension of a Borel set.

Theorem 2.1 (Ren–Wang [15], Orponen–Shmerkin [11])**.** *Let* $\mathcal{P}\subset[0,1]^2$ *and* $\Theta\subset[0,1]$ *be Borel sets. Then*

$$\sup_{\theta\in\Theta}\dim_H \pi_\theta\mathcal{P} \ge \min\{1,\ \dim_H\mathcal{P},\ \frac{\dim_H\mathcal{P}+\dim_H\Theta}{2}\}.$$

Remark 1. The names in the theorem statement reflect the last two contributions in a long line of work. In 2023, Orponen and Shmerkin proved an important special case, the sticky case. Later that year, Ren and Wang identified and solved a second special case, the semi-well-spaced case, and combined this with the sticky case to give the full theorem. The conjecture is due to Oberlin [10], and sharpness of the bound follows from a construction analogous to Example 1.3, due to Wolff [22]. We note that Theorem 2.1 follows from the stronger Furstenberg set conjecture, proved in [15]. See Section 2.2 for more discussion of this story.

Remark 2. The Furstenberg set conjecture has an interesting history. Furstenberg raised a question about continuum incidences in unpublished work, motivated by his ×2×3 conjecture in ergodic theory. In the context of projections, his original conjecture strengthens the $\dim_H\mathcal{P}=1$ case of Theorem 2.1. Wolff [22] refined the conjecture to apply to all dimensions of $\mathcal{P}$, and popularized it in the context of restriction theory and harmonic analysis. The connections between continuum incidences, dynamical systems, and harmonic analysis are very active today.

The two easy bounds from the last section have continuum analogues. First, if $\theta_1\ne\theta_2$, then $\dim_H\mathcal{P}\le\dim_H\pi_{\theta_1}\mathcal{P}+\dim_H\pi_{\theta_2}\mathcal{P}$, giving the Cauchy–Schwarz(Lines) bound

$$\sup_{\theta\in\Theta}\dim_H\pi_\theta\mathcal{P}\ge\frac{1}{2}\dim_H\mathcal{P}\qquad\text{if }\Theta\text{ is not a singleton.}\tag{2.1}$$

Second, Kaufman [9] proved the Cauchy–Schwarz(Points) bound

$$\sup_{\theta\in\Theta}\dim_H\pi_\theta\mathcal{P}\ge\min\{\dim_H\mathcal{P},\dim_H\Theta\}.\tag{2.2}$$

Kaufman's result covers the case of Theorem 2.1 when $\dim_H \Theta \geq \dim_H \mathcal{P}$. Remember that in the discrete setting, we split up the count $\#\{(p_1, p_2, \ell) : p_1, p_2 \in \ell\}$ into two pieces, depending on whether or not $p_1 = p_2$. In the continuous setting, one must split into dyadic pieces, depending on how far p_1 is from p_2.

Fu–Ren [6] proved

$$\sup_{\theta\in\Theta} \dim_H \pi_\theta \mathcal{P} \geq \min\{1, \dim_H \mathcal{P} + \dim_H \Theta - 1\}. \tag{2.3}$$

One can think of this bound as refining (2.2) when $\dim \mathcal{P} \geq 1$. Their bound covers the case of Theorem 2.1 where $\dim_H \mathcal{P} + \dim_H \Theta \geq 2$, i.e., where the right hand side is 1.

2.1. **Why Hausdorff dimension?** In the 1980s, Mattilla, Kaufman, Falconer, and other researchers in fractal geometry studied how the Hausdorff dimension behaves under projection. Furstenberg and later Wolff got interested in this question because of applications to dynamics and harmonic analysis—conjectures that don't have Hausdorff dimension in the statement. Why does Hausdorff dimension come up naturally in dynamics and harmonic analysis?

In analysis, it is realistic to assume we can only measure positions up to some fixed measurement error $\delta > 0$. The cardinality of a finite set in $\mathbb{R}^2$ or $\mathbb{R}$ is no longer a measurable quantity. What we can measure is the δ-covering number,

$$|\mathcal{P}|_\delta = \text{The minimal number of } \delta\text{-balls needed to cover } \mathcal{P}.$$

It is reasonable to ask,

How can we estimate $|\sup_{\theta\in\Theta} \mathcal{P}|_\delta$ in terms of $|\mathcal{P}|_\delta$ and $|\Theta|_\delta$?

Do the Szemeredi–Trotter bounds in Corollary 1.2 hold if we replace cardinality with δ-covering number?

They do not. To see why, let's consider two examples of point sets with the same δ-covering number.

Ex. 1. $\mathcal{P}$ consists of N δ-separated points packed tightly into a square of width $N^{1/2}\delta$.
Ex. 2. $\mathcal{P}$ consists of N well spaced points in the unit square (one in each $N^{-1/2} \times N^{-1/2}$ square).

In the first example, $|\pi_\theta \mathcal{P}|_\delta \approx N^{1/2}$ for all $\theta \in [0, 1]$. But for Example 2, Guth–Solomon–Wang [7] proved that the analogue of Corollary 1.2 does hold,

$$\sup_{\theta\in\Theta} |\pi_\theta \mathcal{P}|_\delta \gtrapprox \min\{\delta^{-1}, |\mathcal{P}|_\delta, |\mathcal{P}|_\delta^{1/2} |\Theta|_\delta^{1/2}\}.$$

Here, $A \lessapprox B$ means $A \leq C_\varepsilon N^\varepsilon B$ for any $\varepsilon > 0$.

The two point sets have the same size when measured by their δ-covering number, but not other covering numbers. Letting $w = N^{1/2}\delta$, in Example 1, $|\mathcal{P}|_w = 1$. This forces the w-covering number of the projection to be 1 as well, which constraints the size of the projection. For this problem, the size of $\mathcal{P}$ at all intermediate scales are relevant, not just scale δ.

Hausdorff dimension is closely related to covering numbers. If $\dim_H \mathcal{P} > t$, then $|\mathcal{P}|_w \gtrapprox w^{-t}$ for all $w > 0$. Example 2 satisfies such a covering number bound, but Example 1 does not. This is why Hausdorff dimension comes up naturally: The right way to measure the size of a set is to look at covering numbers at *all* scales, not just the bottom scale.

Theorem 2.1 follows from a discrete, quantitative version, that is stated using a discrete version of Hausdorff dimension.

2.2. **Ingredients in the proof of Theorem 2.1.** (Note: This section is hasty).

The proof of Theorem 2.1 relies on a sequence of 6 papers. See Figure 2 for the main results and logical dependencies between these papers, and see Figure 3 for log-log plots of various projection bounds from these works.

The first paper in this sequence was written by Bourgain [1] in the late 2000s. He proved an ε-improvement over the Cauchy–Schwarz(Lines) bound,

$$\sup_{\theta\in\Theta}\dim_H \pi_\theta\mathcal{P} \geq \frac{1}{2}\dim_H\mathcal{P} + \varepsilon(\dim_H\Theta), \qquad \varepsilon(\dim_H\Theta) > 0 \text{ for all } \dim_H\Theta > 0.$$

Bourgain's result fails if we work over the complex numbers rather than the real numbers. For example, let $\mathcal{P} = \mathbb{R}^2 \cap B(1) \subset \mathbb{C}^2$, and let $\Theta = \mathbb{R}\cap B(1) \subset \mathbb{C}$. For each $\theta \in \Theta$, $\pi_\theta\mathcal{P} \subset \mathbb{R}\cap B(2)$. Thus, $\dim \pi_\theta \mathcal{P} = \frac{1}{2}\dim\mathcal{P}$ for all $\theta\in\Theta$, so the Cauchy–Schwarz(Lines) bound is sharp, and Theorem 2.1 fails over the complex numbers. Bourgain's theorem is the crucial step that distinguishes the real and complex numbers.

Using Bourgain's theorem, Orponen and Shmerkin [12] proved an ε-improvement over the Cauchy–Schwarz(Points) bound. Together, these two results form the green line in Figure 3.

Subsequent papers proved stronger and stronger results by combining Bourgain's theorem, induction on scales, and the Plünnecke–Ruzsa inequality from combinatorics. Skipping a few steps, this culiminated in Orponen–Shmerkin proving the *sticky* case of Theorem 2.1: If $|\mathcal{P}|_w \approx w^{-\dim_H\mathcal{P}}$ for all $w > 0$, then the result holds.

In a separate line of work, Guth–Solomon–Wang [7] introduced the *high-low* method to prove Theorem 2.1 in the special case where $\mathcal{P}$ is well-spaced, in the sense of Section 1.1. One way to think about the high-low method is as a variant of the Cauchy–Schwarz bound. In the Cauchy–Schwarz bound, one needs to estimate $\|\sum_{T\in\mathbb{T}} 1_T\|^2_{L^2(\mathbb{R})}$, where $\mathbb{T}$ is the set of δ-tubes appearing as fibers over δ-balls. In the high-low method, one splits this L^2 norm into a high frequency part and a low frequency part. If the low frequency part dominates, use induction. If the high frequency part dominates, the bound is better than Cauchy–Schwarz. Fu–Ren built on the high-low method to prove (2.3), see the dotted line in the right graph of Figure 3. In particular, their estimate gives Theorem 2.1 when $\dim_H\mathcal{P} + \dim_H\Theta \geq 2$. Ren–Wang proved an even stronger incidence estimate using the high-low method, and combined their result with the sticky case to prove Theorem 2.1.

In both strands of work, induction on scales plays a crucial role. There are two kinds of induction on scales. In order to describe them, we need to describe the setup: Assume $|\mathcal{P}|_w \gtrsim w^{-t}$ for all $w\in[\delta,1]$; assume $\mathcal{P}$ is pigeonholed, so every $w\times w$ square active in covering $\mathcal{P}$ has roughly the same number of points; and assume $|\Theta|_w \gtrsim w^{-s}$ for all $w\in[\delta,1]$. These are discrete versions of saying, $\dim_H\mathcal{P}\geq t$ and $\dim_H\Theta\geq s$. The goal is to estimate $\sup_{\theta\in\Theta}|\pi_\theta\mathcal{P}|_\delta$.

The first multiscale decomposition is very much like the argument in Section 1.1. I call this the *submodular decomposition* because it is related to the submodularity inequality for functions on posets. Say we want to estimate $|\pi_\theta\mathcal{P}|_\delta$. We first estimate $|\pi_\theta\mathcal{P}|_w$ by applying induction to the thickened set of points in $\mathcal{P}$ and Θ. Then, we estimate $\frac{|\pi_\theta\mathcal{P}|_\delta}{|\pi_\theta\mathcal{P}|_w}$ by zooming into a w-square Q, and estimating $|\pi_\theta(\mathcal{P}\cap Q)|$. This decomposition works well in the sticky case, because $\mathcal{P}\cap Q$ is still t-dimensional. In the general case, $\mathcal{P}\cap Q$ might not be t-dimensional, so this decomposition might not work.

The second multiscale decomposition works no matter what $\mathcal{P}$ looks like. For this reason, I call it the *lossless decomposition.* To estimate $\frac{|\pi_\theta\mathcal{P}|_\delta}{|\pi_\theta\mathcal{P}|_w}$, consider all the δ-tubes going through a point of $\mathcal{P}$, in an angular range w. These tubes sweep out a $w\times 1$ tube, and intersect in a $\delta\times\frac{\delta}{w}$-tubelet. Every point in the big $w\times 1$ tube contributes a $\delta\times\frac{\delta}{w}$-tubelet. If we apply an anisotropic rescaling map that takes the $w \times 1$ tube to a unit square, the $\delta \times \frac{\delta}{w}$-tubelets map to $\frac{\delta}{w}$-squares. We apply induction to this family of $\frac{\delta}{w}$-squares. This appears challenging, because our hypotheses don't tell us anything about the $\delta \times \frac{\delta}{w}$-tubelets. Some of the work of Theorem 2.1 comes down to understanding these tubelets in different special cases. This is also a key part of Wang–Zahl's proof of the Kakeya conjecture.

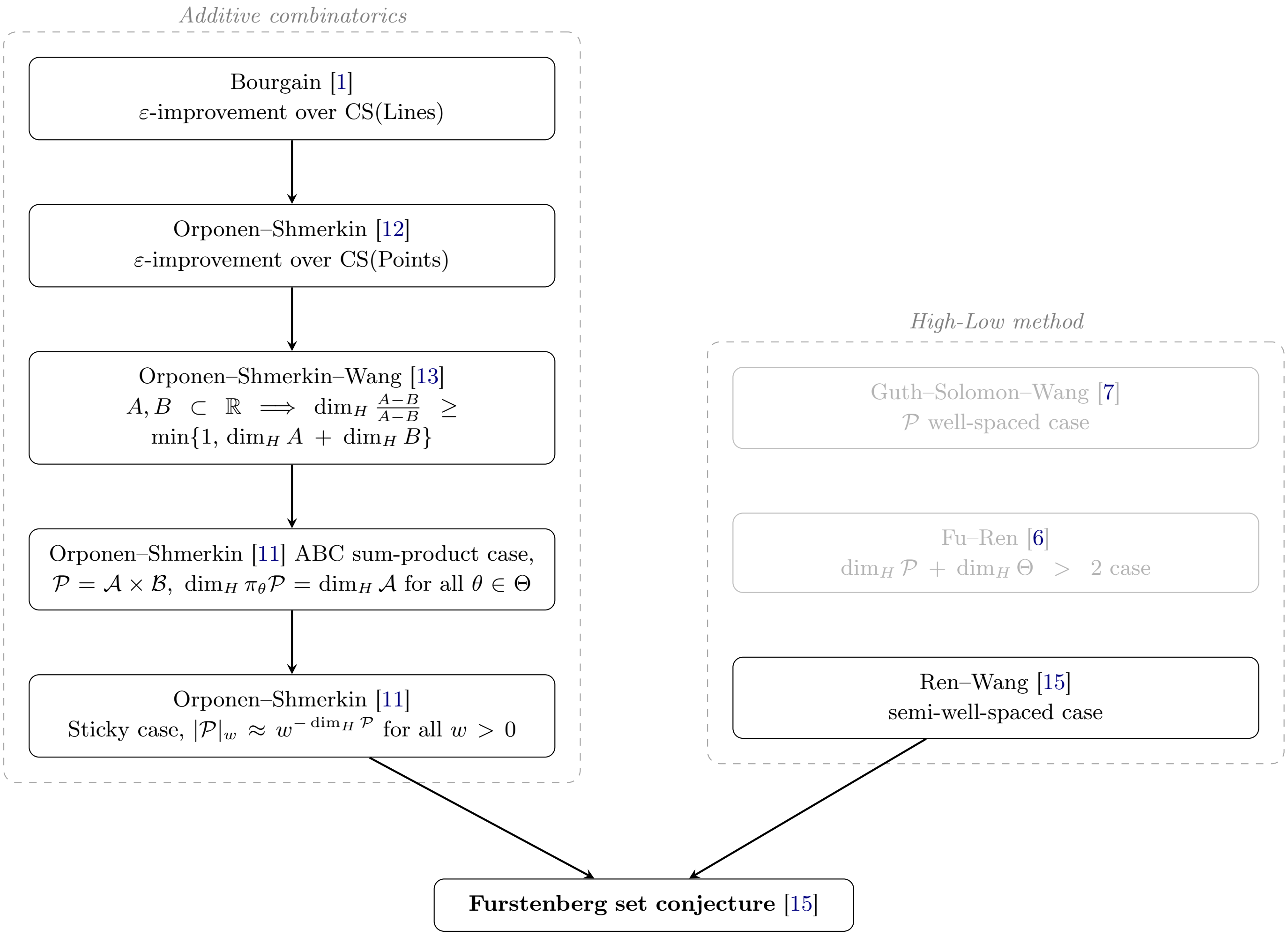

FIGURE 2. Steps in the proof of Theorem 2.1. One strand of work is based on combining Bourgain's projection theorem, additive combinatorics, and induction on scales. The other strand is based on the high-low method and induction on scales. Both strands come together to prove Theorem 2.1.

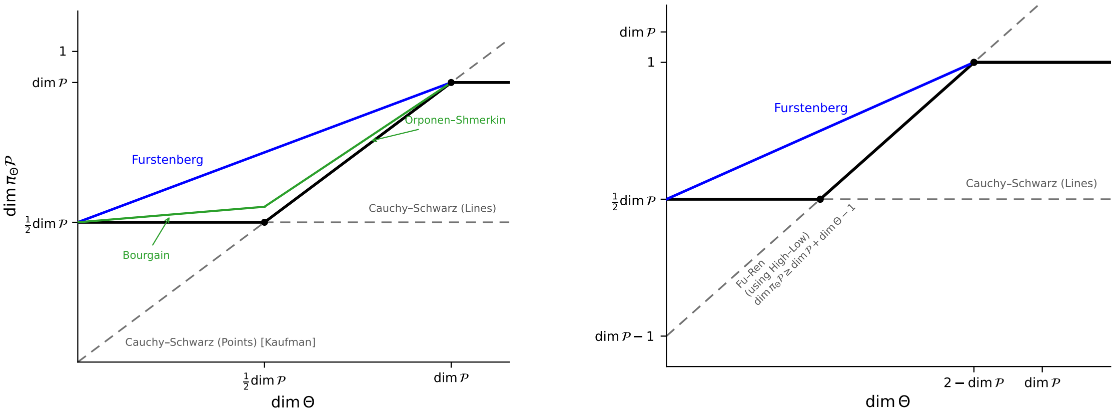

FIGURE 3. Both graphs show projection bounds with $\dim_H \mathcal{P}$ fixed and $\dim_H \Theta$ varying. **Left**: $\dim_H \mathcal{P} \leq 1$. The dimension of the projection tops out at $\dim_H \mathcal{P}$. The two reference bounds are Cauchy–Schwarz(Lines) and Cauchy–Schwarz(Points), with ε-improvements by Bourgain [1] and Orponen–Shmerkin [12]. The ε-improvements crucially distinguish $\mathbb{R}$ from $\mathbb{C}$. **Right**: $\dim_H \mathcal{P} > 1$. The dimension of the projection tops out at 1. Cauchy–Schwarz(Points) is replaced with Fu and Ren's bound [6] using the High–Low method.

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