PROOF OF THE KAKEYA IN $\mathbb{R}^3$ FOLLOWING GUTH–WANG–ZAHL

ALEX COHEN

Under construction. There are some technical issues in Section 5.

Wang and Zahl [5] recently proved the Kakeya conjecture in $\mathbb{R}^3$. We follow Guth–Wang–Zahl's [2] streamlined proof of the Kakeya conjecture, with a slightly different induction setup. We often take text straight from their writeup. Some notation / ideas are taken from Guth's overview [1]. The induction strategy follows the Kakeya lecture notes on my website: `https://cims.nyu.edu/~ac6074/KakeyaLec.pdf`. See Theorem 2.1 for the statement of Wang and Zahl's Kakeya set theorem.

The proof involves two multiscale decompositions. The first I call the submodular decomposition, because it is related to the following inequality. Let $X \subset \mathbb{R}^d$, let U, V be convex sets, and let $|X|_U, |X|_V$ be the minimal covering number by translates of these convex sets. Then

$$|X|_U |X|_V \gtrsim |X|_{U\cap V} |X|_{U+V}.$$

This inequality says that the logarithm of the covering number is submodular on the poset of convex sets. For us, this is the multiscale decomposition where we first look at the δ-tubes inside a ρ-tube, and then look at the collection of ρ-tubes. See Proposition 4.3.

The second I call the lossless decomposition. We study the $\delta \times \delta \times \rho$ tubelets inside a ρ-ball. There are two sub-problems: First, estimate the volume of the union of these tubelets, with a loss coming from the Frostman constant of these tubelets. The Frostman constant is related to the covering number of these tubelets by $a \times b \times \rho$ convex sets. The second step is to estimate the total number of $a \times b \times \rho$ convex sets. See Lemma 6.1.

Acknowledgements

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Outline

- In Section 1, we set up all the relevant definitions and notations, closely following [2].
- In Section 2 we define the intermediate statement $\mathbf{A}(\sigma)$ that we will induct on. This statement says that the main result, Theorem 2.1, holds with a loss factor δ^{σ}. This is a little different [2], because in their intermediate statement, the size of $\mathbb{T}$ plays a role.
- In Section 3 we restate the Cauchy–Schwarz results from [2] on unions of slabs.
- In Section 4 we start by restating the results from [2] on factoring of convex sets, and then prove results on unions of slabs which are a minor modification from [2]. The proofs are a little simpler because the statements are a little weaker.
- In Section 5 we state Wang–Zahl's Sitcky Kakeya theorem and use it to improve on $\mathbf{A}(\sigma)$ when $\mathbb{T}$ either fails to be Frostman or fails to be Katz–Tao. The arguments in this section are a little weaker and a little simpler than the corresponding results in [2]. The approach in this section is similar to that of [4] (thanks to Josh Zahl for pointing this out).

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- In Section 6 we analyze $\delta\times\delta\times\rho$ tubelets inside of a ρ-tube. We prove that if $\sigma>0$, then $\mathbf{A}(\sigma)$ implies $\mathbf{A}(\sigma-\kappa)$ for some $\kappa>0$. The main idea of this section is to estimate the Frostman constant of tubelets inside a ρ-ball by estimating unions of convex sets by induction.

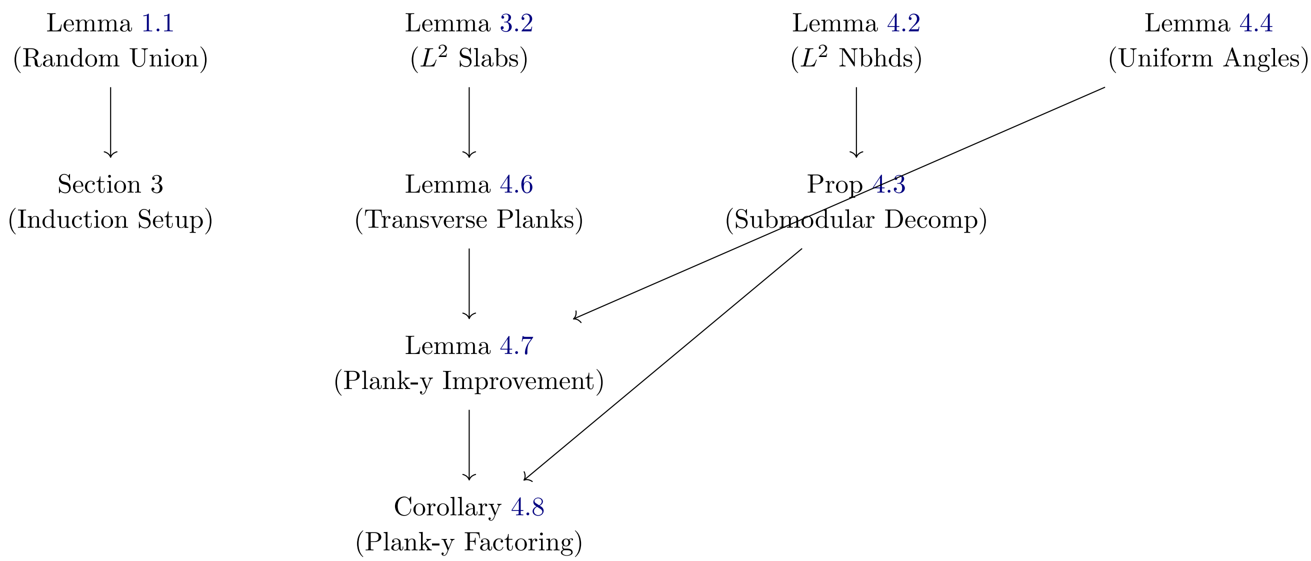

FIGURE 1. Dependency graph of the Kakeya proof lemmas.

1. NOTATION

Notations.

- Blackboard bold denotes a finite family of convex sets and non-bold denote members of the sets, e.g. $\mathbb{V}$ is a collection of convex sets and V_1, V_2, V, V' are members. Furthermore, any blackboard bold family of convex sets should be approximately congruent, meaning each one is contained in a congruent copy of the 100-dilate of the other.
- To avoid ambiguity we use $\mathrm{Vol}(\bullet)$ for volume and $\#(\bullet)$ for cardinality.
- If V is a convex set, the dimensions of V are the lengths of its outer John ellipsoid listed in increasing order. In $\mathbb{R}^3$, a tube is a $\delta\times\delta\times 1$ convex set, a slab is a $\theta\times 1\times 1$ convex set, and a plank is an $a\times b\times 1$ convex set.
- A *shading* of a set V is a measurable set $Y(V)\subset V$. We use $(\mathbb{V}, Y)$ to denote a family of convex sets and an associated shading. We use the notation

$$U(\mathbb{V}, Y) := \bigcup_{\mathbb{V}} Y(V).$$

If the shading is clear from context, we drop Y and just write $U(\mathbb{V})$. We write

$$\lambda(\mathbb{V}, Y) := \frac{1}{\#\mathbb{V}}\sum_{V\in\mathbb{V}}\frac{\mathrm{Vol}(Y(V))}{\mathrm{Vol}(V)}$$

for the average density of the shading. We define the average multiplicity

$$\mu(\mathbb{V}, Y) := \frac{\sum_{V\in\mathbb{V}}\mathrm{Vol}(Y(V))}{\mathrm{Vol}(U(\mathbb{V}, Y))} = \lambda(\mathbb{V}, Y)\frac{\#\mathbb{V}\,\mathrm{Vol}(V)}{\mathrm{Vol}(U(\mathbb{V}, Y))}.$$

- For a finite family of convex sets $\mathbb{W}$ and a convex set K, we denote
$$\mathbb{W}[K] := \{W \in \mathbb{W} : W \subset K\}.$$
- For a shading $(\mathbb{V}, Y)$ and a point $x \in \mathbb{R}^n$, $\mathbb{V}_Y(x) = \{V \in \mathbb{V} : x \in Y(V)\}$. We say $(\mathbb{V}, Y)$ has constant multiplicity if $\#\mathbb{V}_Y(x)$ is roughly constant over $x \in U(\mathbb{V}, Y)$.
- We say $(\mathbb{V}', Y')$ is a refinement of $(\mathbb{V}, Y)$ if $\mathbb{V}' \subset \mathbb{V}$ and $Y'(V) \subset Y(V)$ for each $V \in \mathbb{V}'$. We say it is a c-refinement if $\sum_{\mathbb{V}'} \mathrm{Vol}(Y'(V)) \geq c \sum_{\mathbb{V}} \mathrm{Vol}(Y(V))$.
- $N_\rho(V)$ denotes the ρ-neighborhood of V
- We say two convex sets V_1, V_2 with approximately equal axes are essentially distinct if neither one is contained in the 2-dilate of the other.
- If $\mathbb{T}$ is a set of δ-tubes, $\mathbb{T}_\rho$ denotes a minimal covering of $\mathbb{T}$ by ρ-tubes. The family $\mathbb{T}_\rho$ is essentially distinct in the sense that for each $T_\rho \in \mathbb{T}_\rho$, there are $\lesssim 1$ many tubes T'_ρ that are not essentially distinct from T_ρ.

Frostman and Katz–Tao constants.

- The Katz–Tao constant of a family $\mathbb{V}$ is
$$C_{KT}(\mathbb{V}) := \sup_{K \text{ a convex set}} \#\mathbb{V}[K] \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(K)}.$$
Informally, we say $\mathbb{V}$ is Katz–Tao if $C_{KT}(\mathbb{V}) \lessapprox 1$.
- The Frostman constant of a constant volume family $\mathbb{V}$ living inside a convex set K is
$$C_F(\mathbb{V}, K) := \sup_{K' \subset K} \frac{\#\mathbb{V}[K']}{\#\mathbb{V}\,\mathrm{Vol}(K')/\,\mathrm{Vol}(K)}.$$
If K is the unit ball we use the shorthand $C_F(\mathbb{V})$. Informally, we say $\mathbb{V}$ is Frostman if $C_F(\mathbb{V}) \lessapprox 1$. A common situation is that we have a set of δ-tubes $\mathbb{T}$, and look at $\mathbb{T}[T_\rho]$, the δ-tubes inside a ρ-tube. We write $C_F(\mathbb{T}[T_\rho])$ as a shorthand for $C_F(\mathbb{T}[T_\rho], T_\rho)$.
- Let $\mathbb{V}$ be a family living inside a convex set K. The Frostman and Katz–Tao constants are related by
$$C_{KT}(\mathbb{V}) = C_F(\mathbb{V}, K)\,\#\mathbb{V}\frac{\mathrm{Vol}(V)}{\mathrm{Vol}(K)}. \tag{1.1}$$
- Frostman constant and inflations. Let $\mathbb{V}$ be a family of convex sets. Let w be a scale and consider the family $\mathbb{V}^w = \{N_w(V) : V \in \mathbb{V}\}$. Then $C_F(\mathbb{V}^w) \leq C_F(\mathbb{V})$, as $\#\mathbb{V}^w[K] \leq \#\mathbb{V}[K]$, and $\#\mathbb{V}^w = \#\mathbb{V}$.
- Frostman constant and zooming in. Let $\mathbb{V}$ be a family of convex sets, let $\mathbb{W}$ be another family, and suppose we have a decomposition $\mathbb{V} = \bigsqcup_{W \in \mathbb{W}} \mathbb{V}_W$ where each set in $\mathbb{V}_W$ is contained in W. Suppose in addition that $\#\mathbb{V}_W$ is roughly constant over $W \in \mathbb{W}$. Then

$$C_F(\mathbb{V}_W, W) = \sup_{K \subset W} \frac{\#\mathbb{V}_W[K]}{\#\mathbb{V}_W\,\mathrm{Vol}(K)/\,\mathrm{Vol}(W)} \leq \sup_{K \subset W} \frac{C_F(\mathbb{V})\#\mathbb{V}\,\mathrm{Vol}(K)}{\#\mathbb{V}_W\,\mathrm{Vol}(K)/\,\mathrm{Vol}(W)} \leq C_F(\mathbb{V})\,\#\mathbb{W}\,\mathrm{Vol}(W). \tag{1.2}$$

Pigeonholing.

- If $\mathbb{V}$ is a family of convex sets in $\mathbb{R}^n$ with axes in the range $[\delta, \delta^{-1}]$, then we wish to pigeonhole numbers $v_1 \leq \cdots \leq v_n$ and a subset $\mathbb{V}' \subset \mathbb{V}$ consisting of convex sets whose John ellipsoids have lengths roughly $v_1 \times \cdots \times v_n$. We can do so with $\sum_{\mathbb{V}'} \mathrm{Vol}(V') \gtrsim (\log \delta^{-1})^{-n} \sum_{\mathbb{V}} \mathrm{Vol}(V)$.
- If $(\mathbb{V}, Y)$ is a shading, we wish to find a subset $X \subset \mathbb{R}^n$ and a number ν so that $\#\mathbb{V}_Y(x) \sim \nu$ on X. We can do so with $\nu\,\mathrm{Vol}(X) \gtrsim (\log \#\mathbb{V})^{-1} \sum_{\mathbb{V}} \mathrm{Vol}(Y(V))$.

- If $(\mathbb{V}, Y)$ is a shading, we wish to find a subset $\mathbb{V}'$ along which the shadings have roughly constant measure. If $a = \min_{\mathbb{V}} \mathrm{Vol}(Y(V))$ and $A = \max_{\mathbb{V}} \mathrm{Vol}(Y(V))$, we can find such a set so that $(\mathbb{V}', Y)$ is a $\log(A/a)$-refinement of $(\mathbb{V}, Y)$.

If X and Y are functions of a parameter ρ, then $X \lessapprox Y$ means that for all $\varepsilon > 0$, there is a constant $C(\varepsilon)$ so that $X \le C_\varepsilon \rho^{-\varepsilon} Y$. The quantity ρ is often implicit. Here is the value of ρ in the above examples. In all of these examples, for this choice of ρ, we can replace the loss factor with $\gtrapprox$.

- For inequalities involving a collection $\mathbb{V}$, we define $\rho = \min_{V \in \mathbb{V}} \frac{\text{Smallest axis of } V}{\text{Longest axis of } V}$.
- For inequalities involving the cardinality of a set $\mathbb{V}$, we define $\rho = \#\mathbb{V}^{-1}$.
- For inequalities involving a shading $(\mathbb{V}, Y)$, we take $\rho = \frac{\min_{\mathbb{V}} \mathrm{Vol}(Y(V))}{\max_{\mathbb{V}} \mathrm{Vol}(Y(V))}$.
- For inequalities involving several of the above we take ρ to be a min over several of the above.

In addition, we will often have inequalities that require the shading density to be large. Let X and Y be two quantities involving a shading density λ and a scale parameter ρ. We use $X \lessapprox^* Y$ to denote that, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $X \le C_\varepsilon \rho^{-\varepsilon} \lambda^{C_\varepsilon} Y$.

If $X \lessapprox^* Y$, then for every $\varepsilon > 0$, there is some $\eta, \rho_0 > 0$ such that if $\rho \in (0, \rho_0)$ and $\lambda \ge \rho^\eta$, then $X \le \rho^{-\varepsilon} Y$. Conversely, we are often in a situation where there is an easy bound $X \le C\rho^C Y$ for some $C > 0$. If this holds, then the reverse implication is also true. Suppose that if $\rho \in (0, \rho_0)$ and $\lambda \ge \rho^\eta$, then $X \le \rho^{-\varepsilon} Y$. Then $X \le C\rho^{-\varepsilon} \rho_0^{-C} \lambda^{-C/\eta}$ for any value of ρ and λ. We will pass back and forth between these two interpretations of $\lessapprox^*$.

Uniformity.

- Let $\delta \in (0, 1/3)$, and let $\mathbb{T}$ be a set of δ-tubes in $B_1 \subset \mathbb{R}^n$. We say that $\mathbb{T}$ is *uniform* if for every tube $T \in \mathbb{T}$,

$$\#\mathbb{T}[N_\rho(T)] \gtrsim (\log \log \delta^{-1})^{-20} \max_{T' \in \mathbb{T}} \#\mathbb{T}[N_\rho(T')].$$

Lemma 1.1 ([2, Lemma 3.8])**.** *Suppose that $\mathbb{T}$ is a set of essentially distinct δ-tubes in $B_1 \subset \mathbb{R}^n$. Let $R_1, \dots, R_J$ be random rigid motions of size 1, with $J = C_F(\mathbb{T})$, and let $\mathbb{T}' = \bigcup_{j=1}^J R_j(\mathbb{T})$. Then with high probability the tubes of $\mathbb{T}'$ are essentially distinct up to multiplicity $\lessapprox_\delta 1$ and also $C_F(\mathbb{T}') \lessapprox_\delta 1$.*

Lemma 1.2. *Suppose that $\mathbb{T}$ is a set of δ-tubes in $B_1 \subset \mathbb{R}^n$. Let $\mathbb{T}'$ be a random subset with size $\frac{\#\mathbb{T}}{C_{KT}(\mathbb{T})}$. With high probability, $C_{KT}(\mathbb{T}') \lessapprox_\delta 1$.*

2. Induction setup

$\mathbf{A}(\sigma)$ denotes the statement that for all $\varepsilon > 0$, the following holds for η and δ sufficiently small (in terms of ε and σ). If $\mathbb{T}$ is a set of δ-tubes in B_1 with $C_F(\mathbb{T}) \le \delta^{-\eta}$, and Y is a δ^η-dense shading of $\mathbb{T}$, then $\mathrm{Vol}(U(\mathbb{T}, Y)) \ge \delta^{\sigma+\varepsilon}$.

The main result of [5] is that $\mathbf{A}(0)$ holds. This says the following.

Theorem 2.1 (Kakeya set theorem)**.** *For all $\varepsilon > 0$, if η and δ sufficiently small the following holds. If $\mathbb{T}$ is a set of δ-tubes in B_1 with $C_F(\mathbb{T}) \le \delta^{-\eta}$, and Y is a δ^η-dense shading of $\mathbb{T}$, then $\mathrm{Vol}(U(\mathbb{T}, Y)) \ge \delta^\varepsilon$.*

Now let $(\mathbb{T}, Y)$ be an arbitrary set of tubes (not necessarily Frostman). Let $(\mathbb{T}', Y)$ be the union of $C_F(\mathbb{T})$ random translates of $\mathbb{T}$, along with the induced shading. By Lemma 1.1, if δ is sufficiently small in terms of η, then with high probability $C_F(\mathbb{T}') \le \delta^{-\eta}$, so $\mathrm{Vol}(U(\mathbb{T}', Y)) \ge \delta^{\sigma+\varepsilon}$. On the other hand, $\mathrm{Vol}(U(\mathbb{T}', Y)) \le C_F(\mathbb{T}) \mathrm{Vol}(U(\mathbb{T}, Y))$. Thus $\mathbf{A}(\sigma)$ implies that if δ, η are sufficiently small in terms of ε, σ, and Y is a δ^η-dense shading, then $\mathrm{Vol}(U(\mathbb{T}, Y)) \geq \delta^{\sigma+\varepsilon} \frac{1}{C_F(\mathbb{T})}$. Using our $\gtrapprox^*$ notation, $\mathbf{A}(\sigma)$ says

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \delta^\sigma \frac{1}{C_F(\mathbb{T})}. \tag{2.1}$$

Using the definition $\mu(\mathbb{T}, Y) = \frac{\#\mathbb{T}\,\mathrm{Vol}(T)}{\mathrm{Vol}(U(\mathbb{T},Y))}\lambda(\mathbb{T}, Y)$, and the relation $C_{KT}(\mathbb{T}) = C_F(\mathbb{T})\#\mathbb{T}\,\mathrm{Vol}(T)$, we can rewrite (2.1) as an upper bound on multiplicity,

$$\mu(\mathbb{T}, Y) \lessapprox^* \delta^{-\sigma} C_{KT}(\mathbb{T}). \tag{2.2}$$

3. The L^2 method

Let $(\mathbb{S}, Y)$ be a collection of $\delta \times 1 \times 1$ slabs along with a shading. Let

$$\mathrm{Tri}(\mathbb{S}) = \{(x, S_1, S_2) \,:\, x \in Y(S_1) \cap Y(S_2)\}.$$

This is a subset of $\mathbb{R}^3 \times \mathbb{S}$ and comes with a natural measure, which is the product of Lebesgue on $\mathbb{R}^3$ and the counting measure on $\mathbb{S}$. We define

$$\mathrm{Tri}_\theta(\mathbb{S}) = \{(x, S_1, S_2) \in \mathrm{Tri}(\mathbb{S}) \,:\, \angle(S_1, S_2) \sim \theta\}.$$

Definition 3.1. We say that θ is a typical angle of intersection for $(\mathbb{S}, Y)$ if $\mathrm{Msr}(\mathrm{Tri}_\theta(\mathbb{S})) \approx \mathrm{Msr}(\mathrm{Tri}(\mathbb{S}))$ up to $\log \delta^{-1}$ factors. If $\mu(\mathbb{S}, Y) \geq 2$, then $\mathbb{S}$ has a typical intersection angle $\theta \in [\delta, 1]$.

Lemma 3.2 ([2, Lemma 6.8])**.** *Suppose $(\mathbb{S}, Y)$ is a shaded set of $\delta \times 1 \times 1$ slabs contained in $\widetilde{S}$, a $\theta \times 1 \times 1$ slab, with $\theta \in [\delta, 1]$. Equip $\mathbb{S}$ with a shading Y. If θ is a typical intersection angle for $\mathbb{S}$, then*

$$\frac{\mathrm{Vol}(U(\mathbb{S}, Y))}{\mathrm{Vol}(\tilde{S})} \gtrsim \lambda(\mathbb{S}, Y)^2.$$

Note that $\mathbb{S}$ does not need to be essentially distinct.

4. Factoring and plank-y improvement

Lemma 4.1 (Maximal density factoring lemma [2, Lemma 4.1])**.** *Let $\mathbb{V}$ be a family of convex subsets of $\mathbb{R}^n$. Then there is a set $\mathbb{V}' \subset \mathbb{V}$ with $\#\mathbb{V}' \gtrapprox \#\mathbb{V}$; a family $\mathbb{W}$ of convex subsets in $\mathbb{R}^n$, each of approximately the same dimensions; and a partition*

$$\mathbb{V}' = \bigsqcup_{W \in \mathbb{W}} \mathbb{V}_W$$

where each set in $\mathbb{V}_W$ is contained in W. This partition has the following properties.

(i) *$\mathbb{W}$ is Katz–Tao, i.e. $C_{KT}(\mathbb{W}) \sim 1$. Furthermore, $C_F(\mathbb{V}') \approx C_F(\mathbb{W}) \approx (\#\mathbb{W}\,\mathrm{Vol}(W))^{-1}$.*
(ii) *$\mathbb{V}_W$ is Frostman in W, i.e. $C_F(\mathbb{V}_W, W) \sim 1$. Furthermore, $C_{KT}(\mathbb{V}') \approx C_{KT}(\mathbb{V}_W) \approx \#\mathbb{V}_W \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(W)}$.*

Lemma 4.2 ([2, Lemma 5.9])**.** *Let $\mathbb{V}$ be a collection of $v_1 \times v_2 \times v_3$ convex sets inside a $w_1 \times w_2 \times w_3$ convex set W, and let Y be a shading on $\mathbb{V}$. We have*

$$\frac{\mathrm{Vol}(N_{w_1}(U(\mathbb{V}, Y)))}{\mathrm{Vol}(W)} \gtrapprox \frac{1}{C_F(\mathbb{V}, W)} \lambda(\mathbb{V}, Y)^2.$$

Proposition 4.3 (Submodular decomposition, c.f. [2, Proposition 5.1])**.** *Let $\mathbb{V} = \bigsqcup_{\mathbb{W}} \mathbb{V}_W$, where $\mathbb{V}$ and $\mathbb{W}$ are families of congruent convex sets in $\mathbb{R}^3$. Let Y be a shading on $\mathbb{V}$.*

There is a $\gtrapprox 1$ refinement $(\mathbb{V}', Y')$ and a subset $\mathbb{W}' \subset \mathbb{W}$ such that the following holds. Define $\mathbb{V}'_W = \mathbb{V}_W \cap \mathbb{V}'$, so that $\mathbb{V}' = \bigsqcup_{W \in \mathbb{W}'} \mathbb{V}'_W$. Define an induced shading on $\mathbb{W}'$ by

$$Y_{\mathbb{W}'}(W) = W \cap N_{w_1}(U(\mathbb{V}_W, Y')), \qquad \text{where } w_1 \text{ is the shortest dimension of } W.$$

(1) $\#\mathbb{V}'_W$ is roughly constant over $W' \in \mathbb{W}'$.
(2) $\lambda(\mathbb{V}'_W, Y')$ is roughly constant over $W \in \mathbb{W}'$.
(3) For all $W \in \mathbb{W}'$, we have

$$\mathrm{Vol}(U(\mathbb{V}, Y)) \geq \mathrm{Vol}(U(\mathbb{V}', Y')) \gtrapprox \mathrm{Vol}(U(\mathbb{W}', Y_{\mathbb{W}'})) \frac{\mathrm{Vol}(U(\mathbb{V}_W, Y'))}{\mathrm{Vol}(W)}. \tag{4.1}$$

If the w_1-neighborhoods of $\mathbb{V}_W$ are C-Frostman inside of each W, then by Lemma 4.2, $\lambda(\mathbb{W}', Y_{\mathbb{W}'}) \gtrapprox C^{-1}\lambda(\mathbb{V}, Y)^2$.

If we write (4.1) in terms of multiplicity, we find

$$\mu(\mathbb{V}, Y) \frac{1}{\#\mathbb{V}\,\mathrm{Vol}(V)\lambda(\mathbb{V}, Y)} \lessapprox \mu(\mathbb{W}', Y_{\mathbb{W}'}) \frac{1}{\#\mathbb{W}'\,\mathrm{Vol}(W)\lambda(\mathbb{W}', Y_{\mathbb{W}'})} \mu(\mathbb{V}'_W, Y') \frac{\#\mathbb{W}'\,\mathrm{Vol}(W)}{\#\mathbb{V}'\,\mathrm{Vol}(V)\lambda(\mathbb{V}', Y)}$$

which rearranges to

$$\mu(\mathbb{V}, Y) \lessapprox \mu(\mathbb{W}', Y_{\mathbb{W}'})\, \mu(\mathbb{V}'_W, Y') \frac{1}{\lambda(\mathbb{W}', Y_{\mathbb{W}'})}. \tag{4.2}$$

The factor $\frac{1}{\lambda(\mathbb{W}', Y_{\mathbb{W}'})}$ is an artifact of our proof and should not be taken seriously.

Proof. First, pigeonhole to a subfamily of $\mathbb{W}$ so that $\lambda(\mathbb{V}_W, Y)$ is roughly constant over $W \in \mathbb{W}$. Next, for each W, pass to a $\gtrapprox 1$ refinement $(\mathbb{V}''_W, Y'')$ of $(\mathbb{V}_W, Y)$ with constant multiplicity. Let $\mathbb{B}_W$ be a disjoint family of w_1-balls such that $\alpha_W = \frac{\mathrm{Vol}(B \cap U(\mathbb{V}''_W, Y''))}{\mathrm{Vol}(B)}$ is roughly constant over $B \in \mathbb{B}_W$, and $\#\mathbb{B}_W \mathrm{Vol}(B)\alpha_W \gtrapprox \mathrm{Vol}(U(\mathbb{V}''_W, Y''))$. Define Y' to be the refined shading that only includes points landing in $\cup\mathbb{B}_W$. Because Y'' has constant multiplicity, this is a $\gtrapprox 1$ refinement. Pass to a subfamily $\mathbb{W}'$ covering a $\gtrapprox 1$ portion of $\mathbb{V}$ such that α_W and $\#\mathbb{V}''_W$ is roughly constant over $W \in \mathbb{W}'$. Define $\mathbb{V}' = \bigsqcup_{W \in \mathbb{W}'} \mathbb{V}''_W$. Define $Y_{\mathbb{W}'}$ as in the Proposition statement.

It remains to establish condition (3). Notice that

$$\alpha_W \gtrsim \frac{\mathrm{Vol}(U(\mathbb{V}'_W, Y'))}{\#\mathbb{B}_W \mathrm{Vol}(B)} \gtrsim \frac{\mathrm{Vol}(U(\mathbb{V}'_W, Y'))}{\mathrm{Vol}(W)}.$$

Let $\mathbb{B}$ be a maximal disjoint family from $\bigcup_{W \in \mathbb{W}'} \mathbb{B}_W$. The set $U(\mathbb{W}', Y_{\mathbb{W}'})$ is contained in the $10w_1$-neighborhood of $\mathbb{B}$, and vice-versa, so $\#\mathbb{B}\,\mathrm{Vol}(B) \approx \mathrm{Vol}(U(\mathbb{W}', Y_{\mathbb{W}'}))$. Thus

$$\mathrm{Vol}(U(\mathbb{V}', Y')) \gtrsim \#\mathbb{B}\,\mathrm{Vol}(B)\, \alpha_W \gtrsim \mathrm{Vol}(U(\mathbb{W}', Y_{\mathbb{W}'})) \frac{\mathrm{Vol}(U(\mathbb{V}_W, Y'))}{\mathrm{Vol}(W)}$$

as desired. □

Lemma 4.4 ([2, Lemma 6.11])**.** *Let $(\mathbb{P}, Y)$ be a family of $a \times b \times 1$ planks and their associated shading. Let $\varepsilon > 0$ and suppose that $\mu(\mathbb{P}, Y) \geq \delta^{-\varepsilon}$. Then there is an ≈ 1 refinement $(\mathbb{P}, Y')$ of $(\mathbb{P}, Y)$ and a number $\theta \in [a/b, 1]$ with the following properties. In what follows, let $A = e^{\sqrt{\log a^{-1}}}$*

- *$(\mathbb{P}, Y')$ has constant multiplicity on $U(\mathbb{P}, Y')$ up to a factor ~ 1.*
- $\max_{P_1, P_2 \in \mathbb{P}_{Y'}(x)} \angle(TP_1, TP_2) \sim \theta$.

- *If* $\mathbb{P}_{Y''}(x) \subset \mathbb{P}_{Y'}(x)$ *with* $\#\mathbb{P}_{Y''}(x) \geq \frac{1}{A}\#\mathbb{P}_{Y'}(x)$*, then*
$$\max_{P_1,P_2 \in \mathbb{P}_{Y''}(x)} \angle(TP_1, TP_2) \approx \theta.$$

Definition 4.5 ([2, Definition 6.12])**.** We say θ is a typical angle of intersection for $(\mathbb{P}, Y)$ if $(\mathbb{P}, Y)$ satisfies the conclusions of Lemma 4.4 for some $A \geq e^{(\log a^{-1})^{1/4}}$.

Lemma 4.6 (Transverse planks)**.** *Assume* $\mathbf{A}(\sigma)$ *holds. If* $\mathbb{P}$ *is a collection of* $\delta \times b \times 1$ *planks and* 1 *is a typical intersection angle of* $(\mathbb{P}, Y)$*, then*
$$\mathrm{Vol}(U(\mathbb{P}, Y)) \gtrapprox^* b^\sigma \frac{1}{C_F(\mathbb{P})}.$$

Note: In this Lemma and elsewhere, the implicit scale parameter in $\gtrapprox^*$ is δ, not b.

Proof. Let $\lambda(\mathbb{P}, Y) \geq \delta^\eta$. Let $\mathbb{B}$ be a minimal covering of $U(\mathbb{P}, Y)$ by b-balls. For each $B \in \mathbb{B}$, let ψ_B be the affine map taking $2B$ to the unit ball, and define the rescaled collection of slabs
$$\mathbb{S}_B = \psi_B(\{P \cap 2B : P \in \mathbb{P} \text{ intersects } B\}), \qquad Y_{\mathbb{S}_B}(S) = \psi_B(P \cap B).$$
This is a collection of $\frac{\delta}{b} \times 1 \times 1$ slabs with a shading, and 1 is a typical intersection angle. Call B *dense* if $\lambda(Y_{\mathbb{S}_B}) \geq \delta^{2\eta}$. If B is dense, then according to Lemma 3.2, $\mathrm{Vol}(U(\mathbb{S}_B, Y_{\mathbb{S}_B})) \gtrapprox \delta^{4\eta}$. Because $\mathbb{B}$ is a minimal covering, it is finitely overlapping. Thus
$$\mathrm{Vol}(U(\mathbb{P}, Y)) \gtrsim \sum_{B \in \mathbb{B}_{dense}} \mathrm{Vol}(U(\mathbb{P}, Y) \cap B) \gtrapprox \delta^{4\eta} \#\mathbb{B}_{dense} \gtrsim \delta^{4\eta}\,\mathrm{Vol}(\cup \mathbb{B}_{dense}).$$
If we can show $\mathrm{Vol}(\cup\mathbb{B}_{dense}) \gtrapprox^* b^\sigma \frac{1}{C_F(\mathbb{P})}$, that will establish the result.

Define a refined shading by
$$Y_1(P) := Y(P) \cap (\cup \mathbb{B}_{dense}).$$
We estimate the density of this shading by upper bounding its complement,
$$\begin{aligned}\lambda(\mathbb{P}, Y) - \lambda(\mathbb{P}, Y_1) &\leq \frac{1}{\#\mathbb{P}\,\mathrm{Vol}(P)} \sum_{B \in \mathbb{B} \text{ is not dense}} \sum_{P \in \mathbb{P}} \mathrm{Vol}(Y(P) \cap B) \\ &\leq \frac{1}{\#\mathbb{P}\,\mathrm{Vol}(P)} \sum_{B \in \mathbb{B} \text{ is not dense}} \delta^{2\eta} b\,\mathrm{Vol}(P) \sum_{P \in \mathbb{P}} 1_{P \text{ intersects } B} \\ &\lesssim \delta^{2\eta}.\end{aligned}$$
In the last line, we used that each $P \in \mathbb{P}$ has $\lesssim b^{-1}$ balls intersecting it. Thus, for δ sufficiently small, $\lambda(\mathbb{P}, Y_1) \geq \frac{1}{2}\delta^\eta$.

For $P \in \mathbb{P}$, let P_b be the $10b$-neighborhood, and let $\mathbb{T}_b$ be this collection of $10b$-tubes. As described in the introductory section, $C_F(\mathbb{T}_b) \leq C_F(\mathbb{P})$. If B intersects P then it is contained in P_b, so we may define a shading
$$Y_{\mathbb{T}_b}(P_b) := \cup\{B \in \mathbb{B} : B \text{ is dense and intersects } P\}.$$
By definition, $\mathrm{Vol}(U(\mathbb{T}_b, Y_{\mathbb{T}_b})) = \mathrm{Vol}(\cup\mathbb{B}_{dense})$. We have
$$\frac{\mathrm{Vol}(Y_b(P_b))}{\mathrm{Vol}(P_b)} \sim |Y_1(P)|_b b \gtrsim \frac{\mathrm{Vol}(Y_1(P))}{\mathrm{Vol}(P)},$$
so $\lambda(\mathbb{T}_b, Y_b) \gtrsim \lambda(\mathbb{P}, Y_1) \gtrsim \delta^{2\eta}$. Thus by $\mathbf{A}(\sigma)$, if δ and η are sufficiently small, $\mathrm{Vol}(U(\mathbb{T}_b, Y_{\mathbb{T}_b})) \geq \delta^\varepsilon b^\sigma \frac{1}{C_F(\mathbb{P})}$ as desired. □

Lemma 4.7 (Plank-y Improvement)**.** *Assume $\mathbf{A}(\sigma)$ holds. Then for any collection $\mathbb{P}$ of $\delta \times b \times 1$ planks,*

$$\mathrm{Vol}(U(\mathbb{P}, Y)) \gtrapprox^* b^\sigma \frac{1}{C_F(\mathbb{P})}.$$

Proof. It suffices to assume $C_F(\mathbb{P}) \leq \delta^{-\eta}$, and show that for δ and η small enough, $\mathrm{Vol}(U(\mathbb{P}, Y)) \geq \delta^\varepsilon b^\sigma$.

If $\mu(\mathbb{P}, Y) \leq \delta^{-\varepsilon/2}$, then $\mathrm{Vol}(U(\mathbb{P}, Y)) \geq \delta^{\varepsilon/2+\eta} \#\mathbb{P}\,\mathrm{Vol}(P) \gtrsim \delta^{\varepsilon/2+2\eta}$. Thus we may assume $\mu(\mathbb{P}, Y) \geq \delta^{-\varepsilon/2}$. After passing to $\gtrapprox 1$ refinements, we may assume $(\mathbb{P}, Y)$ satisfies the conclusions of Lemma 4.4.

Let $\theta \in [\delta/b, 1]$ be the output of that Lemma. We decompose $\mathbb{P}$ into a collection $\mathbb{S}$ of $\theta \times 1 \times 1$ slabs satisfying

$$\mathbb{P} = \bigsqcup_{S \in \mathbb{S}} \mathbb{P}_S, \qquad \text{each } P \in \mathbb{P}_S \text{ is contained in } S \text{ and } \angle(P, S) \lesssim \theta.$$

After passing to a $\gtrapprox 1$ refinement, we may assume $\#\mathbb{P}_S$ is roughly constant over $S \in \mathbb{S}$. We have

$$C_F(\mathbb{P}_S, S) = \sup_{K \subset S} \frac{\#\mathbb{P}_S[K]}{\#\mathbb{P}_S \mathrm{Vol}(K)/\mathrm{Vol}(S)} \leq \sup_{K \subset S} \frac{C_F(\mathbb{P})\#\mathbb{P}\,\mathrm{Vol}(K)}{\#\mathbb{P}_S \mathrm{Vol}(K)/\mathrm{Vol}(S)} \leq C_F(\mathbb{P}) \frac{\#\mathbb{P}}{\#\mathbb{P}_S} \mathrm{Vol}(S).$$

After passing to a $\gtrapprox 1$ refinement, we may assume $\#\mathbb{P}_S$ is constant over S. Thus $\frac{\#\mathbb{P}}{\#\mathbb{P}_S} \approx \#\mathbb{S}$, and $C_F(\mathbb{P}_S, S) \lessapprox C_F(\mathbb{P})\#\mathbb{S}\,\mathrm{Vol}(S)$.

Due to the angular condition in Lemma 4.4, for each $x \in U(\mathbb{P}, Y)$, there are $\lesssim 1$ slabs S that have planks contributing to $\mathbb{P}_Y(x)$. Thus, for each $x \in U(\mathbb{P}, Y)$, there is some $S \in \mathbb{S}$ contributing a $\gtrsim 1$ portion of the planks in $\mathbb{P}_Y(x)$. Define $S(x)$ to be such a slab, and refine the shading so that $x \in Y(P)$ only if $P \in \mathbb{P}_{S(x)}$. This new shading still satisfies the conclusions of Lemma 4.4, and in addition satisfies

$$\mathrm{Vol}(U(\mathbb{P}, Y)) = \sum_{S \in \mathbb{S}} \mathrm{Vol}(U(\mathbb{P}_S, Y)) \tag{4.3}$$

Fix $S \in \mathbb{S}$, let ψ_S be an affine transformation mapping to the unit box, and let $\widetilde{\mathbb{P}} = \psi_S(\mathbb{P}_S)$. Let $\widetilde{Y}$ be the associated shading. This is a collection of $\frac{\delta}{\theta} \times b \times 1$ planks with $C_F(\widetilde{\mathbb{P}}) \lessapprox C_F(\mathbb{P})\#\mathbb{S}\,\mathrm{Vol}(S)$. In addition, $(\widetilde{\mathbb{P}}, \widetilde{Y})$ satisfies the conclusions of Lemma 4.4 with $\theta = 1$. Thus, by Lemma 4.6, $\mathrm{Vol}(U(\widetilde{\mathbb{P}}, \widetilde{Y})) \gtrapprox^* \frac{1}{C_F(\mathbb{P})\#\mathbb{S}\,\mathrm{Vol}(S)} b^\sigma$. It follows from (4.3) that $\mathrm{Vol}(U(\mathbb{P}, Y)) \gtrapprox^* \frac{1}{C_F(\mathbb{P})} b^\sigma$. □

Corollary 4.8. *Assume $\mathbf{A}(\sigma)$ holds. For all $\varepsilon > 0$, the following holds for δ and η sufficiently small. Let $(\mathbb{T}, Y)$ be a collection of δ-tubes in $B_1 \subset \mathbb{R}^3$ with a δ^η-dense shading, and suppose there is a family $\mathbb{V}$ of $a \times b \times 1$ convex sets, and a decomposition $\mathbb{T} = \bigsqcup_{V \in \mathbb{V}} \mathbb{T}_V$ where each tube in $\mathbb{T}_V$ is contained in V. Assume in addition that $\#\mathbb{T}_V$ is roughly constant over $V \in \mathbb{V}$. Let $(\mathbb{T}_V)_a$ be the a-neighborhood of tubes in $\mathbb{T}_V$, and suppose $C_F((\mathbb{T}_V)_a, V) \leq \delta^\eta$ for each $V \in \mathbb{V}$. Then*

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \big(\delta \frac{b}{a}\big)^\sigma \frac{1}{C_F(\mathbb{V})} \frac{1}{\sup_{V \in \mathbb{V}} C_F(\mathbb{T}_V, V)}. \tag{4.4}$$

Equivalently,

$$\mu(\mathbb{T}, Y) \lessapprox^* \big(\delta \frac{b}{a}\big)^{-\sigma} C_{KT}(\mathbb{V}) \sup_{V \in \mathbb{V}} C_{KT}(\mathbb{T}_V). \tag{4.5}$$

Proof. By the submodular decomposition Proposition 4.3, after passing to $\gtrapprox 1$ refinements,

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox \mathrm{Vol}(U(\mathbb{V}, Y_{\mathbb{V}})) \frac{\mathrm{Vol}(U(\mathbb{T}_V, Y))}{\mathrm{Vol}(V)} \qquad \text{for each } V \in \mathbb{V},$$

where $\lambda(Y_{\mathbb{V}}) \gtrapprox \delta^\eta \lambda(Y)^2$. The output of Proposition 4.3 also guarantees that $\#\mathbb{T}_V$ is roughly constant over $V \in \mathbb{V}$.

By Lemma 4.7, $\mathrm{Vol}(U(\mathbb{V}, Y_{\mathbb{V}})) \gtrapprox^* b^\sigma \frac{1}{C_F(\mathbb{V})}$. Next, perform a rescaling map taking V to a $1 \times 1 \times 1$ convex set and $\mathbb{T}_V$ to a collection of $\frac{\delta}{b} \times \frac{\delta}{a} \times 1$ planks. By Lemma 4.7 applied to these planks, $\frac{\mathrm{Vol}(U(\mathbb{T}_V, Y))}{\mathrm{Vol}(V)} \gtrapprox^* (\delta/a)^\sigma \frac{1}{\sup_{V \in \mathbb{V}} C_F(\mathbb{T}_V, V)}$. Combining these two pieces gives (4.4). If we rewrite (4.4) in terms of multiplicities, we get

$$\mu(\mathbb{T}, Y) \frac{1}{\lambda(\mathbb{T}, Y)} \frac{1}{\#\mathbb{T}\,\mathrm{Vol}(T)} \lessapprox^* \big(\delta \frac{b}{a}\big)^{-\sigma} C_F(\mathbb{V}) \sup_{V \in \mathbb{V}} C_F(\mathbb{T}_V, V)$$
$$= \big(\delta \frac{b}{a}\big)^{-\sigma} \frac{C_{KT}(\mathbb{V})}{\#\mathbb{V}\,\mathrm{Vol}(V)} \sup_{V \in \mathbb{V}} \frac{C_{KT}(\mathbb{T}_V, V)}{\#\mathbb{T}_V\,\mathrm{Vol}(T)/\mathrm{Vol}(V)}.$$

Since $\#\mathbb{T}_V$ is roughly constant, $\#\mathbb{V}\#\mathbb{T}_V \approx \#\mathbb{T}$, leading to (4.5). □

5. Sticky Kakeya and improvement lemma

Definition 5.1 ([2, Definition 7.1]). Let $\mathbb{T}$ be a set of δ-tubes in $\mathbb{R}^n$.

(A) We say $\mathbb{T}$ is Frostman at every scale with error C if for every scale $\rho \in [\delta, 1]$, we have

$$C_F(\mathbb{T}[T_\rho], T_\rho) \leq C \qquad \text{for every } T_\rho \in \mathbb{T}_\rho.$$

(B) We say $\mathbb{T}$ is Katz–Tao at every scale with error C if for every scale $\rho \in [\delta, 1]$, we have

$$C_{KT}(\mathbb{T}_\rho) \leq C.$$

Wang and Zahl proved a Sticky Kakeya theorem in [3]. The following statement, adapted to Frostman–Wolff constants, is a slight variation on [4, Theorem 5.2]. The statement is taken from [2, Theorem 7.3].

Theorem 5.2 (Sticky Kakeya). *For all $\varepsilon > 0$, there exists $\eta, \delta_0 > 0$ so that the following holds for all $\delta \in (0, \delta_0]$. Let $(\mathbb{T}, Y)$ be a uniform set of δ-tubes in $B_1 \subset \mathbb{R}^3$ with a δ^η-dense shading.*

(A) If $\mathbb{T}$ is Frostman at every scale with error $\delta^{-\eta}$, then

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \geq \delta^\varepsilon.$$

(B) If $\mathbb{T}$ is Katz–Tao at every scale with error $\delta^{-\eta}$, then

$$\mu(\mathbb{T}, Y) \leq \delta^{-\varepsilon}.$$

Proposition 5.3 (Nowhere sticky reduction [2, Lemma 7.7]). *Let $N \geq 1$ be an integer. Define $\varepsilon = 1/\sqrt{N}$, and let $\kappa \leq \kappa_1 \leq \cdots \leq \kappa_N \leq \varepsilon$. Let $\delta > 0$ and let $\mathbb{T}$ be a uniform set of δ-tubes.*

(A) If $C_F(\mathbb{T}) \leq \delta^{-\kappa}$, then at least one of the following holds.

- *(i) $\mathbb{T}$ is $\lessapprox \delta^{-5\varepsilon}$-Frostman at every scale.*
- *(ii) There are scales $\delta \leq \tau \leq \theta \leq 1$ with $\tau \leq \delta^\varepsilon \theta$, and an integer $1 \leq j \leq N$ so that the following holds:*
  - *$C_F(\mathbb{T}[T_\tau], T_\tau) \lessapprox (\tau/\delta)^{\kappa_{j-1}}$ for all $T_\tau \in \mathbb{T}_\tau$.*
  - *$C_F(\mathbb{T}_\tau[T_\theta], T_\theta) \lessapprox (\theta/\tau)^{\kappa_{j-1}}$ for all $T_\theta \in \mathbb{T}_\theta$.*
  - *For all $\rho \in [\tau(\theta/\tau)^\varepsilon, \theta(\tau/\theta)^\varepsilon]$, we have $C_F(\mathbb{T}_\tau[T_\rho]) \geq (\rho/\tau)^{\kappa_j}$ for all $T_\rho \in \mathbb{T}_\rho$.*

(B) If $C_{KT}(\mathbb{T}) \leq \delta^{-\kappa}$, at least one of the following holds.

- *(i) $\mathbb{T}$ is $\lessapprox \delta^{-\varepsilon}$-Katz–Tao at every scale.*

(ii) *There are scales* $\delta \leq \tau \leq \theta \leq 1$ *with* $\tau \leq \delta^{\varepsilon}\theta$ *and an integer* $1 \leq j \leq N$ *so that the following holds:*
   - $C_{KT}(\mathbb{T}_\theta) \lessapprox \theta^{-\kappa_{j-1}}$.
   - $C_{KT}(\mathbb{T}_\tau[T_\theta]) \lessapprox (\theta/\tau)^{\kappa_{j-1}}$ *for all* $T_\theta \in \mathbb{T}_\theta$.
   - *For all* $\rho \in [\tau(\theta/\tau)^{\varepsilon}, \tau(\tau/\theta)^{\varepsilon}]$, *we have* $C_{KT}(\mathbb{T}_\rho[T_\theta]) \geq (\theta/\rho)^{\kappa_j}$ *for all* $T_\theta \in \mathbb{T}_\theta$.

Let $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta,\kappa)$ denote the assertion that for sufficiently small η and δ the following holds. Let $\mathbb{T}$ be a set of essentially distinct δ-tubes in B_1 and let Y be a δ^η-dense shading. If $C_F(\mathbb{T}) \leq \delta^{-\kappa}$ and $\#\mathbb{T}\,\mathrm{Vol}(T) \geq \delta^{-\zeta}$, then $\mathrm{Vol}(U(\mathbb{T},Y)) \geq \delta^{\sigma-\kappa}$. This statement can be upgraded to not have a Frostman hypothesis. An arbitrary set of essentially distinct tubes satisfies

$$\#\mathbb{T}\,\mathrm{Vol}(T) \geq \delta^{-\zeta} \Longrightarrow \mathrm{Vol}(U(\mathbb{T},Y)) \gtrapprox^* \frac{1}{C_F(\mathbb{T})}\delta^{\sigma-\kappa}.$$

Similarly, let $\mathbf{A}_{\mathrm{LDL}}(\sigma,\zeta,\kappa)$ denote the assertion that for sufficiently small η and δ the following holds. Let $\mathbb{T}$ be a set of δ-tubes in B_1 and let Y be a δ^η-dense shading. If $C_{KT}(\mathbb{T}) \leq \delta^{-\kappa}$ and $\#\mathbb{T}\,\mathrm{Vol}(T) \leq \delta^{\zeta}$, then $\mu(\mathbb{T},Y) \leq \delta^{-\sigma+\kappa}$. This statement can be upgraded to not have a Frostman hypothesis. An arbitrary set of tubes satisfies

$$\#\mathbb{T}\,\mathrm{Vol}(T) \leq \delta^{\zeta} \Longrightarrow \mu(\mathbb{T},Y) \lessapprox^* C_{KT}(\mathbb{T})\,\delta^{-\sigma+\kappa}.$$

Lemma 5.4 (High / low density lemma)**.** *Assume* $\mathbf{A}(\sigma)$ *holds with* $\sigma > 0$. *For all* $\zeta > 0$, *there exists* $\kappa > 0$ *such that* $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta,\kappa)$ *holds and* $\mathbf{A}_{\mathrm{LDL}}(\sigma,\zeta,\kappa)$ *holds.*

The proof of the following lemma follows a similar strategy to [4, Section 4].

Proof of high density lemma. Since $\#\mathbb{T} \lessapprox \delta^{-4}$ and $\mathrm{Vol}(T) \sim \delta^2$, we have $\#\mathbb{T}\,\mathrm{Vol}(T) \lessapprox \delta^{-2}$, so the hypothesis $\#\mathbb{T}\,\mathrm{Vol}(T) \geq \delta^{-\zeta}$ is vacuous for $\zeta > 2$ and δ sufficiently small. Thus $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta,\kappa)$ holds trivially for all $\kappa > 0$ when $\zeta > 2$. Define

$$\zeta = \sup\{\zeta' : \mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta',\kappa) \text{ fails for all } \kappa > 0\}.$$

Assume by way of contradiction that $\zeta > 0$. The set of ζ' for which $\mathbf{A}(\sigma,\zeta',\kappa)$ holds for some $\kappa > 0$ is open, so it suffices to show $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta,\kappa)$ holds for some $\kappa > 0$.

We plan to apply Proposition 5.3. Select parameters in the following order.

(1) Choose $\varepsilon \in (0, \zeta/10)$ small enough that Theorem 5.2 says: If $\mathbb{T}$ is $\delta^{-2\varepsilon}$-Frostman at every scale with δ^ε dense shading, then $\mathrm{Vol}(U(\mathbb{T},Y)) \geq \delta^{\sigma/2}$. Set $N = \lceil 25/\varepsilon^2 \rceil$.
(2) Set $\zeta' = \zeta + \varepsilon^2$. Choose $\kappa' > 0$ so that $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta',\kappa')$ holds.
(3) Set $\kappa_N = \frac{1}{10}\kappa'\varepsilon^2$. Choose $\kappa_j \leq \frac{1}{10}\varepsilon^2\sigma\kappa_{j+1}$ for $j = N-1$ through $j = 0$. Set $\kappa = \kappa_0$. Thus we may take $\kappa = (\varepsilon^2\sigma/10)^{\lceil 25/\varepsilon^2 \rceil + 1}\kappa'$. We also need $\kappa < \sigma/2$ for the sticky Kakeya case of Proposition 5.3.
(4) Choose η and δ sufficiently small for all applications below.

Let $\mathbb{T}$ be an essentially distinct set of δ-tubes with $C_F(\mathbb{T}) \leq \delta^{-\kappa}$ and $\#\mathbb{T}\,\mathrm{Vol}(T) \geq \delta^{-\zeta}$, and let Y be a δ^η-dense shading. Apply Proposition 5.3 (A) with parameters $\kappa \leq \kappa_1 \leq \cdots \leq \kappa_N \leq \varepsilon$.

If we land in the first alternative, $\mathbb{T}$ is $\lessapprox \delta^{-\varepsilon}$-Frostman at every scale. Theorem 5.2 implies $\mathrm{Vol}(U(\mathbb{T},Y)) \geq \delta^{\sigma/2} \geq \delta^{\sigma-\kappa}$.

Suppose we land in the second alternative with scales $\delta \leq \tau \leq \theta \leq 1$ with $\tau \leq \delta^\varepsilon\theta$, and an integer $1 \leq j \leq N$. Apply the submodular decomposition Proposition 4.3 in two steps: First to $\mathbb{T}[T_\theta]$ at scale

τ, and then to $\mathbb{T}_\tau$ at scale θ. We pass to $\gtrapprox 1$ refinements, but keep using the same letters by abuse of notation, and estimate

$$\mathrm{Vol}(U(\mathbb{T},Y)) \gtrapprox \mathrm{Vol}(U(\mathbb{T}_\theta, Y_{\mathbb{T}_\theta}))\,\frac{\mathrm{Vol}(U(\mathbb{T}_\tau[T_\theta], Y_{\mathbb{T}_\tau}))}{\mathrm{Vol}(T_\theta)}\,\frac{\mathrm{Vol}(U(\mathbb{T}[T_\tau],Y))}{\mathrm{Vol}(T_\tau)}. \tag{5.1}$$

Here, Y is implicitly a refinement of our original shading; $Y_{\mathbb{T}_\tau}$ is the induced shading on $\mathbb{T}_\tau$, as in Proposition 4.3; and $Y_{\mathbb{T}_\theta}$ is the induced shading on $\mathbb{T}_\theta$. By $\mathbf{A}(\sigma)$,

$$C_F(\mathbb{T}_\theta) \lessapprox \delta^{-\kappa}, \qquad \mathrm{Vol}(U(\mathbb{T}_\theta, Y_{\mathbb{T}_\theta})) \gtrapprox^* \delta^{\kappa}\theta^{\sigma}, \tag{5.2}$$

$$C_F(\mathbb{T}_\tau[T_\theta]) \lessapprox (\theta/\tau)^{\kappa_{j-1}}, \qquad \frac{\mathrm{Vol}(U(\mathbb{T}_\tau[T_\theta], Y_{\mathbb{T}_\tau}))}{\mathrm{Vol}(T_\theta)} \gtrapprox^* (\tau/\theta)^{\kappa_{j-1}}\,(\tau/\theta)^{\sigma} \tag{5.3}$$

$$C_F(\mathbb{T}[T_\tau]) \lessapprox (\tau/\delta)^{\kappa_{j-1}}, \qquad \frac{\mathrm{Vol}(U(\mathbb{T}[T_\tau], Y))}{\mathrm{Vol}(T_\tau)} \gtrapprox^* (\delta/\tau)^{\kappa_{j-1}}\,(\delta/\tau)^{\sigma}. \tag{5.4}$$

In these equations and elsewhere, $\gtrapprox^*$ uses the δ and η parameters we started with, not the new scale and shading parameters for each sub-problem. $A \gtrapprox^* B$ means $A \geq C_\varepsilon \delta^{\varepsilon + C_\varepsilon \eta} B$ for all $\varepsilon > 0$. Multiplying these three inequalities gives

$$\mathrm{Vol}(U(\mathbb{T},Y)) \gtrapprox^* \delta^{\kappa+\kappa_{j-1}}\,\delta^{\sigma}. \tag{5.5}$$

We aim to improve on one of the three inequalities by a factor of $\delta^{-5\kappa_{j-1}}$, which will prove the result. If

$$\theta \leq \delta^{\varepsilon^2} \text{ and } \#\mathbb{T}_\theta\,\mathrm{Vol}(T_\theta) \geq \theta^{-\zeta'} \tag{5.6}$$

then we apply $\mathbf{A}_{\mathrm{HDL}}(\sigma, \zeta', \kappa')$. We check $C_F(\mathbb{T}_\theta) \lessapprox \delta^{-\kappa} \leq \delta^{-\kappa'}$, and the loss from the shading density gets absorbed into $\gtrapprox^*$ notation. $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta',\kappa')$ improves over (5.2) by a factor $\gtrapprox^* \theta^{-\kappa'} \geq \delta^{-\varepsilon^2\kappa'} \geq \delta^{-5\kappa_N}$. If

$$\tau \geq \delta^{1-\varepsilon^2} \text{ and } \frac{\#\mathbb{T}\,\mathrm{Vol}(T)}{\#\mathbb{T}_\tau\,\mathrm{Vol}(T_\tau)} \geq (\delta/\tau)^{-\zeta'} \tag{5.7}$$

then $\mathbf{A}_{\mathrm{HDL}}(\sigma,\zeta',\kappa')$ improves over (5.4) by a factor $\gtrapprox^* (\delta/\tau)^{-\kappa'} \geq \delta^{-\varepsilon^2\kappa'} \geq \delta^{-5\kappa_N}$.

If neither (5.6) nor (5.7) holds, then

$$\#\mathbb{T}_\theta\,\mathrm{Vol}(T_\theta) \leq \delta^{-2\varepsilon^2}\theta^{-\zeta} \qquad \text{and} \qquad \frac{\#\mathbb{T}\,\mathrm{Vol}(T)}{\#\mathbb{T}_\tau\,\mathrm{Vol}(T_\tau)} \leq \delta^{-2\varepsilon^2}(\delta/\tau)^{-\zeta}.$$

Let $\widetilde{\mathbb{T}}$ be obtained by rescaling $\mathbb{T}_\tau[T_\theta]$ so that T_θ becomes a tube of width 1 and T_τ becomes a tube of width $\tilde{\delta} = \tau/\theta$. Let $\widetilde{Y}$ be the rescaling of $Y_{\mathbb{T}_\tau}$. By uniformity of $\mathbb{T}$,

$$\#\widetilde{\mathbb{T}}\,\mathrm{Vol}(\widetilde{T}) \approx \frac{\#\mathbb{T}_\tau\,\mathrm{Vol}(T_\tau)}{\#\mathbb{T}_\theta\,\mathrm{Vol}(T_\theta)} \geq \delta^{4\varepsilon^2}\delta^{\zeta'-\zeta}(\tau/\theta)^{-\zeta'} \geq \tilde{\delta}^{-\zeta'+\frac{\zeta'-\zeta}{\varepsilon}+4\varepsilon} = \tilde{\delta}^{-\zeta'+5\varepsilon} \geq \tilde{\delta}^{-\zeta'/2}.$$

Thus $\widetilde{\mathbb{T}}$ is also a high-density set of tubes, although not quite as high-density as $\mathbb{T}$. We know

$$C_F(\widetilde{\mathbb{T}}[T_\rho]) \geq (\rho/\tilde{\delta})^{\kappa_j} \qquad \text{for all } \rho \in [\tilde{\delta}^{1-\varepsilon}, \tilde{\delta}^{\varepsilon}] \text{ and } T_\rho \in \mathbb{T}_\rho. \tag{5.8}$$

Choose $\rho = \tilde{\delta}^{\varepsilon}$. Factor $\widetilde{\mathbb{T}}[T_\rho]$ into convex sets. After pigeonholing and passing to a $\gtrapprox 1$ portion of $\mathbb{T}_\rho$, we may assume that each $\widetilde{\mathbb{T}}[T_\rho]$ factors into convex sets of the same dimensions. Thus there is a collection $\mathbb{V}$ of $a \times b \times 1$ sets and a decomposition $\widetilde{\mathbb{T}} = \bigsqcup_{V\in\mathbb{V}} \widetilde{\mathbb{T}}_V$ so that every tube in $\widetilde{\mathbb{T}}_V$ is contained in V, and $C_F(\widetilde{\mathbb{T}}_V, V) \lessapprox 1$. Moreover, because this is a max density factoring,

$$C_{KT}(\mathbb{V}[T_\rho]) \lessapprox 1, \qquad C_{KT}(\widetilde{\mathbb{T}}[T_\rho]) \approx C_{KT}(\widetilde{\mathbb{T}}_V) \approx \#\widetilde{\mathbb{T}}_V \frac{\mathrm{Vol}(\widetilde{T})}{\mathrm{Vol}(V)}.$$

The second equation above implies the tubes in $\widetilde{\mathbb{T}}_V$ must be high density,

$$\#\widetilde{\mathbb{T}}_V \frac{\operatorname{Vol}(\widetilde{T})}{\operatorname{Vol}(V)} \approx C_{KT}(\widetilde{\mathbb{T}}[T_\rho]) \geq \frac{\#\widetilde{\mathbb{T}} \operatorname{Vol}(\widetilde{T})}{\#\widetilde{\mathbb{T}}_\rho \operatorname{Vol}(\widetilde{T}_\rho)} \geq \rho^2 \tilde{\delta}^{-\zeta/2} \geq \delta^{-\zeta/2+2\varepsilon}.$$

By essential distinctness, $\#\widetilde{\mathbb{T}}_V \frac{\operatorname{Vol}(\widetilde{T})}{\operatorname{Vol}(V)} \lesssim \left(\frac{\operatorname{Vol}(\widetilde{T})}{\operatorname{Vol}(V)}\right)^{-1}$, implying $b^2 \geq \operatorname{Vol}(V) \gtrapprox \tilde{\delta}^{2-\zeta/2+2\varepsilon}$. Thus $b \geq \tilde{\delta}^{1-\varepsilon}$ for δ sufficiently small.

Let T_b be a $b \times b \times 1$ tube covering V. We estimate

$$C_F(\widetilde{\mathbb{T}}[T_b]) = \sup_{K \subset T_b} \frac{\#\widetilde{\mathbb{T}}[K]}{\#\widetilde{\mathbb{T}}[T_b] \operatorname{Vol}(K)/\operatorname{Vol}(T_b)} \leq \frac{\sum_{V \in \mathbb{V}[T_b]} \#\mathbb{T}_V[K]}{\sum_{V \in \mathbb{V}[T_b]} \#\mathbb{T}_V \operatorname{Vol}(K)/\operatorname{Vol}(T_b)}$$

$$\lessapprox \frac{\sum_{V \in \mathbb{V}[T_b]} \#\mathbb{T}_V \operatorname{Vol}(K \cap V)/\operatorname{Vol}(V)}{\sum_{V \in \mathbb{V}[T_b]} \#\mathbb{T}_V \operatorname{Vol}(K)/\operatorname{Vol}(T_b)} \leq (b/a).$$

On the other hand, by (5.8), $C_F(\widetilde{\mathbb{T}}[T_b], T_b) \gtrapprox (b/\tilde{\delta})^{-\kappa_j}$. Thus $b/a \gtrapprox (b/\tilde{\delta})^{-\kappa_j} \geq \tilde{\delta}^{-\varepsilon\kappa_j} \geq \delta^{-\varepsilon^2\kappa_j}$. By Corollary 4.8, (5.3) can be improved by a factor $\gtrapprox^* (b/a)^\sigma \gtrapprox \delta^{-\varepsilon^2\kappa_j\sigma} \geq \tilde{\delta}^{-5\kappa_{j-1}}$ as desired. □

The proof that $\mathbf{A}_{\mathrm{LDL}}(\sigma, \zeta, \kappa)$ holds is analogous, but we use Proposition 5.3 (B) and we replace volume estimates with multiplicity estimates.

Proof of low density lemma. Since $\#\mathbb{T} \operatorname{Vol}(T) \gtrapprox \delta^2$, the hypothesis is vacuous for $\zeta > 2$ and δ sufficiently small, so $\mathbf{A}_{\mathrm{LDL}}(\sigma, \zeta, \kappa)$ holds trivially for all $\kappa > 0$ when $\zeta > 2$. Define

$$\zeta = \sup\{\zeta' : \mathbf{A}_{\mathrm{LDL}}(\sigma, \zeta', \kappa) \text{ fails for all } \kappa > 0.\}.$$

Assume for contradiction that $\zeta > 0$. It suffices to show $\mathbf{A}_{\mathrm{LDL}}(\sigma, \zeta', \kappa)$ holds for some $\kappa > 0$.

We plan to apply Proposition 5.3 (B). Select parameters in the following order.

(1) Choose $\varepsilon \in (0, \zeta/20)$ small enough that Theorem 5.2 says: If $\mathbb{T}$ is $\delta^{-2\varepsilon}$-Katz-Tao at every scale with δ^ε-dense shading, then $\mu(\mathbb{T}, Y) \leq \delta^{-\sigma/2}$. Set $N = \lceil 25/\varepsilon^2 \rceil$.
(2) Set $\zeta' = \zeta + \varepsilon^2$. Choose $\kappa' > 0$ so that $\mathbf{A}_{\mathrm{LDL}}(\sigma, \zeta', \kappa')$ holds.
(3) Set $\kappa_N = \frac{1}{10}\kappa'\varepsilon^2$. Choose $\kappa_j \leq \frac{1}{10}\varepsilon^2\sigma\kappa_{j+1}$ for $j = N-1$ through $j = 0$. Set $\kappa = \kappa_0$. We also need $\kappa < \sigma/2$ for the sticky Kakeya case of Proposition 5.3.
(4) Choose η and δ sufficiently small for all applications below.

Let $\mathbb{T}$ be a set of δ-tubes in B_1 with $C_{KT}(\mathbb{T}) \leq \delta^{-\kappa}$ and $\#\mathbb{T} \operatorname{Vol}(T) \leq \delta^\zeta$, and let Y be a δ^η-dense shading. Apply Proposition 5.3 (B) with parameters $\kappa \leq \kappa_1 \leq \cdots \leq \kappa_N \leq \varepsilon$.

If we land in the first alternative, $\mathbb{T}$ is $\lessapprox \delta^{-\varepsilon}$-Katz–Tao at every scale. Theorem 5.2 gives $\mu(\mathbb{T}, Y) \leq \delta^{-\sigma/2} \leq \delta^{-\sigma+\kappa}$.

Suppose we land in the second alternative with scales $\delta \leq \tau \leq \theta \leq 1$ with $\tau \leq \delta^\varepsilon\theta$, and an integer $1 \leq j \leq N$. Using the multiplicity version of submodularity, we find (after harmless refinements)

$$\mu(\mathbb{T}, Y) \lessapprox^* \mu(\mathbb{T}_\theta, Y_{\mathbb{T}_\theta})\, \mu(\mathbb{T}_\tau[T_\theta], Y_{\mathbb{T}_\tau})\, \mu(\mathbb{T}[T_\tau]).$$

By $\mathbf{A}(\sigma)$,

$$C_{KT}(\mathbb{T}_\theta) \lessapprox \delta^{-\kappa}, \qquad \mu(\mathbb{T}_\theta, Y_{\mathbb{T}_\theta}) \lessapprox^* \delta^{-\kappa}\theta^{-\sigma} \tag{5.9}$$

$$C_{KT}(\mathbb{T}_\tau[T_\theta]) \lessapprox (\theta/\tau)^{\kappa_{j-1}}, \qquad \mu(\mathbb{T}_\tau[T_\theta], Y_{\mathbb{T}_\tau}) \lessapprox^* (\tau/\theta)^{-\kappa_{j-1}}\,(\tau/\theta)^{-\sigma} \tag{5.10}$$

$$C_{KT}(\mathbb{T}[T_\tau]) \lessapprox (\tau/\delta)^{\kappa_{j-1}}, \qquad \mu(\mathbb{T}[T_\tau], Y) \lessapprox^* (\delta/\tau)^{-\kappa_{j-1}}\,(\delta/\tau)^{-\sigma}. \tag{5.11}$$

Multiplying these three inequalities gives

$$\mu(\mathbb{T}, Y) \lessapprox^* \delta^{-\kappa-\kappa_{j-1}}\,\delta^{-\sigma}. \tag{5.12}$$

We will aim to improve on one of these three inequalities by a factor of $\delta^{5\kappa_{j-1}}$, which will prove the result. The Katz–Tao constants of each sub-problem are small enough to apply $\mathbf{A}_{\mathrm{LDL}}(\sigma,\zeta',\kappa')$. If

$$\theta \le \delta^{\varepsilon^2} \text{ and } \#\mathbb{T}_\theta \operatorname{Vol}(T_\theta) \le \theta^{\zeta'} \tag{5.13}$$

then $\mathbf{A}_{\mathrm{LDL}}(\sigma,\zeta',\kappa')$ improves over (5.9) by a factor $\lessapprox^* \delta^{\varepsilon^2\kappa'} \le \delta^{5\kappa_N}$. If

$$\tau \ge \delta^{1-\varepsilon^2} \text{ and } \frac{\#\mathbb{T}\operatorname{Vol}(T)}{\#\mathbb{T}_\tau \operatorname{Vol}(T_\tau)} \le (\delta/\tau)^{\zeta'} \tag{5.14}$$

then $\mathbf{A}_{\mathrm{LDL}}(\sigma,\zeta',\kappa')$ improves over (5.11) by a factor $\lessapprox^* \delta^{\varepsilon^2\kappa'} \le \delta^{5\kappa_N}$.

If neither (5.13) nor (5.14) holds, then

$$\#\mathbb{T}_\theta \operatorname{Vol}(T_\theta) \ge \delta^{2\varepsilon^2}\theta^{\zeta'} \qquad \text{and} \qquad \frac{\#\mathbb{T}\operatorname{Vol}(T)}{\#\mathbb{T}_\tau \operatorname{Vol}(T_\tau)} \ge \delta^{2\varepsilon^2}(\delta/\tau)^{\zeta'}.$$

Let $\widetilde{\mathbb{T}}$ be obtained by rescaling $\mathbb{T}_\tau[T_\theta]$ so that T_θ becomes a tube of width 1 and T_τ becomes a tube of width $\tilde{\delta} = \tau/\theta$. Let $\widetilde{Y}$ be the rescaling of $Y_{\mathbb{T}_\tau}$. By uniformity of $\mathbb{T}$,

$$\#\widetilde{\mathbb{T}}\operatorname{Vol}(\widetilde{T}) \approx \frac{\#\mathbb{T}_\tau \operatorname{Vol}(T_\tau)}{\#\mathbb{T}_\theta \operatorname{Vol}(T_\theta)} \le \delta^{-4\varepsilon^2}\delta^{-(\zeta'-\zeta)}(\tau/\theta)^{\zeta'} \le \tilde{\delta}^{\zeta'-\frac{\zeta'-\zeta}{\varepsilon}-4\varepsilon} = \tilde{\delta}^{\zeta'-5\varepsilon} \le \tilde{\delta}^{\zeta'/2}.$$

Thus $\widetilde{\mathbb{T}}$ is also a low-density set of tubes, though not quite as low-density as $\mathbb{T}$. We know

$$C_{KT}(\widetilde{\mathbb{T}}_\rho) \ge \rho^{-\kappa_j} \qquad \text{for } \rho \in [\tilde{\delta}^{1-\varepsilon}, \tilde{\delta}^{\varepsilon}].$$

Choose $\rho = \tilde{\delta}^{1-\varepsilon}$, and factor $\widetilde{\mathbb{T}}_\rho$ into $a \times b \times 1$ convex sets $\mathbb{V}$ with $\rho \le a \le b \le 1$. The max density factoring property gives

$$C_{KT}(\mathbb{V}) \lessapprox 1, \qquad C_F(\widetilde{\mathbb{T}}_\rho) \approx C_F(\mathbb{V}) \approx (\#\mathbb{V}\operatorname{Vol}(V))^{-1}.$$

Since

$$C_F(\widetilde{\mathbb{T}}_\rho) \gtrsim (\#\widetilde{\mathbb{T}}_\rho \operatorname{Vol}(T_\rho))^{-1} \gtrsim (\#\widetilde{\mathbb{T}}\operatorname{Vol}(\widetilde{T}))^{-1}(\tilde{\delta}/\rho)^2 \gtrapprox \tilde{\delta}^{-\zeta'/2+2\varepsilon} \ge \tilde{\delta}^{-\zeta'/4},$$

and $\#\mathbb{V}\operatorname{Vol}(V) \ge a^2$ by essential distinctness, we get $a \lessapprox \tilde{\delta}^{\zeta'/8}$.

We estimate the Katz–Tao constant of $\mathbb{T}_b$ in terms of $\mathbb{V}$,

$$C_{KT}(\mathbb{T}_b) \lessapprox C_{KT}(\mathbb{V})\frac{\operatorname{Vol}(T_b)}{\operatorname{Vol}(V)} \lessapprox (b/a).$$

If $b \ge \tilde{\delta}^{\varepsilon}$, then $b/a \ge \tilde{\delta}^{-\zeta'/8+\varepsilon} \ge \tilde{\delta}^{-\varepsilon}$. If $b \le \tilde{\delta}^{\varepsilon}$, then $C_{KT}(\mathbb{T}_b) \ge b^{-\kappa_j}$, so $b/a \gtrapprox \tilde{\delta}^{-\varepsilon\kappa_j}$. By Corollary 4.8, (5.10) can be improved by a factor $\lessapprox^* (a/b)^\sigma \lessapprox \delta^{\varepsilon^2\kappa_j} \le \delta^{5\kappa_{j-1}}$ as desired. ☐

The following lemma improves on $\mathbf{A}(\sigma)$ if $\mathbb{T}$ either fails to be Frostman or fails to be Katz–Tao.

Lemma 5.5 (Sticky Improvement Lemma)**.** *Assume $\mathbf{A}(\sigma)$ with $\sigma > 0$. For any $\zeta > 0$, there exists $\kappa(\zeta,\sigma) > 0$ such that the following holds for δ and η sufficiently small. If $\mathbb{T}$ is an essentially distinct set of δ-tubes in B_1 with a δ^η-dense shading, then*

$$C_F(\mathbb{T}) \ge \delta^{-\zeta} \text{ or } C_{KT}(\mathbb{T}) \ge \delta^{-\zeta} \Longrightarrow \operatorname{Vol}(U(\mathbb{T},Y)) \ge \delta^{\sigma-\kappa}\frac{1}{C_F(\mathbb{T})}. \tag{5.15}$$

There is a useful Corollary. Suppose $\max\{C_F(\mathbb{T}), C_{KT}(\mathbb{T})\} \ge M \ge \delta^{-\zeta}$. Because $\max\{C_F(\mathbb{T}), C_{KT}(\mathbb{T})\} \lesssim \delta^{-2}$, this implies $\delta \lesssim M^{-1/2}$, so

$$\max\{C_F(\mathbb{T}), C_{KT}(\mathbb{T})\} \ge M \ge \delta^{-\zeta} \Longrightarrow \operatorname{Vol}(U(\mathbb{T},Y)) \gtrsim M^{\kappa/2}\,\delta^\sigma\frac{1}{C_F(\mathbb{T})}. \tag{5.16}$$

Proof. Given $\sigma,\zeta > 0$, choose parameters as follows.

(1) Choose $\kappa_0 > 0$ so that $\mathbf{A}_{\mathrm{LDL}}(\sigma,\zeta,\kappa_0)$ holds.

(2) Choose $\kappa_1 > 0$ so that $\mathbf{A}_{\mathrm{HDL}}(\sigma, \kappa_0/2, \kappa_1)$ holds.
(3) Set $\kappa = \frac{1}{8}\kappa_0\kappa_1\sigma$.
(4) Choose η and δ sufficiently small for all applications below.

Let $(\mathbb{T}, Y)$ be an essentially distinct set of δ-tubes in B_1 along with a δ^η-dense shading. We aim to show $\mathrm{Vol}(U(\mathbb{T}, Y)) \geq \delta^{\sigma-\kappa}\frac{1}{C_F(\mathbb{T})}$. If $C_{KT}(\mathbb{T}) \leq \delta^{-\kappa_0}$, apply $\mathbf{A}_{\mathrm{LDL}}(\sigma, \zeta, \kappa_0)$ to prove the result.

From here on, assume $C_{KT}(\mathbb{T}) \geq \delta^{-\kappa_0}$. Apply Lemma 4.1 to $(\mathbb{T}, Y)$. After passing to $\gtrapprox 1$ refinements, we get a family $\mathbb{W}$ of $a \times b \times 1$ convex sets, and a partition $\mathbb{T} = \bigsqcup_{W \in \mathbb{W}} \mathbb{T}_W$ where each tube in $\mathbb{T}_W$ is contained in W. For all $W \in \mathbb{W}$,

$$C_F(\mathbb{T}_W, W) \approx 1 \qquad \text{and} \qquad C_{KT}(\mathbb{T}_W) \approx \#\mathbb{T}_W \frac{\mathrm{Vol}(T)}{\mathrm{Vol}(W)} \approx C_{KT}(\mathbb{T}) \geq \delta^{-\kappa_0}. \tag{5.17}$$

By Corollary 4.8, $\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox (b/a)^\sigma \delta^\sigma \frac{1}{C_F(\mathbb{T})}$. If $b/a \geq \delta^{-2\kappa/\sigma}$, then (5.15) holds.

Thus we may assume $b/a \leq \delta^{-2\kappa/\sigma}$. Apply the submodular decomposition (by abuse of notation, replacing $\mathbb{T}$, Y, etc. with refinements)

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox \mathrm{Vol}(U(\mathbb{W}, Y_{\mathbb{W}})) \frac{\mathrm{Vol}(U(\mathbb{T}_W, Y))}{\mathrm{Vol}(W)}.$$

By Lemma 4.7,

$$\mathrm{Vol}(U(\mathbb{W}, Y_{\mathbb{W}}) \gtrapprox^* b^\sigma \frac{1}{C_F(\mathbb{W})} \gtrapprox^* b^\sigma \frac{1}{C_F(\mathbb{T})}. \tag{5.18}$$

Next, fix any $W \in \mathbb{W}$ and let T_b be a $b \times b \times 1$ tube containing W. Let $\widetilde{\mathbb{T}}$ be obtained by rescaling $\mathbb{T}_W$ so that T_b becomes a tube of width 1 and δ-tubes become $\tilde{\delta} = \delta/b$ tubes. By (5.17),

$$\#\widetilde{\mathbb{T}}\,\mathrm{Vol}(\widetilde{T}) = \#\mathbb{T}_W \frac{\mathrm{Vol}(T)}{\mathrm{Vol}(W)} \frac{\mathrm{Vol}(W)}{\mathrm{Vol}(T_b)} \gtrapprox \delta^{-\kappa_0 + 2\kappa/\sigma} \geq \delta^{-\kappa_0/2}.$$

By essential distinctness, $\#\widetilde{\mathbb{T}}\,\mathrm{Vol}(\widetilde{T}) \lesssim \tilde{\delta}^{-2}$, so $\tilde{\delta} \lesssim \delta^{\kappa_0/4}$. For sufficiently small δ this gives

$$C_F(\widetilde{\mathbb{T}}) = C_F(\mathbb{T}_W, T_b) \lessapprox b/a \leq \delta^{-2\kappa/\sigma} \leq \tilde{\delta}^{-\kappa_1}, \qquad \#\widetilde{\mathbb{T}}\,\mathrm{Vol}(\widetilde{T}) \geq \tilde{\delta}^{-\kappa_0/2}.$$

Apply $\mathbf{A}_{\mathrm{HDL}}(\sigma, \kappa_0/2, \kappa_1)$ to obtain $\mathrm{Vol}(U(\widetilde{\mathbb{T}}, \widetilde{Y})) \geq \tilde{\delta}^{\sigma-\kappa_1}$. Combining with (5.18), we get $\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \tilde{\delta}^{-\kappa_1}\delta^\sigma \frac{1}{C_F(\mathbb{T})}$. For δ sufficiently small, the improvement factor is $\geq \delta^{-\frac{1}{8}\kappa_0\kappa_1}$, as needed. □

6. Lossless decomposition

Let $\mathbb{T}$ be a collection of δ-tubes and let $\mathbb{V}$ be a collection of $a \times b \times \rho$ convex sets. We say T is incident to V if $\mathrm{Vol}(T \cap V) \approx \delta^2\rho$. If this holds, then $T \cap V$ is a $\delta \times \delta \times \rho$ tubelet. We define a *shading by convex sets* to be a subset $\mathcal{G} \subset \mathbb{T} \times \mathbb{V}$ supported on incident pairs. By analogy with usual shadings, we define

$$\text{For } T \in \mathbb{T}, \qquad \mathcal{G}(T) = \{V \in \mathbb{V} : (T, V) \in \mathcal{G}\},$$
$$\text{For } V \in \mathbb{V}, \qquad \mathbb{T}_{\mathcal{G}}(V) = \{T \in \mathbb{T} : (T, V) \in \mathcal{G}\}.$$

There is an induced shading $Y_{\mathcal{G}}$ on $\mathbb{T}$, given by

$$Y_{\mathcal{G}}(T) = T \cap (\cup \mathcal{G}(T)).$$

We define the density using the induced shading, $\lambda(\mathcal{G}) := \lambda(Y_{\mathcal{G}})$.

Lemma 6.1 (Lossless Decomposition). *Assume $\mathbf{A}(\sigma)$. Let $\mathbb{T}$ be a set of δ-tubes in B_1 with $C_F(\mathbb{T}) \leq \delta^{-\eta}$, let Y be a δ^η-dense shading, and let $\rho \in [\delta, 1]$ be an intermediate scale. The following holds for δ and η sufficiently small in terms of ε. There exist scales $\delta \leq a \leq b \leq \rho$, a collection $\mathbb{V}$ of essentially distinct $a \times b \times \rho$ convex bodies, and a $\delta^{2\eta}$-dense shading $\mathcal{G}$ of $\mathbb{T}$ by $\mathbb{V}$, such that*

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \geq \delta^\varepsilon \big(\frac{\delta}{\rho}\frac{b}{a}\big)^\sigma \#\mathbb{V}\,\mathrm{Vol}(V).$$

Proof. Let $\mathbb{B}$ be a minimal cover of $U(\mathbb{T}, Y)$ by ρ-balls. For each $B \in \mathbb{B}$, define $\mathbb{T}_B = \{T \cap 2B : T \in \mathbb{T}$ intersects $B\}$, and $Y_{\mathbb{T}_B}(T \cap B) = Y(T) \cap B$.

Apply factoring to $\mathbb{T}_B$. This produces a $\gtrapprox 1$ subcollection $\mathbb{T}'_B$; a family of $a \times b \times \rho$ convex set $\mathbb{V}_B$; and a decomposition $\mathbb{T}'_B = \bigsqcup_{V \in \mathbb{V}_B} (\mathbb{T}_{B'})_V$, where each tubelet in $(\mathbb{T}_{B'})_V$ is contained in V. Moreover, $C_F(\mathbb{T}'_B) \approx (\#\mathbb{V}_B \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(B)})^{-1}$. By $\mathbf{A}(\sigma)$,

$$\frac{\mathrm{Vol}(U(\mathbb{T}, Y) \cap B)}{\mathrm{Vol}(B)} = \frac{\mathrm{Vol}(U(\mathbb{T}_B, Y_{\mathbb{T}_B}))}{\mathrm{Vol}(B)} \gtrapprox \frac{1}{C_F(\mathbb{T}'_B, 2B)} (\frac{\delta}{\rho}\frac{b}{a})^\sigma \gtrapprox (\#\mathbb{V}_B \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(B)}) (\frac{\delta}{\rho}\frac{b}{a})^\sigma.$$

After a $\gtrapprox 1$ refinement we may assume the balls of $\mathbb{B}$ are disjoint and the dimensions $a \times b \times \rho$ are roughly constant over $B \in \mathbb{B}$. Let $\mathbb{V} = \bigcup_B \mathbb{V}_B$. With this definition, $\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox \#\mathbb{V}\,\mathrm{Vol}(V)(\frac{\delta}{\rho}\frac{b}{a})^\sigma$.

There is a natural shading $\mathcal{G}$ between $\mathbb{T}$ and $\mathbb{V}$. For each $B \in \mathbb{B}$, $V \in \mathbb{V}_B$, and $T \in (\mathbb{T}'_B)_V$, we include the pair (T, V) in $\mathcal{G}$. It remains to check the density of this shading.

For $T \in \mathbb{T}$, the convex sets in $\mathcal{G}(T)$ are roughly disjoint. Thus $\lambda(\mathcal{G}) \approx \frac{1}{\#\mathbb{T}}\rho\#\mathcal{G}$. We have

$$\#\mathcal{G} = \sum_{B \in \mathbb{B}} \#\mathbb{T}'_B \gtrapprox \sum_{B \in \mathbb{B}} \#\mathbb{T}_B \gtrapprox \sum_{T \in \mathbb{T}} \#\{B \in \mathbb{B} \text{ intersecting } T\} \gtrapprox \sum_{T \in \mathbb{T}} |Y(T)|_\rho \gtrapprox \#\mathbb{T}\delta^\eta\rho^{-1}$$

as needed. □

Lemma 6.2 (Shadings by convex sets). *Assume $\mathbf{A}(\sigma)$. Let $\mathbb{T}$ be a collection of δ-tubes in B_1 with $C_F(\mathbb{T}) \leq \delta^{-\eta}$, let $\mathbb{V}$ be a collection of $a \times b \times \rho$ convex sets, and let $\mathcal{G}$ be a δ^η-dense shading of $\mathbb{T}$ by $\mathbb{V}$. If δ, η are small enough in terms of ε, then $\#\mathbb{V}\,\mathrm{Vol}(V) \geq \delta^\varepsilon \big(\rho\frac{a}{b}\big)^\sigma$.*

Moreover, for all $\zeta > 0$, there exists $\kappa = \kappa(\zeta, \sigma) > 0$ such that the following holds. Assume $\mathbb{T}$ is uniform, and assume that in addition to the above, either $\#\mathbb{T}_a \mathrm{Vol}(T_a) \geq \delta^{-\zeta}$ or $\#\mathbb{T}_{b/\rho} \mathrm{Vol}(T_{b/\rho}) \geq \delta^{-\zeta}$. Then $\#\mathbb{V}\,\mathrm{Vol}(V) \geq \delta^{-\kappa}(\rho\frac{a}{b})^\sigma$.

Proof. After a harmless refinement, we may assume $\mathbb{T}$ is uniform. All the action happens between scales a and b/ρ.

Fix a b/ρ tube $T_{b/\rho} \in \mathbb{T}_{b/\rho}$. Define $\mathbb{V}_{T_{b/\rho}}$ to be the union of all the convex sets $\mathcal{G}(T)$ over $T \in \mathbb{T}[T_{b/\rho}]$. If $V \in \mathbb{V}_{T_{b/\rho}}$, that means the b-neighborhood of V, which is a $b \times b \times \rho$ tubelet, is angularly aligned with $T_{b/\rho}$. So, each $V \in \mathbb{V}$ is contained in $\lesssim 1$ of the families $\mathbb{V}_{T_{b/\rho}}$. Thus

$$\#\mathbb{V}\,\mathrm{Vol}(V) \gtrsim \mathrm{Vol}(T_{b/\rho}) \sum_{T_{b/\rho} \in \mathbb{T}_{b/\rho}} \#\mathbb{V}_{T_{b/\rho}} \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(T_{b/a})}. \tag{6.1}$$

Next, define a shading on $\mathbb{T}_a$ by

$$Y_{\mathbb{T}_a}(T_a) = T_a \cap (\bigcup_{T \in \mathbb{T}[T_a]} \mathcal{G}(T)).$$

Because the convex sets in $\mathbb{V}$ have thickness a, the shading density is lower bounded by

$$\mathrm{Vol}(Y_{\mathbb{T}_a}(T_a)) \gtrsim \sup_{T \in \mathbb{T}[T_a]} |Y_{\mathcal{G}}(T)|_\rho \rho \gtrsim \lambda(\mathbb{T}[T_a], \mathcal{G}),$$

and thus $\lambda(Y_{\mathbb{T}_a}) \gtrsim \lambda(\mathbb{T}, \mathcal{G}) \geq \delta^\eta$. We have $U(\mathbb{T}_a[T_{b/\rho}], Y_{\mathbb{T}_a}) \subset \cup \mathbb{V}_{T_{b/\rho}}$, so by $\mathbf{A}(\sigma)$,

$$\#\mathbb{V}_{T_{b/\rho}} \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(T_{b/a})} \geq \frac{U(\mathbb{T}_a[T_{b/\rho}], Y_{\mathbb{T}_a})}{\mathrm{Vol}(T_{b/a})} \gtrapprox^* \big(\rho \frac{a}{b}\big)^\sigma \frac{1}{C_F(\mathbb{T}_a[T_{b/\rho}])}. \tag{6.2}$$

Thus, by (6.1),

$$\#\mathbb{V} \,\mathrm{Vol}(V) \gtrapprox^* \big(\rho \frac{a}{b}\big)^\sigma \#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho}) \frac{1}{\sup_{T_{b/\rho} \in \mathbb{T}_{b/\rho}} C_F(\mathbb{T}_a[T_{b/\rho}]).} \tag{6.3}$$

Let us estimate the Frostman constant. Using that $C_F(\mathbb{T}_a) \lesssim C_F(\mathbb{T})$,

$$\begin{aligned} C_F(\mathbb{T}_a[T_{b/\rho}], T_{b/\rho}) = \sup_{T_a \subset K \subset T_b} \frac{\#\mathbb{T}_a[K]}{\#\mathbb{T}_a[T_{b/\rho}] \,\mathrm{Vol}(K)/\,\mathrm{Vol}(T_{b/\rho})} &\lesssim C_F(\mathbb{T}) \frac{\#\mathbb{T}_a}{\#\mathbb{T}_a[T_{b/\rho}]} \,\mathrm{Vol}(T_{b/\rho}) \\ &\lesssim C_F(\mathbb{T}) \,\#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho}). \end{aligned} \tag{6.4}$$

If we plug this bound into (6.3), the term $\#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho})$ cancels, so $\#\mathbb{V} \,\mathrm{Vol}(V) \gtrapprox^* (\rho \frac{a}{b})^\sigma \frac{1}{C_F(\mathbb{T})}$.

We move on to discussing improved bounds. Set $\zeta' = \zeta/4$, and let $\kappa' > 0$ be the output of Lemma 5.5. Set $\kappa = \min\{\kappa'/2, \zeta/4\}$. Fix $T_{b/\rho} \in \mathbb{T}_{b/\rho}$. We get an improvement for (6.2), and then sum over all $T_{b/\rho} \in \mathbb{T}_{b/\rho}$.

By (5.16), if $\#\mathbb{T}_a[T_{b/\rho}] \frac{\mathrm{Vol}(T_a)}{\mathrm{Vol}(T_{b/\rho})} \geq \delta^{-\zeta'}$, then our final bound improves by a factor of $\delta^{-\kappa'/2}$. If not, that implies $\#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho}) \geq \delta^{\zeta'} \#\mathbb{T}_a \,\mathrm{Vol}(T_a)$.

Also by (5.16), if $C_F(\mathbb{T}_a[T_{b/\rho}], T_{b/\rho}) \geq \delta^{-\zeta'}$, then our final bound improves by a factor of $\delta^{-\kappa'/2}$. If not, our bound for the Frostman constant in (6.4) is suboptimal by a factor of

$$\frac{C_F(\mathbb{T}) \,\#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho})}{C_F(\mathbb{T}_a[T_{b/\rho}], T_{b/\rho})} \gtrapprox \delta^{-\eta} \delta^{\zeta'} \#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho}) \gtrapprox \delta^{-\eta} \delta^{2\zeta'} \#\mathbb{T}_a \,\mathrm{Vol}(T_a),$$

so the bound improves by this factor. Overall,

$$\#\mathbb{V} \ \mathrm{Vol}(V) \gtrapprox^* (\rho \tfrac{a}{b})^\sigma \ \min\{\delta^{-\kappa'/2}, \max\{\delta^{2\zeta'} \#\mathbb{T}_a \,\mathrm{Vol}(T_a), \delta^{\zeta'} \#\mathbb{T}_{b/\rho} \,\mathrm{Vol}(T_{b/\rho})\}\} \gtrapprox^* (\rho \tfrac{a}{b})^\sigma \delta^{-\kappa}$$

as desired. □

Lemma 6.3. *Assume $\mathbf{A}(\sigma)$ holds. Set $\nu = \min\{\sigma/20, 1/20\}$. For any $\zeta > 0$, there exists $\kappa = \kappa(\sigma, \zeta) > 0$ such that the following holds for δ and η sufficiently small.*

Let $\mathbb{T}$ be a uniform collection of δ-tubes in B_1 with $C_F(\mathbb{T}) \leq \delta^{-\eta}$ and let Y be a δ^η-dense shading. If $\#\mathbb{T}_w \,\mathrm{Vol}(T_w) \geq \delta^{-\zeta}$ for all $w \in [\delta^{1-\nu}, \delta^\nu]$, then $\mathrm{Vol}(U(\mathbb{T}, Y)) \geq \delta^{\sigma - \kappa}$.

Proof. Choose $\kappa \in (0, \sigma/2)$ so that Lemma 6.2 applies with parameters ζ and κ. Also enforce $\kappa \leq \sigma/2$. Set $\rho = \delta^\nu$.

Apply Lemma 6.1 at scale ρ to produce scales $\delta \leq a \leq b \leq \rho$, a collection $\mathbb{V}$ of essentially distinct $a \times b \times \rho$ convex bodies, and a $\delta^{2\eta}$-dense shading $\mathcal{G}$ of $(\mathbb{T}, \mathbb{V})$, such that

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \big(\frac{\delta}{\rho} \frac{b}{a}\big)^\sigma \#\mathbb{V} \,\mathrm{Vol}(V). \tag{6.5}$$

We automatically know $a \leq \delta^\nu$ and $b/\rho \geq \delta^{1-\nu}$. If $a \geq \delta^{1-\nu}$ or $b/\rho \leq \delta^\nu$, then Lemma 6.2 applies and gives $\#\mathbb{V} \,\mathrm{Vol}(V) \gtrapprox^* \delta^{-\kappa} \big(\rho \frac{a}{b}\big)^\sigma$, which gives $\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \delta^{\sigma - \kappa'}$ as desired.

In the remaining case, $a \leq \delta^{1-\nu}$ and $b \geq \rho \delta^\nu$. The tubes incident to $V \in \mathbb{V}$ are all contained in an $\frac{a}{\rho} \times \frac{b}{\rho} \times 1$ convex body $\bar{V}$. Thus

$$\#\mathbb{T}_{\mathcal{G}}(V) \leq \#\mathbb{T}[\bar{V}] \leq C_F(\mathbb{T}) \,\mathrm{Vol}(\bar{V}) \#\mathbb{T} \lessapprox^* \#\mathbb{T} \,\mathrm{Vol}(V) \,\rho^{-2}.$$

Summing over all $V \in \mathbb{V}$ gives $\#\mathbb{T} \lessapprox^* \#\mathbb{T}\#\mathbb{V}\,\mathrm{Vol}(V)\rho^{-2}$, so $\#\mathbb{V}\,\mathrm{Vol}(V) \gtrapprox^* \rho^2$. Plugging into (6.5) gives

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \delta^{2\nu\sigma}\rho^2 \gtrapprox^* \delta^{\sigma/5}$$

as needed. □

Theorem 6.4 (Inductive step)**.** *Assume* $\mathbf{A}(\sigma)$ *holds. Then* $\mathbf{A}(\sigma - \kappa)$ *holds for some* $\kappa > 0$*.*

Proof. We will apply Proposition 5.3 (A) with parameters $\kappa \le \kappa_1 \le \cdots \le \kappa_N \le \varepsilon$. Choose parameters as follows.

1. Choose ε small enough that Theorem 5.2 gives: If $\mathbb{T}$ is $\le \delta^{-2\varepsilon}$-Frostman at every scale, then $\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \delta^{\sigma/2}$. Additionally, make sure $\varepsilon \le \min\{\sigma/20, 1/20\}$ for the application of Lemma 6.2. Set $N = \lceil 25/\varepsilon^2 \rceil$.
2. Set $\kappa_N = \varepsilon$. Iteratively define κ_{j-1} in terms of κ_j as follows.
   Select $r_j > 0$ so that Lemma 6.3 applies with $(\nu, \zeta, \kappa) = (\varepsilon, \frac{1}{2}\varepsilon\kappa_j, r_j)$. Then, select $\kappa_{j-1} \le \min\{\frac{1}{3}\varepsilon\kappa_j, \frac{1}{5}\varepsilon r_j\}$. Continue down to $j = 0$, and set $\kappa = \kappa_0$. Also enforce $\kappa < \sigma/2$.

Let $\mathbb{T}$ be an essentially distinct set of tubes in B_1 with $C_F(\mathbb{T}) \le \delta^{-\eta}$, and let Y be a δ^η-dense shading. Apply Proposition 5.3 (A).

If we land in the first alternative, $\mathbb{T}$ is $\lessapprox \delta^{-\varepsilon}$-Frostman at every scale. Theorem 5.2 implies $\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox^* \delta^{\sigma/2} \ge \delta^{\sigma-\kappa}$.

Suppose we land in the second alternative with scales $\delta \le \tau \le \theta \le 1$ with $\tau \le \delta^\varepsilon\theta$, and an integer $1 \le j \le N$. We know

- $C_F(\mathbb{T}[T_\tau], T_\tau) \lessapprox (\tau/\delta)^{\kappa_{j-1}}$ for all $T_\tau \in \mathbb{T}_\tau$.
- $C_F(\mathbb{T}_\tau[T_\theta], T_\theta) \lessapprox (\theta/\tau)^{\kappa_{j-1}}$ for all $T_\theta \in \mathbb{T}_\theta$.
- For all $\rho \in [\tau(\theta/\tau)^\varepsilon, \theta(\tau/\theta)^\varepsilon]$, we have $C_F(\mathbb{T}_\tau[T_\rho]) \ge (\rho/\tau)^{\kappa_j}$ for all $T_\rho \in \mathbb{T}_\rho$.

Apply the submodular decomposition Proposition 4.3. After passing to $\gtrapprox 1$ refinements,

$$\mathrm{Vol}(U(\mathbb{T}, Y)) \gtrapprox \mathrm{Vol}(U(\mathbb{T}_\theta, Y_{\mathbb{T}_\theta}))\,\frac{\mathrm{Vol}(U(\mathbb{T}_\tau[T_\theta], Y_{\mathbb{T}_\tau}))}{\mathrm{Vol}(T_\theta)}\,\frac{\mathrm{Vol}(U(\mathbb{T}[T_\tau], Y))}{\mathrm{Vol}(T_\tau)}.$$

By $\mathbf{A}(\sigma)$, these three terms are bounded below by $\gtrapprox^* \delta^{\kappa_{j-1}}\theta^\sigma$, $\gtrapprox^* \delta^{\kappa_{j-1}}(\tau/\theta)^\sigma$, and $\gtrapprox^* \delta^{\kappa_{j-1}}(\delta/\tau)^\sigma$ respectively. We focus on the second term.

Let $\widetilde{\mathbb{T}}$ be obtained by rescaling $\mathbb{T}_\tau[T_\theta]$ so that T_θ becomes a tube of width 1 and T_τ becomes a tube of width $\tilde{\delta} = \tau/\theta$. Let $\tilde{Y}$ be the rescaling of $Y_{\mathbb{T}_\tau}$. We have

$$C_F(\widetilde{\mathbb{T}}[T_\rho]) \ge (\rho/\tilde{\delta})^{\kappa_j} \qquad \text{for all } \rho \in [\tilde{\delta}^{1-\varepsilon}, \tilde{\delta}^\varepsilon].$$

On the other hand, by the same calculation as (6.4),

$$C_F(\widetilde{\mathbb{T}}[T_\rho]) \lesssim C_F(\widetilde{\mathbb{T}})\,\#\widetilde{\mathbb{T}}_\rho\,\mathrm{Vol}(\widetilde{T}_\rho),$$

so

$$\#\widetilde{\mathbb{T}}_\rho\,\mathrm{Vol}(\widetilde{T}_\rho) \gtrsim \tilde{\delta}^{-\varepsilon\kappa_j+\kappa_{j-1}} \ge \tilde{\delta}^{-\frac{1}{2}\varepsilon\kappa_j} \qquad \text{for all } \rho \in [\tilde{\delta}^{1-\varepsilon}, \tilde{\delta}^\varepsilon].$$

Apply Lemma 6.3 to improve overall by a factor of $\tilde{\delta}^{-r_j} \ge \delta^{-\varepsilon r_j} \ge \delta^{-5\kappa_{j-1}}$. □

Proof of Theorem 2.1. The set of $\sigma > 0$ for which $\mathbf{A}(\sigma)$ holds is a closed set of the form $[\sigma^*, \infty)$. If $\sigma^* > 0$, then Theorem 6.4 provides a contradiction. Thus, $\mathbf{A}(0)$ holds. □

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