KAKEYA MINICOURSE PART PROBLEM SET

ALEX COHEN

1. The L^2 method

The goal of this problem is to explore the L^2 method and pigeonholing. First recall some definitions.

- $\mathbb{T}$ is a collection of essentially distinct $\delta \times 1$ tube segments in $B(10) \subset \mathbb{R}^2$, meaning
$$\mathrm{Vol}(T_1 \cap T_2) \leq \frac{1}{2}\mathrm{Vol}(T_1) \qquad \text{for all distinct } T_1, T_2 \in \mathbb{T}.$$
- For any convex set U,
$$\mathbb{T}[U] = \{T \in \mathbb{T} : T \subset U\}.$$
- The *Frostman constant* is defined by
$$C_F(\mathbb{T}) = \sup_{U \text{ is a convex set}} \frac{\#\mathbb{T}[U]}{\#\mathbb{T}\,\mathrm{Vol}(U)}.$$

Start by proving the following Lemma using the L^2 method.

Lemma 1.1. *We have*
$$\mathrm{Vol}(\cup\mathbb{T}) \gtrsim (\log \delta^{-1})^{-1}\frac{1}{C_F(\mathbb{T})}.$$

Hint: Estimate $\int\big(\sum_{T\in\mathbb{T}} 1_T\big)^2 dx$ using the Frostman constant. Then apply Cauchy–Schwarz,
$$\int \sum_{T\in\mathbb{T}} 1_T\, dx \leq \Big(\int \big(\sum_{T\in\mathbb{T}} 1_T\big)^2 dx\Big)^{1/2} (\mathrm{Vol}(\cup\mathbb{T}))^{1/2}.$$

Next, use maximum density sets and pigeonholing to prove the following consequence.

Theorem 1.2. *Let $\mathbb{T}$ be a collection of essentially distinct $\delta\times 1$ tube segments contained in $B(10) \subset \mathbb{R}^2$. Then there exists*

- *A subcollection $\mathbb{T}'' \subset \mathbb{T}$ with $\#\mathbb{T}'' \gtrsim (\log\delta^{-1})^{-10}\#\mathbb{T}$,*
- *A scale $w \in [\delta, 1]$, and*

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- *a collection $\mathbb{T}^w$ of essentially distinct $w \times 10$ tubes such that every $T \in \mathbb{T}''$ is contained in some $T^w \in \mathbb{T}^w$,*

for which

$$\mathrm{Vol}(\cup \mathbb{T}''[T^w]) \gtrsim (\log \delta^{-1})^{-1} \mathrm{Vol}(T^w) \qquad \textit{for each } T^w \in \mathbb{T}^w, \tag{1.1}$$

and

$$\mathrm{Vol}(\cup \mathbb{T}^w) \gtrsim (\log \delta^{-1})^{-1} \# \mathbb{T}^w \, \mathrm{Vol}(T^w). \tag{1.2}$$

Here is a suggested proof outline.

(a) **Decomposition into maximal-density sets.**. Start with $\mathbb{T}_1 = \mathbb{T}$. Let T^{w_1} be a $w_1 \times 10$ tube maximizing the density

$$\frac{\#\mathbb{T}_1[T^{w_1}]}{\#\mathbb{T}_1 \, \mathrm{Vol}(T^{w_1})}.$$

Then set $\mathbb{T}_2 = \mathbb{T}_1 \setminus \mathbb{T}[T^{w_1}]$, and continue in this way until $\#\mathbb{T}_m \leq \frac{1}{2}\#\mathbb{T}$. Set

$$\mathbb{T}' = \bigsqcup_{j=1}^{m-1} \mathbb{T}_j[T^{w_j}].$$

(b) **Dyadic pigeonholing.** Use dyadic pigeonholing to find a scale $w \in [\delta, 1]$, a number $r \in [1, \#\mathbb{T}]$, and an index set $\mathcal{J} \subset \{1, \dots, m-1\}$ satisfying the following. For each $j \in \mathcal{J}$,

$$w_j \in [w/2, w] \qquad \text{and} \qquad \#\mathbb{T}_j[T^{w_j}] \in [r/2, r].$$

Moreover, letting

$$\mathbb{T}'' = \bigsqcup_{j \in \mathcal{J}} \mathbb{T}_j[T^{w_j}],$$

$\#\mathbb{T}'' \gtrsim (\log \delta^{-1})^{-10} \#\mathbb{T}$. In the theorem statement, set

$$\mathbb{T}^w = \{\text{A } w\text{-tube containing } T^{w_j} \text{ for each } j \in \mathcal{J}\}.$$

(c) **Application of the L^2 bound.** Apply Lemma 1.1 to each $\mathbb{T}''[T^w]$ to deduce (1.1), and to $\mathbb{T}^w$ to deduce (1.2).

2. Problem 2

This is an exploratory problem that should be graded for completion, not correctness.

Consider applying the lossless decomposition. Suppose

- We choose a scale $\rho \in [\delta, 1]$.
- The tubelet set $\mathbb{T}_B$ inside a ρ-ball has $\delta \times b \times \rho$ planks for the maximum density convex set, where $b \in (\delta, \rho)$.
- These $\delta \times b \times \rho$ planks intersect tangentially (c.f. Lec 4).

Describe the Kakeya sub-problems that come up when we apply the lossless decomposition. Don't worry about getting a gain over $\mathbf{A}(\sigma)$ using high density or low density lemma. And don't worry about being rigorous—just explore how we make new tube sets from these configurations of tubes and planks.