THE PROOF OF KAKEYA, FOLLOWING WANG–ZAHL

ALEX COHEN

1. Lecture 1: Introduction

These are lecture notes from a two week minicourse at MIT about Wang and Zahl's [5] proof of the Kakeya conjecture. Some other useful resources are

- Wang and Zahl's article [5],
- Guth's introduction to the proof [2],
- Guth's proof outline [1].

Before getting into the details, we establish some notation.

1.1. Notation and convention.

- $\mathbb{T}$ is a set of $\delta \times \delta \times 1$ tube segments in $B(10) \subset \mathbb{R}^3$
- The tubes in $\mathbb{T}$ are essentially distinct: For any $T_1, T_2 \in \mathbb{T}$,
$$\mathrm{Vol}(T_1 \cap T_2) \leq \frac{1}{2} \mathrm{Vol}(T_1).$$
- $\mathbb{T}^\rho$ is the set of ρ-tubes involved in covering $\mathbb{T}$. For $E \subset \mathbb{R}^3$, E^ρ is the ρ-neighborhood of E.
- For a convex set U, $\mathbb{T}[U] = \{T \in \mathbb{T} : T \subset U\}$.
- Shadings: Each tube $T \in \mathbb{T}$ is equipped with a subset $Y(T) \subset T$ with
$$\mathrm{Vol}(Y(T)) \geq \delta^\eta \mathrm{Vol}(T),$$
where $\eta > 0$ is a small parameter. We will ignore the shading in our notation: when we write
$$\cup \mathbb{T} \quad \text{or} \bigcup_{T \in \mathbb{T}} T,$$
we really mean
$$\bigcup_{T \in \mathbb{T}} Y(T)$$
for some δ^η-dense shading $Y(T)$.
- $A \gtrsim B$ means $A \geq \frac{1}{C} B$ for some absolute constant C. $A \gg B$ means A is much greater than B. This is an informal statement, but generally one should think $A \geq \delta^{-\kappa} B$ for some fixed $\kappa > 0$.

Date: December 1, 2025.

- In the proofs, η is a small parameter controlling the shading density, Ahlfors–David regularity, and possibly other things. $A \gtrapprox_\eta B$ means that for any $\varepsilon > 0$, $A \geq c_\varepsilon \delta^{\varepsilon + C\eta}$. Here C is a universal constant. We will sometimes drop the subscript η and just write $A \gtrapprox B$.
- We make a standing assumption that quantities have been pigeonholed so that they are essentially constant. In particular, each ρ-tube $T^\rho \in \mathbb{T}^\rho$ contains roughly the same number of δ-tubes, and for any ρ-ball $B = B(x_0, \rho)$ centered at a point of $E = \cup\mathbb{T}$, the measure of $E \cap B$ is approximately uniform.

1.2. **The Frostman Condition.** The *Frostman–Wolff constant* $C_F(\mathbb{T})$ measures the extend to which $\mathbb{T}$ concentrates in convex sets,

$$C_F(\mathbb{T}) = \sup_U \frac{\#\mathbb{T}[U]}{\mathrm{Vol}(U)\#\mathbb{T}}.$$

Equivalently, it is the smallest constant such that

$$\#\mathbb{T}[U] \leq C_F(\mathbb{T})\ \mathrm{Vol}(U)\#\mathbb{T}$$

for all convex U. It suffices to consider rectangular prisms of size $a \times b \times 100$ with $\delta \leq a \leq b \leq 100$. We sometimes write $a \times b \times 1$ in place of $a \times b \times 100$.

Taking $U = \delta \times \delta \times 1$ gives $\#\mathbb{T} \geq \frac{1}{C_F(\mathbb{T})}\delta^{-2}$. More generally, the Frostman condition can be expressed in terms of covering numbers. For fixed $\delta \leq a \leq b \leq 1$, let $\mathcal{U}$ be the smallest possible family of $a \times b \times 1$ prisms covering $\mathbb{T}$, i.e.

$$\text{each } T \in \mathbb{T} \text{ is contained in some } U \in \mathcal{U}.$$

Then

$$\#\mathbb{T} \leq \sum_{U \in \mathcal{U}} \#\mathbb{T}[U] \leq C_F(\mathbb{T})\,\mathrm{Vol}(U)\#\mathcal{U}\,\#\mathbb{T},$$

so

$$\#\mathcal{U} \geq \frac{1}{C_F(\mathbb{T})}\ \mathrm{Vol}(U)^{-1}.$$

Thus, the Frostman condition implies that the number of ρ-tubes needed to cover $\mathbb{T}$ is at least ρ^{-2}; the number of $a \times 1 \times 1$ slabs is at least a^{-1}; and the number of $a \times b \times 1$ prisms is at least $(ab)^{-1}$.

Under our pigeonholing assumptions, we can reverse this logic. Let $\#(a \times b \times 1)$ denote the minimal number of such prisms needed to cover $\mathbb{T}$. Then

$$\text{(1.1)}\quad \text{Assuming } \mathbb{T} \text{ is well pigeonholed,} \qquad \frac{1}{C_F(\mathbb{T})} = \inf_{\delta \leq a \leq b \leq 1} \#(a\times b\times 1)\,\mathrm{Vol}(a\times b\times 1).$$

A technical note: In general, it is too much to ask for that every $a \times b \times 1$ prism involved in covering $\mathbb{T}$ has the same number of tubes in it. But if $a \times b \times 1$ is extremal for the Frostman constant, this can be justified.

1.3. **Introduction.** The goal of these lectures is to prove the Kakeya theorem [5] using the Sticky Kakeya theorem [6] as an ingredient.

Theorem 1.1 (Kakeya)**.** *Let $\mathbb{T}$ be an essentially distinct set of $\delta \times \delta \times 1$ tubes with $C_F(\mathbb{T}) \lesssim 1$. For each $T \in \mathbb{T}$, let*

$$Y(T) \subset T$$

be a shading with $\mathrm{Vol}(Y(T)) \geq \delta^\eta \,\mathrm{Vol}(T)$. *For any $\varepsilon > 0$,*

$$\mathrm{Vol}\Big(\bigcup_{T\in\mathbb{T}} Y(T)\Big) \gtrapprox_\eta 1.$$

Recall that by our notation, this means that for any $\varepsilon > 0$,

$$\mathrm{Vol}\Big(\bigcup_{T\in\mathbb{T}} Y(T)\Big) \geq c_\varepsilon \delta^{\varepsilon + C\eta}.$$

Going forward, we will often suppress the shading $Y(T)$ and the subscript η in $\gtrapprox_\eta$.

The Sticky Kakeya theorem deals with the special case when the tubes are Ahlfors–David regular, or sticky.

Theorem 1.2 (Sticky Kakeya)**.** *Let $\mathbb{T}$ be an essentially distinct set of $\delta \times \delta \times 1$ tubes with $C_F(\mathbb{T}) \lesssim 1$ and*

$$\#\mathbb{T}^\rho \approx_\eta \rho^{-2} \qquad \text{for } \rho \in [\delta, 1]. \tag{1.2}$$

Then

$$\mathrm{Vol}\Big(\bigcup_{T\in\mathbb{T}} Y(T)\Big) \gtrapprox 1.$$

Due to pigeonholing, the sticky hypothesis (1.2) is equivalent to

$$\#\mathbb{T}[T^\rho] \approx_\eta (\delta/\rho)^{-2} \qquad \text{for all } \rho \in [\delta, 1] \text{ and } T^\rho \in \mathbb{T}^\rho.$$

Our goal is to prove Theorem 1.1 using Theorem 1.2.

1.4. **The challenge of non-sticky.** Let us make an example by placing down

$$\begin{aligned} &A(\delta^{1/2})^{-2} \text{ many } \delta^{1/2}\text{-tubes, and} \\ &\frac{1}{A}(\delta^{1/2})^{-2} \text{ many } \delta\text{-tubes inside each } \delta^{1/2}\text{-tube.} \end{aligned} \tag{1.3}$$

Here $A \in [1, (\delta^{1/2})^{-2}]$ is a parameter controlling the degree of stickiness at scale $\delta^{1/2}$. If $A = 1$ then $\mathbb{T}$ is sticky at scale $\delta^{1/2}$, and if $A = (\delta^{1/2})^{-2}$ then $\mathbb{T}$ is very far from sticky—there is just one δ-tube inside of each $\delta^{1/2}$-tube.

It is natural to consider how the $\delta^{1/2}$-tubes of $\mathbb{T}^{\delta^{1/2}}$ are arranged, and how the δ-tubes inside each $\delta^{1/2}$-tube are arranged. The best we could hope for is that the union of the $\delta^{1/2}$-tubes cover the entire unit ball, and the δ-tubes inside each $\delta^{1/2}$-tube are essentially disjoint:

$$\begin{aligned} &\mathrm{Vol}(\cup\mathbb{T}^{\delta^{1/2}}) \approx 1, \quad \text{and} \\ &\frac{\mathrm{Vol}(\cup\mathbb{T}[T^{\delta^{1/2}}])}{\mathrm{Vol}(T^{\delta^{1/2}})} \approx \#(\delta\text{-tubes in each } \delta^{1/2}\text{-tube})\frac{\mathrm{Vol}(\delta\text{-tube})}{\mathrm{Vol}(\delta^{1/2}\text{-tube})} = \frac{1}{A}. \end{aligned} \tag{1.4}$$

The *lossy multiscale inequality* relates these two volumes to the volume of the whole Kakeya set.

Proposition 1.3 (Lossy Multiscale Inequality)**.** *For $\rho \in [\delta, 1]$,*

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrsim \mathrm{Vol}(\cup\mathbb{T}^\rho)\frac{\mathrm{Vol}(\cup\mathbb{T}[T^\rho])}{\mathrm{Vol}(T^\rho)}.$$

If we apply this Proposition to our example at scale $\delta^{1/2}$, we find

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrsim \frac{1}{A},$$

which is not efficient if $A \gg 1$. That's why we call it lossy.

Why is this proposition so inefficient? Let us recall a proof. Let

$$E = \cup\mathbb{T},$$

and decompose E into a union of ρ-balls covering it. One of our pigeonholing hypotheses is that for any ρ-ball B involved in covering E, the volume of $E \cap B$ is roughly constant. Thus we can decompose the volume of E into a piece above scale ρ and a piece below scale ρ:

$$\mathrm{Vol}(E^\rho) = \text{Volume of the } \rho\text{-neighborhood of } E,$$

$$\frac{\mathrm{Vol}(E \cap B)}{\mathrm{Vol}(B)} = \text{Density of } E \text{ inside a } \rho\text{-ball},$$

$$\mathrm{Vol}(E) \approx \mathrm{Vol}(E^\rho)\frac{\mathrm{Vol}(E \cap B)}{\mathrm{Vol}(B)}. \tag{1.5}$$

The first term is equal to $\mathrm{Vol}(\cup\mathbb{T}^\rho)$, which we have a good estimate for in (1.4). The second term is trickier. To prove Proposition 1.3, we only consider the contribution to $E \cap B$ from a single ρ-tube T^ρ entering B,

$$\frac{\mathrm{Vol}(E \cap B)}{\mathrm{Vol}(B)} \geq \frac{\mathrm{Vol}(\cup\mathbb{T}[T^\rho] \cap B)}{\mathrm{Vol}(B)}.$$

The density of $\mathbb{T}[T^\rho]$ inside B is equal to the density inside T^ρ,

$$\frac{\mathrm{Vol}(\cup\mathbb{T}[T^\rho] \cap B)}{\mathrm{Vol}(B)} = \frac{\mathrm{Vol}(\cup\mathbb{T}[T^\rho])}{\mathrm{Vol}(T^\rho)},$$

yielding Proposition 1.3.

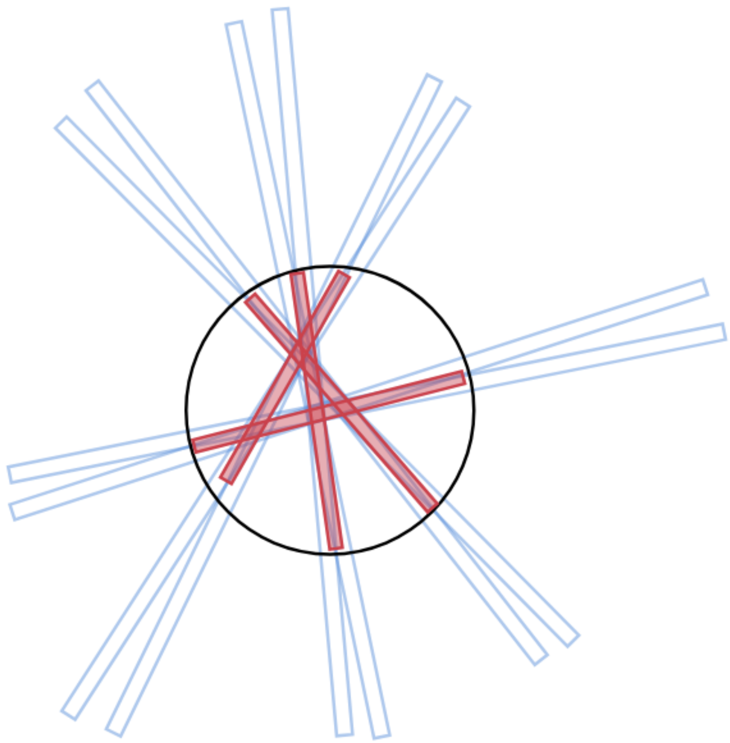

FIGURE 1. The tubelets $\mathbb{T}_B$ inside a ρ-ball, figure from Guth [1, Figure 1].

When $\mathbb{T}$ is sticky, it is okay to just consider the contribution to $E \cap B$ from a single ρ-tube, because we expect each ρ-ball to only have one ρ-tube through it. But when $\mathbb{T}$ is not sticky, this is very lossy. Focus on a $\delta^{1/2}$-ball B in our example. We expect A-many $\delta^{1/2}$-tubes through it, each of which contributes a set of density $\frac{1}{A}$ to B. In Proposition 1.3, we allow for the possibility that these A-many sets overlap each other perfectly, leading to a total density of $\frac{1}{A}$. In order to prove Kakeya, we need to prove the exact opposite: We need to prove these sets barely overlap at all, leading to a total density of 1.

To solve this problem, we can think about how tubes of $\mathbb{T}$ intersect B. Let

$$\mathbb{T}_B = \{T \cap B \,:\, T \in \mathbb{T}\}$$

be the set of $(\delta \times \delta \times \delta^{1/2})$-tubelets active inside of B. See Guth's figure Figure 1.

We can write $E \cap B$ as a union of these tubelets,

$$E \cap B = \cup \mathbb{T}_B.$$

In general, it isn't clear how many tubelets there are in $\mathbb{T}_B$. But in our special example, we can estimate the number of these using Assumption (1.4): There are A-many $\delta^{1/2}$

tubes through B, each of which contributes $\frac{1}{A}(\delta^{1/2})^{-2}$ many tubelets, giving a total count of $(\delta^{1/2})^{-2}$ many tubelets. If the tubelets $\mathbb{T}_B$ happen to be Frostman–Wolff, we are in luck: we have a new problem of the same type inside B, and we can use induction to show

$$\frac{\mathrm{Vol}(\cup \mathbb{T}_B)}{\mathrm{Vol}(B)} \gtrsim 1.$$

If the tubelets of $\mathbb{T}_B$ are not Frostman–Wolff, it isn't clear what to do. To prove Kakeya, Wang and Zahl figured out what to do if $\mathbb{T}_B$ is not Frostman–Wolff.

In order to deal with $\mathbb{T}_B$, Wang and Zahl prove a structural theorem describing the union of any set of tubes, Frostman or not. Let $\mathbb{T}$ be an essentially distinct set of δ-tubes, and let U be the maximum density convex set in the definition of the Frostman constant, meaning the supremum in

$$C_F(\mathbb{T}) = \sup_{U'} \frac{\#\mathbb{T}[U']}{\mathrm{Vol}(U')\#\mathbb{T}}$$

is achieved at U. After some pigeonholing, we can assuming $\mathbb{T}$ is covered by a collection of essentially disjoint translated and rotated copies of U, each of which have the same density of tubes—call this collection $\mathcal{U}$.

Wang and Zahl proved that $E = \cup \mathbb{T}$ fills out each $U \in \mathcal{U}$, and that the convex sets $\mathcal{U}$ are essentially disjoint:

$$\begin{aligned} \mathrm{Vol}(E \cap U) &\approx \mathrm{Vol}(U) \qquad \text{for each } U \in \mathcal{U}, \quad \text{and} \\ \mathrm{Vol}(E) &\approx \#\mathcal{U}\ \mathrm{Vol}(U). \end{aligned} \tag{1.6}$$

This is a precise description of what $\cup \mathbb{T}$ looks like. In order to study $\mathbb{T}_B$, we apply the description in (1.6) and split into different cases depending on what U looks like.

1.5. **A non-sticky example.** I would like to describe a particular example that is helpful to keep in mind during the proof.

We choose $A = (\delta^{1/2})^{-1}$ in (1.3), meaning we place down

$$\begin{aligned} &(\delta^{1/2})^{-3} \text{ many } \delta^{1/2}\text{-tubes, and} \\ &(\delta^{1/2})^{-1} \text{ many } \delta\text{-tubes inside each } \delta^{1/2}\text{-tube.} \end{aligned}$$

Inside each $\delta^{1/2}$-tube, the δ-tubes are arranged in a regulus.

To make a regulus, arrange the $\delta^{1/2}$ tube so it points vertically, and take a bottom slice and a top slice, each of which are $\delta^{1/2}$ discs. Place a parameterized line segment in each disk,

$$\begin{aligned} &t \mapsto \gamma_1(t) \in \text{Bottom Disk is a line segment,} \\ &t \mapsto \gamma_2(t) \in \text{Top Disk is a line segment.} \end{aligned}$$

For each t, we include the tube $T_{\gamma_1(t),\gamma_2(t)}$ connecting the bottom point to the top point. In total, our tube set is

$$\mathbb{T}[T^{\delta^{1/2}}] = \{T_{\gamma_1(t),\gamma_2(t)} : t = \{0, \delta, 2\delta, \ldots, \delta^{1/2}\}.$$

The union of these tubes sweeps out a two dimensional surface inside of $T^{\delta^{1/2}}$ called a regulus. If we intersect this regulus with a $\delta^{1/2}$-ball, we get a $\delta \times \delta^{1/2} \times \delta^{1/2}$ slab. As we move along the tube, these slabs rotate.

Inside of each $\delta^{1/2}$-ball B there are $\delta^{-1/2}$ incoming tubes, each of which contribute a $\delta \times \delta^{1/2} \times \delta^{1/2}$-slab. We want to show these slabs are essentially disjoint, so that E fills out B. But what is stopping several tubes from contributing the same slab?

Actually, Katz and Zahl [3] found an example of this form over the ring $\mathbb{F}_p[x]/x^2$ where there is just one slab inside B—total overlap—giving E a density of $\delta^{1/2}$. So there was good reason to be scared of this example.

I said before that the strategy is to study $\mathbb{T}_B$, but that seems useless in this example. In the worst-case scenario that there is one slab inside B, what is there to say? The description (1.6) tells us what we already know: $E \cap B$ looks like a single filled out slab.

The trick is to study $\mathbb{T}_B$ where B has diameter ρ close to 1. This is not possible in Katz and Zahl's example, as the ring $\mathbb{F}_p[x]/x^2$ has only two scales. If B has diameter close to 1 and (1.6) tells us $E \cap B$ is a union of filled out slabs, we have made progress: we know the Kakeya set is close to a union of filled out slabs. The Frostman hypothesis directly tells us there are lots of these, and we can analyze the union of those slabs using L^2 arguments. There are several other cases depending on what U looks like. For this reason, I cannot describe exactly how Wang and Zahl deal with this example, because I don't know what $\mathbb{T}_B$ looks like when ρ is close to 1. It depends on the details of how the reguli are laid out in each $\delta^{1/2}$ tube, and how the $\delta^{1/2}$ tubes are laid out.

1.6. **Bird's eye view: the multiscale strategy.** Let $\mathbf{A}(\sigma)$ denote the assertion

$$\mathbb{C}_F(\mathbb{T}) \lesssim 1 \Longrightarrow \mathrm{Vol}(\bigcup_{T\in\mathbb{T}} Y(T)) \gtrapprox \delta^{\sigma}.$$

Our goal is to show $\mathbf{A}(0)$ holds. Let

$$\sigma = \inf\{\sigma' : \mathbf{A}(\sigma') \text{ holds}\}, \tag{1.7}$$

and assume by way of contradiction that $\sigma > 0$.

A *multiscale decomposition* is a way of relating a Kakeya problem to several smaller Kakeya problems. The lossy decomposition (Proposition 1.3) is efficient if $\mathbb{T}[T^{\rho}]$ is Frostman, but not in general. In general, we analyze $\mathbb{T}_B$ using the description (1.6) to break an arbitrary Kakeya problem into several smaller sub-problems. Here is a precise bird's eye view description of the proof strategy.

To prove Theorem 1.1, take a Frostman set of tubes $\mathbb{T}$ which is a worst scenario for $\mathbf{A}(\sigma)$:

$$\mathrm{Vol}(\cup\mathbb{T}) \approx \delta^{\sigma}. \tag{1.8}$$

We will produce a finite list of related collections

$$\mathbb{T}_j \text{ is a collection of } w_j \text{ tubes for } j = 1 \dots j_0$$

such that

- $C_F(\mathbb{T}_j) \lessapprox 1$,
- $w_1 \dots w_{j_0} = \delta$,
- $\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \prod_{j=1}^{j_0} \mathrm{Vol}(\cup\mathbb{T}_j)$. This is the main point. By applying $\mathbf{A}(\sigma)$ to each term on the right hand side, we learn
$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \prod w_j^{\sigma} = \delta^{\sigma}, \tag{1.9}$$
which lets us recover $\mathbf{A}(\sigma)$ from itself.
- One of the sub-problems, say $\mathbb{T}_j$, is sticky. That means we can apply the Sticky Kakeya theorem rather than $\mathbf{A}(\sigma)$ to estimate $\mathrm{Vol}(\cup\mathbb{T}_j)$, which improves Equation (1.9) to
$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \prod_{j' \neq j} w_{j'}^{\sigma} = \delta^{\sigma} w_j^{-\sigma}.$$
As long as w_j is sufficiently small, this represents an improvement over the $\mathbf{A}(\sigma)$ bound and contradicts (1.8).

We won't always explicitly write down these sub-problems. They appear implicitly in lemmas.

2. Lecture 2: High density lemma and nowhere sticky reduction

The main goal of this lecture is to prove the following high density lemma, which improves on $\mathbf{A}(\sigma)$ when $\#\mathbb{T} \gg \delta^{-2}$.

Lemma 2.1 (High Density Lemma)**.**

$$C_F(\mathbb{T}) \lessapprox 1 \Longrightarrow \mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \delta^{\sigma}(\delta^2\#\mathbb{T})^{\sigma/2}.$$

The proof of this lemma uses Sticky Kakeya and the Lossy Decomposition (Proposition 1.3). It is helpful to rethink when we can apply Sticky Kakeya. The sticky hypothesis (1.2) says

$$\#\mathbb{T}^{\rho} \approx \rho^{-2} \qquad \text{for } \rho \in [\delta, 1].$$

This is a useful hypothesis because it guarantees that after rescaling, the δ-tubes inside each ρ-tube are themselves Convex Frostman. We can make this the hypothesis instead. Let T^{ρ} be a ρ-tube, and let $\psi^{T^{\rho}} : T^{\rho} \to [0,1]^3$ be an affine rescaling map to the unit cube.

Proposition 2.2 (New Sticky Kakeya). *Assume that*

$$C_F(\psi^{T^\rho}(\mathbb{T}[T^\rho])) \lesssim \delta^{-\eta} \qquad \textit{for all } \rho \in [\delta, 1] \textit{ and } T^\rho \in \mathbb{T}^\rho. \tag{2.1}$$

Then

$$\mathrm{Vol}(\cup \mathbb{T}) \gtrapprox_\eta 1.$$

If (2.1) holds, we say $\mathbb{T}$ is η-*sticky*. If $\mathbb{T}$ is η-sticky, one can show that it contains a subset $\mathbb{T}'$ satisfying the hypotheses of Theorem 1.2. So, we can prove Proposition 2.2 from Theorem 1.2.

Our strategy is to find a large number of scales for which (2.1) holds. Roughly speaking, we will use Sticky Kakeya near the scales where (2.1) holds, and use $\mathbf{A}(\sigma)$ elsewhere.

Equation (2.1) is hard to verify because we need to look at every convex set inside a ρ-tube. It would be easier if we just needed to worry about tubes, and could forget about other convex sets. To be precise, we would like the following.

Tube-y Assumption. If the number of w-tubes contained inside a ρ-tube of $\mathbb{T}^\rho$ satisfies

$$\#(w\text{-tubes in a } \rho\text{-tube}) \gtrsim (w/\rho)^{-2} \quad \text{for all } w \in [\delta, 1],$$

then $\psi^{T^\rho}(\mathbb{T}[T^\rho])$ is Convex Frostman.

It turns out that the High Density Lemma can be reduced to this special case. Here is a very brief sketch. If the **Tube-y Assumption** fails, that implies that for some ρ-tube T^ρ, the convex set inside of T^ρ maximizing the Frostman density

$$\frac{\#\mathbb{T}[U]}{\#\mathbb{T}[T^\rho]\,\mathrm{Vol}(U)/\,\mathrm{Vol}(T^\rho)}$$

is some $a \times b \times 1$ prism that is not a tube. The fact that U is maximum density implies that $\mathbb{T}[U]$ is Frostman relative to U, in the following sense:

$$\text{For any } V \subset U,\ \#\mathbb{T}[V] \le \#\mathbb{T}[U]\frac{\mathrm{Vol}(V)}{\mathrm{Vol}(U)}.$$

At this point it is natural to study two problems: The union of the tubes inside U, and the union of congruent copies of U covering $\mathbb{T}$. These are convex set variants of the original Kakeya problem. Wang and Zahl study these convex set variants. They prove a Kakeya type theorem for tubes inside an arbitrary convex set, and for unions of arbitrary convex sets. There is a high density lemma in this larger context, which is proved by induction. Within this larger context, the case that $\mathbb{T}$ is Frostman inside U is good for induction. It remains to prove the high density lemma under the **Tube-y Assumption.**

From now on I will make the **Tube-y Assumption**. We want to understand the set of *good scales* ρ such that

$$\#(w\text{-tubes inside each }\rho\text{-tube}) \gtrsim (w/\rho)^{-2} \qquad \text{for } w \in [\delta, \rho].$$

To do so, we make a log-log plot describing how tubes are distributed. Define the branching function $f : [0,1] \to [0,4]$ by

$$\delta^{-f(x)} := \#\mathbb{T}^{\delta^x}.$$

Analyzing branching functions is an important idea in fractal geometry—this idea was introduced by Keleti and Shmerkin [4]. Some examples:

- If $\mathbb{T}$ is sticky, then $\#\mathbb{T}^\rho \approx \rho^{-2}$, and $f(x) = 2x$.
- If δ^{-2} tubes are placed down uniformly at random, then all $(\delta^{1/2})^{-4}$ many distinct $\delta^{1/2}$ tubes will be active, and each will have just one δ-tube inside of them. Thus
$$\#\mathbb{T}^\rho = \begin{cases} \rho^{-4} & \text{if } \rho \in [\delta^{1/2}, 1], \\ \delta^{-2} & \text{if } \rho \in [\delta, \delta^{1/2}] \end{cases}$$
and
$$f(x) = \begin{cases} 4x & \text{if } x \in [0, 1/2], \\ 2 & \text{if } x \in [1/2, 1]. \end{cases}$$
- If we place down $\delta^{-2-\zeta}$ tubes uniformly at random, what changes? The tubes are well spaced, meaning that for some scale ρ_0, every ρ_0 tube is active and there is one δ-tube per ρ_0 tube. To calculate ρ_0, we use the equation
$$\rho_0^{-4} = \delta^{-2-\zeta} \implies \rho_0 = \delta^{-1/2-\zeta/4}.$$
The part of the branching function with slope 4 gets expanded a bit, and the part of slope 0 gets contracted a bit. The new branching function is
$$f(x) = \begin{cases} 4x & \text{if } x \in [0, 1/2 + \zeta/4], \\ 2 + \zeta & \text{if } x \in [1/2 + \zeta/4, 1]. \end{cases}$$

We can find good scales using the branching function. The scale $\rho = \delta^x$ is good if and only if

$$f(x+a) \geq f(x) + 2a \quad \text{for } a \in [0, 1-x].$$

Geometrically, consider the graph of f, and make a line of slope 2 eminating from the point $(x, f(x))$. If this line lies below the graph of f, then δ^x is a good scale.

The higher density $\mathbb{T}$ is, the more good scales there are. Let

$$\#\mathbb{T} = \delta^{-2-\zeta} \qquad \text{for } \zeta > 0.$$

Let $y \in [0, \zeta]$, and consider the line

$$\{(t, 2t + y) : t \in [0, 1]\}.$$

The graph of f starts out below this line and ends up above it, so they have to cross at some point. Let $G(y)$ be the supremum x-value where f lies below this line,

$$G(y) = \sup\{x : f(x) \leq 2x + y\}.$$

Equality must hold at $G(y)$,

$$f(G(y)) = 2G(y) + y.$$

The supremum in the definition implies that $G(y)$ is a good point. For any $a \in [0, 1 - G(y)]$,

$$f(G(y) + a) \geq 2(G(y) + a) + y = f(G(y)) + 2a.$$

Next we show that $y \mapsto G(y)$ increases in a quantitative way. Since the branching function f is 4-Lipschitz and satisfies $f(G(y)) = 2G(y) + y$, for any $y_1 < y_2$ we have

$$|f(G(y_2)) - f(G(y_1))| = 2|G(y_2) - G(y_1)| + |y_2 - y_1|.$$

Applying the Lipschitz bound gives

$$2(G(y_2) - G(y_1)) + (y_2 - y_1) \leq 4(G(y_2) - G(y_1)).$$

Rearranging yields

$$|G(y_2) - G(y_1)| \geq \tfrac{1}{2}|y_2 - y_1|.$$

In particular, the set of good scales obtained from G,

$$\mathcal{G} := G([0, \zeta]),$$

has Lebesgue measure at least $\zeta/2$.

Consider our third example above, where we placed down $\delta^{-2-\zeta}$ tubes randomly. One can check that $\mathcal{G} = [0, \zeta/2]$ in this example, matching our measure lower bound.

Suppose that $\mathcal{G}$ is a union of a constant number of intervals,

$$\mathcal{G} = I_1 \sqcup \cdots \sqcup I_m.$$

The rest of $[0, 1]$ is also broken up into intervals,

$$[0, 1] \setminus \mathcal{G} = F_1 \cup \cdots \cup F_{m'}.$$

Each interval corresponds to some range of scales in $[\delta, 1]$. For instance, if $I_j = [a, b]$, this interval corresponds to the range of scales $[\delta^b, \delta^a]$, and we can look at the set of δ^b-tubes inside a δ^a-tube:

$$\mathbb{T}_{I_j} = \psi(\mathbb{T}^{\delta^b}[T^{\delta^a}]), \qquad \psi \text{ is an affine rescaling map.}$$

The fact that all of I_j is good implies that $\mathbb{T}_{I_j}$ is sticky in the sense of (2.1). With this notation, the Lossy Decomposition (Proposition 1.3) implies

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \prod_{j=1}^{m} \mathrm{Vol}(\cup\mathbb{T}_{I_j}) \prod_{j'=1}^{m'} \mathrm{Vol}(\cup\mathbb{T}_{F'_j}).$$

We use Sticky Kakeya for the good I_j-intervals, and $\mathbf{A}(\sigma)$ for the remaining F_j intervals. Note that the left endpoint of each F_j interval lies in $\mathcal{G}$, so the tube sets $\mathbb{T}_{F_j}$ are Frostman. These estimates imply

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrsim \delta^{\sigma \mathrm{Leb}([0,1]\setminus\mathcal{G})} = \delta^{\sigma}\delta^{-\sigma\mathrm{Leb}(\mathcal{G})} \geq \delta^{\sigma}\delta^{-\sigma\zeta/2} = \delta^{\sigma}(\delta^2\#\mathbb{T})^{\sigma/2}$$

as desired.

With some more work, this argument can be generalized to work for all $\mathcal{G}$ (not necessarily a finite union of intervals).

3. Lecture 3: The L^2 method

3.1. **Tubes.** Let $\mathbb{T}$ be a set of tubes in $\mathbb{R}^2$, and let

$$E = \cup\mathbb{T}.$$

For $p \in E$, let $\mathbb{T}(p)$ be the tubes through p. By pigeonholing, we hypothesize that $\#\mathbb{T}(p) \sim$ const. as p varies over E.

The L^2 method involves computing $\int \left(\sum_{T\in\mathbb{T}} 1_T\right)^2 dx$ in two different ways. First we put the sum inside the integral,

$$\int_E \sum_{T_1,T_2\in\mathbb{T}} 1_{T_1}1_{T_2}\,dx = \int_E \#\mathbb{T}(p)^2\,dx = \mathrm{Vol}(E)\#\mathbb{T}(p)^2. \tag{3.1}$$

Next, we put the sum outside the integral,

$$\sum_{T_1,T_2\in\mathbb{T}} \int_{[0,1]^2} 1_{T_1}1_{T_2}\,dx = \sum_{T_1,T_2\in\mathbb{T}} \mathrm{Vol}(T_1\cap T_2). \tag{3.2}$$

Comparing these two, we find

$$\mathrm{Vol}(E)\#\mathbb{T}(p)^2 \lesssim \sum_{T_1,T_2\in\mathbb{T}} \mathrm{Vol}(T_1\cap T_2). \tag{3.3}$$

On the other hand, $\mathrm{Vol}(E) = \frac{\#\mathbb{T}\,\mathrm{Vol}(T)}{\#\mathbb{T}(p)}$. Comparing these two expressions gives

$$\mathrm{Vol}(E) \gtrsim \frac{(\#\mathbb{T}\,\mathrm{Vol}(T))^2}{\sum_{T_1,T_2\in\mathbb{T}}\mathrm{Vol}(T_1\cap T_2)}. \tag{3.4}$$

The denominator has several contributions, depending on the angle between T_1 and T_2. In order to make the denominator simpler, it is helpful to assume $\mathbb{T}(p)$ is *broad*, meaning a large portion of pairs of tubes in $\mathbb{T}(p)$ are 1-separated in angle,

$$\#\{T_1,T_2\in\mathbb{T}(p) : \theta(T_1,T_2)\sim 1\} \approx \#\mathbb{T}(p)^2.$$

Due to the broadness hypotheses, (3.1) becomes

$$\begin{aligned}\int_E \sum_{T_1,T_2\in\mathbb{T}} 1_{T_1}1_{T_2}\,dx &\approx \int_E \sum_{T_1,T_2\in\mathbb{T},\theta(T_1,T_2)\sim 1} 1_{T_1}1_{T_2}\,dx\\ &= \sum_{T_1,T_2\in\mathbb{T},\theta(T_1,T_2)\sim 1} \mathrm{Vol}(T_1\cap T_2)\\ &\lesssim \#\mathbb{T}^2\,\mathrm{Vol}(T)^2.\end{aligned}$$

In (3.4), $\sum_{T_1,T_2\in\mathbb{T}}\mathrm{Vol}(T_1\cap T_2)$ gets replaced with $\#\mathbb{T}^2\,\mathrm{Vol}(T)^2$, giving the estimate

$$\mathrm{Vol}(E)\gtrsim 1.$$

This is a funny result. We didn't assume anything about the tube set $\mathbb{T}$, just that $\mathbb{T}(p)$ is broad. The number of tubes appeared in our equations, but canceled out so that the final result, $\mathrm{Vol}(E)\gtrsim 1$, did not depend on the total number of tubes.

An arbitrary set of tubes may not be broad, but we can still use the L^2 method to understand their union. After a bunch of pigeonholing, we find that there is a scale $w\in[\delta,1]$ such that for each $p\in E$

- $\mathbb{T}(p)$ is w-*narrow*, meaning all the tubes of $\mathbb{T}(p)$ are contained in one $w\times 1$ rectangle, and
- $\mathbb{T}(p)$ is w-*broad*, meaning a large portion of pairs of tubes in $\mathbb{T}(p)$ are w-separated in angle,

$$\#\{T_1,T_2\in\mathbb{T}(p)\,:\,\theta(T_1,T_2)\sim w\}\approx\#\mathbb{T}(p)^2.$$

The broadness hypothesis is a robust way of saying that the tubes $\mathbb{T}(p)$ are not concentrating at a smaller scale than w.

Let $\mathbb{T}^w$ be the set of w-tubes active in covering $\mathbb{T}$. For $T^w\in\mathbb{T}^w$, let

$$E_{T^w}=\cup\mathbb{T}[T^w]$$

be the union of δ-tubes inside of there. By the w-narrow hypothesis, each $p\in E$ belongs to just one of the sets E_{T_w}. We may write

$$E=\bigsqcup_{T^w\in\mathbb{T}^w}E_{T^w}.$$

Now apply a rescaling to map $T^w\to$ Unit Cube. Due to the w-broadness hypothesis, this rescaling maps $\mathbb{T}[T^w]$ to a broad tube set. By the L^2 method,

$$\mathrm{Vol}(E_{T^w})\gtrsim\mathrm{Vol}(T^w).$$

In other words, E fills out every w-tube active in covering $\mathbb{T}$. Overall,

$$\mathrm{Vol}(E)\gtrsim\#\mathbb{T}^w\ \mathrm{Vol}(T^w). \tag{3.5}$$

We may write this equation as the approximate equality of sets

$$E \approx \sqcup \mathbb{T}^w.$$

Notice that if we assume $\mathbb{T}$ is Frostman, then the right hand side is $\gtrsim 1$ for any $w \in [\delta, 1]$, implying $\mathrm{Vol}(E) \gtrsim 1$.

3.2. **Slabs.** Something similar works with a set of slabs instead of a set of tubes. Let $\mathcal{S}$ be a set of $\delta \times 1 \times 1$ slabs, and let

$$E = \cup \mathcal{S}.$$

For $p \in E$, let $\mathcal{S}(p)$ be the set of slabs throgh p. For a slab S, $\theta(S) \in S^2$ denotes the normal vector of S.

Suppose that a typical $p \in E$ is *broad*, meaning

$$\#\{S_1, S_2 \in \mathcal{S}(p) : |\theta(S_1) - \theta(S_2)| \sim 1\} \approx \#\S(p)^2.$$

If a typical $p \in E$ is broad, then

$$\mathrm{Vol}(E)\#\mathcal{S}(p)^2 \lesssim \#\mathcal{S}^2 \ \mathrm{Vol}(S_1 \cap S_2) = (\#\mathcal{S}\, \mathrm{Vol}(S))^2.$$

On the other hand,

$$\mathrm{Vol}(E) = \frac{\#\mathcal{S}\, \mathrm{Vol}(S)}{\#\mathcal{S}(p)} \lesssim \mathrm{Vol}(E)^{1/2}$$

implying

$$\mathrm{Vol}(E) \gtrsim 1.$$

We can define w-broadness and w-narrowness in a similar way. If every $\mathcal{S}(p)$ is w-broad, then every $w \times 1 \times 1$ slab involved in covering $\mathcal{S}$ is totally filled out. If $\mathcal{S}(p)$ is w-narrow, then these $w \times 1 \times 1$ are essentially disjoint. Thus if a typical p is w-broad and w-narrow,

$$E \approx \sqcup \mathcal{S}^w$$

where $\mathcal{S}^w$ is the set of $w \times 1 \times 1$ slabs involved in covering $\mathcal{S}$.

4. Lecture 4: Union of Planks

In the introduction section, in (1.6), we described the structure of an arbitrary union of tubes. To recap, let U be the maximum density convex set for $\mathbb{T}$, and let $\mathcal{U}$ be a minimal number of congruent copies of U to cover $\mathbb{T}$. Wang–Zahl proved

$$\mathrm{Vol}(E \cap U) \approx \mathrm{Vol}(U) \qquad \text{for each } U \in \mathcal{U}, \quad \text{and}$$
$$\mathrm{Vol}(E) \approx \#\mathcal{U}\ \mathrm{Vol}(U).$$

By the covering number description of the Frostman constant (1.1), $\#\mathcal{U}\,\mathrm{Vol}(U) = \frac{1}{C_F(\mathbb{T})}$. So, it follows from Wang and Zahls's description that

$$\mathrm{Vol}(E) \gtrapprox \frac{1}{C_F(\mathbb{T})}.$$

Actually, there is an easy argument to prove this directly from Theorem 1.1, which is robust enough to apply to $\mathbf{A}(\sigma)$.

Lemma 4.1. *Suppose* $\mathbf{A}(\sigma)$ *holds. Then for any set of tubes* $\mathbb{T}$,

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \frac{1}{C_F(\mathbb{T})}\,\delta^{\sigma}.$$

Proof. Let

$$\mathbb{T}' = \text{A union of } C_F(\mathbb{T})\text{-many randomly translated and rotated copies of } \mathbb{T}.$$

Because the copies are all thrown down randomly,

$$\#(a \times b \times 1 \text{ in } \mathbb{T}') \gtrsim C_F(\mathbb{T})\,\#(a \times b \times 1 \text{ in } \mathbb{T}).$$

Thus

$$\frac{1}{C_F(\mathbb{T}')} = \inf_{a\times b\times 1} \#(a\times b\times 1 \text{ in } \mathbb{T}')\,\mathrm{Vol}(a\times b\times 1) \gtrsim C_F(\mathbb{T})\#(a\times b\times 1 \text{ in } \mathbb{T})\,\mathrm{Vol}(a\times b\times 1) \gtrsim 1.$$

By $\mathbf{A}(\sigma)$,

$$\mathrm{Vol}(\cup\mathbb{T}') \gtrsim \delta^{\sigma}.$$

On the other hand,

$$\mathrm{Vol}(\cup\mathbb{T}') \lesssim C_F(\mathbb{T})\,\mathrm{Vol}(\mathbb{T}),$$

so we find

$$\mathrm{Vol}(\mathbb{T}) \gtrsim \frac{1}{C_F(\mathbb{T})}\delta^{\sigma}.$$

∎

Let $\mathbb{P}$ be an arrangement of $\delta \times b \times 1$ planks. The Frostman constant is defined by

$$C_F(\mathbb{P}) = \sup_{U \supset \delta\times b\times 1} \frac{\#\mathbb{P}[U]}{\#\mathbb{P}\,\mathrm{Vol}(U)},$$

which is equivalent by pigeonholing hypotheses to

$$\frac{1}{C_F(\mathbb{P})} = \inf_{U \supset \delta\times b\times 1} \#\mathcal{U}\,\mathrm{Vol}(U).$$

Lemma 4.2 (Union of Planks)**.** *Assume* $\mathbf{A}(\sigma)$*. Let* $\mathbb{P}$ *be an arrangement of* $\delta \times b \times 1$ *planks with* $C_F(\mathbb{P}) \lesssim 1$*. Then*

$$\mathrm{Vol}(\cup\mathbb{P}) \gtrapprox b^{\sigma}.$$

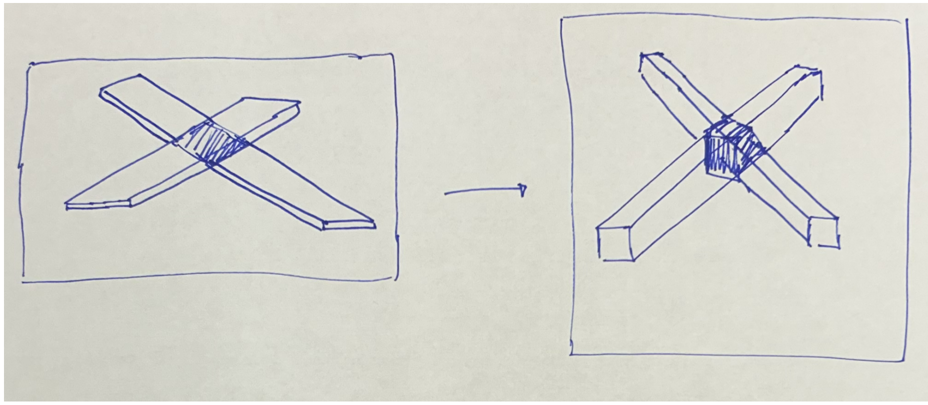

FIGURE 2. Planks intersecting tangentially in a slab are anisotropically rescaled to tubes intersecting in a ball

If $b = \delta$, then $\mathbb{P}$ is an arrangement of tubes, and this Lemma restates $\mathbf{A}(\sigma)$. If $b = 1$, then $\mathbb{P}$ is a Frostman arrangement of slabs. In this case, we can use the L^2 method. Let $\mathcal{S}$ be the arrangement of slabs, and assume $\mathcal{S}(p)$ is w-broad and w-narrow. By the L^2 method,

$$\cup\mathcal{S} \approx \sqcup\mathcal{S}^w$$

where $\mathcal{S}^w$ is the set of $w \times 1 \times 1$ slabs active in covering $\mathcal{S}$. In particular,

$$\mathrm{Vol}(\cup\mathcal{S}) \approx \#\mathcal{S}^w \,\mathrm{Vol}(\mathcal{S}^w).$$

By the Frostman hypothesis on $\mathcal{S}$, the right hand side is $\gtrsim 1$.

When $b \in [\delta, 1]$ is arbitrary, we can find a slabby sub-problem by zooming into a b-ball B. Each incoming plank intersects B in a $\delta \times b \times b$ slab. Let $w \in [\delta, b]$ be smallest such that all the $\delta \times b \times b$ slabs through a point are contained in a $w \times b \times b$ slab. We are assuming that the $\delta \times b \times b$ slabs through a point are w-broad and w-narrow.

By the L^2 method, every $w \times b \times b$ slab active in covering E is completely filled out. By w-narrowness, the $w \times b \times b$ slabs are disjoint. Thus

$$\mathrm{Vol}(E) \gtrsim \#(w \times b \times b)\,\mathrm{Vol}(w \times b \times b).$$

First suppose $w = b$. In this case, the right hand side above is the union of b-tubes containing each plank, and

$$\mathrm{Vol}(E) \approx \mathrm{Vol}(\cup\mathbb{T}^b) \gtrapprox b^\sigma$$

By $\mathbf{A}(\sigma)$.

Next suppose $w = \delta$. If you look at all the $\delta \times b \times 1$ planks through a point, they all contain one fixed $\delta \times b \times b$ slab. In this case, we say $\delta \times b \times 1$ planks intersect *tangentially*, like in Figure 2. In the tangential case, we can zoom into a $\frac{\delta}{b} \times 1 \times 1$ slab and consider all the $\delta \times b \times 1$ planks inside of it. After anisotropically rescaling,

$$\frac{\delta}{b} \times 1 \times 1\text{-slabs} \mapsto \text{Unit cube},$$
$$\delta \times b \times 1\text{-planks} \mapsto b\text{-tubes},$$
$$\delta \times b \times b\text{-slabs} \mapsto b\text{-balls},$$

By $\mathbf{A}(\sigma)$,

(4.1)

$$\#(\delta \times b \times b \text{ inside each } \frac{\delta}{b} \times 1 \times 1) \frac{\mathrm{Vol}(\delta \times b \times b)}{\mathrm{Vol}(\frac{\delta}{b} \times 1 \times 1)} \gtrapprox b^\sigma \frac{1}{C_F(\delta \times b \times 1\text{-planks inside } \frac{\delta}{b} \times 1 \times 1)}.$$

We can estimate the Frostman constant on the right hand side in terms of the Frostman constant of $\mathbb{T}$. We have

$$\begin{aligned} \frac{1}{C_F(\delta \times b \times 1\text{-planks inside } \frac{\delta}{b} \times 1 \times 1)} &= \inf_{U \subset \frac{\delta}{b} \times 1 \times 1} \frac{\#(\mathcal{U} \text{ to cover } \mathbb{T}) \, \mathrm{Vol}(U)}{\#(\frac{\delta}{b} \times 1 \times 1) \, \mathrm{Vol}(\frac{\delta}{b} \times 1 \times 1)} \\ &\gtrsim \frac{1}{C_F(\mathbb{T})} \frac{1}{\#(\frac{\delta}{b} \times 1 \times 1) \, \mathrm{Vol}(\frac{\delta}{b} \times 1 \times 1)}. \end{aligned} \tag{4.2}$$

The more $\frac{\delta}{b} \times 1 \times 1$ slabs there are, the smaller the loss factor is in (4.1). On the other hand, when we compute the total number of $\delta \times b \times b$ slabs active in all of E, we have to sum over all the big $\frac{\delta}{b} \times 1 \times 1$ slabs. These two factors perfectly cancel,

$$\mathrm{Vol}(E) \gtrsim \#(\delta \times b \times b) \, \mathrm{Vol}(\delta \times b \times b) \gtrapprox b^\sigma \frac{1}{C_F(\mathbb{T})}$$

as desired.

If $w \in (\delta, b)$, we can do something similar. By the L^2 method, we know every w-ball of E is filled out. Thus we can consider $w \times b \times 1$ planks instead of $\delta \times b \times 1$ planks. To estimate the number of $w \times b \times b$ slabs, we zoom into $\frac{w}{b} \times 1 \times 1$ slabs and repeat the argument above.

Lemma 4.2 lets us give the following improvement over Lemma 4.1.

Lemma 4.3 (Improvement in plank-y case)**.** *Assume $\mathbf{A}(\sigma)$. Let $\mathbb{T}$ be a set of tubes whose maximum density convex set is an $a \times b \times 1$ prism. Then*

$$\mathrm{Vol}(\cup \mathbb{T}) \gtrapprox \frac{1}{C_F(\mathbb{T})} (\delta \frac{b}{a})^\sigma.$$

Proof of Lemma 4.3. Consider a fixed $a \times b \times 1$ plank. Anisotropically rescale so it becomes the unit cube. Tubes are mapped to $\frac{\delta}{b} \times \frac{\delta}{a} \times 1$ planks which are arranged in a Frostman way. By Lemma 4.2,

$$\frac{\mathrm{Vol}(E \cap a \times b \times 1)}{\mathrm{Vol}(a \times b \times 1)} \gtrapprox (\delta/a)^\sigma.$$

The density inside of a a-ball is at least as large,

$$\frac{\mathrm{Vol}(E \cap B(x_0, a)}{\mathrm{Vol}(B(x_0, a))} \gtrapprox (\delta/a)^\sigma.$$

Now consider the set of a-tubes $\mathbb{T}^a$. By maximum density, the a-tubes inside an $a \times b \times 1$ plank are arranged in a Frostman way. By the L^2 method for tubes in $\mathbb{R}^2$, they fill out the $a \times b \times 1$ planks, implying

$$\mathrm{Vol}(E^a \cap a \times b \times 1) \gtrsim \mathrm{Vol}(a \times b \times 1).$$

Let $\mathbb{P}$ denote the collection of $a \times b \times 1$ planks. We would like to apply Lemma 4.2 to $\mathbb{P}$. We use the same trick as Lemma 4.1 and take $C_F(\mathbb{T})$ random copies to find

$$\mathrm{Vol}(\cup \mathbb{P}) \gtrapprox \#\mathbb{P}\,\mathrm{Vol}(P) b^\sigma = \frac{1}{C_F(\mathbb{T})} b^\sigma.$$

Combining these two densities,

$$\mathrm{Vol}(E) \gtrapprox \frac{1}{C_F(\mathbb{T})} \big(\delta \tfrac{b}{a}\big)^\sigma$$

as desired. □

5. Lecture 5: The lossless decomposition and proof of Kakeya

We are ready to finish the proof of Kakeya. This section is a bit different from [5], although the ingredients are mostly the same.

Let

$$\sigma = \inf\{\sigma' \,:\, \mathbf{A}(\sigma') \text{ holds}\}.$$

Assume by way of contradiction that $\sigma > 0$. Here are our ingredients.

- If $\mathbb{T}$ is a set of δ-tubes, the *maximum density convex set* is the $U = a \times b \times 1$ maximizing the density
$$\frac{\#\mathbb{T}[U]}{\#\mathbb{T}\,\mathrm{Vol}(U)}.$$
Equivalently, letting $\mathcal{U}$ be the minimal collection of congruent copies to cover $\mathbb{T}$, U minimizes
$$\#\mathcal{U}\,\mathrm{Vol}(U).$$
The Frostman constant is
$$C_F(\mathbb{T}) = \frac{\#\mathbb{T}[U]}{\#\mathbb{T}\,\mathrm{Vol}(U)} \qquad \text{and} \qquad \frac{1}{C_F(\mathbb{T})} = \#\mathcal{U}\,\mathrm{Vol}(U).$$
- The high density lemma says that if $\#\mathbb{T} \gg \delta^{-2}$, there is a gain over $\mathbf{A}(\sigma)$. The proof uses the lossy decomposition along with Sticky Kakeya. We need a slightly refined version that allows for an arbitrary set of tubes rather than a Frostman set of tubes.

Lemma 5.1 (Refined high density)**.** *Assume* $\mathbf{A}(\sigma)$*. For any set* $\mathbb{T}$ *of* δ*-tubes,*

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \frac{1}{C_F(\mathbb{T})}\delta^\sigma(\delta^2 C_F(\mathbb{T})\#\mathbb{T})^{\sigma/2}.$$

As a consequence, if we know $\frac{1}{C_F(\mathbb{T})} \geq \lambda$*, then*

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \lambda\delta^\sigma\big(\frac{\delta^2\#\mathbb{T}}{\lambda}\big)^{\min\{\sigma/2,1\}}.$$

Proof. Make a new tube set $\mathbb{T}'$ by taking $C_F(\mathbb{T})$-many random translations and rotations of $\mathbb{T}$. As discussed in Lemma 4.1, $C_F(\mathbb{T}') \lesssim 1$. By the high density lemma Lemma 2.1,

$$\mathrm{Vol}(\cup\mathbb{T}') \gtrapprox \delta^\sigma(\delta^2\#\mathbb{T}')^\sigma = \delta^\sigma(\delta^2 C_F(\mathbb{T})\#\mathbb{T})^{\sigma/2}.$$

On the other hand, $\mathrm{Vol}(\cup\mathbb{T}') \lesssim C_F(\mathbb{T})\,\mathrm{Vol}(\cup\mathbb{T})$. Thus

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \frac{1}{C_F(\mathbb{T})}\delta^\sigma(\delta^2 C_F(\mathbb{T})\#\mathbb{T})^{\sigma/2}.$$

Now, let $\frac{1}{C_F(\mathbb{T})} = A\lambda$ with $A \geq 1$. Then

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \lambda\delta^\sigma\big(\frac{\delta^2\#\mathbb{T}}{\lambda}\big)^{\sigma/2}A^{1-\sigma/2} \gtrapprox \lambda\delta^\sigma\big(\frac{\delta^2\#\mathbb{T}}{\lambda}\big)^{\min\{\sigma/2,1\}}.$$

□

- The low density lemma is a variant of the high density lemma. The proof is analagous—we won't discuss it.

Lemma 5.2 (Low Density Lemma)**.** *Assume* $\mathbf{A}(\sigma)$*. Let* $\mathbb{T}$ *be a set of* δ*-tubes whose maximum density convex set is a* δ*-tube. This is equivalent to saying* $\#\mathbb{T}[U] \lesssim \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(T)}$*. Then*

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \#\mathbb{T}\,\mathrm{Vol}(T)\,(\delta^2\#\mathbb{T})^{-\sigma/2}.$$

The factor $\#\mathbb{T}\,\mathrm{Vol}(T)$ is equal to $\frac{1}{C_F(\mathbb{T})}$. The assumption implies $\#\mathbb{T} \lesssim \delta^{-2}$, so the improvement factor $(\delta^2\#\mathbb{T})^{-\sigma/2}$ is always $\gtrsim 1$, and it is $\gg 1$ if $\#\mathbb{T} \ll \delta^{-2}$.

- The next lemma, the nowhere sticky reduction, will help us apply the high density lemma inside the proof.

Lemma 5.3 (Nowhere Sticky Reduction)**.** *There exists a worst-case-scenario set of tubes* $\mathbb{T}$*, meaning* $C_F(\mathbb{T}) \sim 1$ *and*

$$\mathrm{Vol}(\cup\mathbb{T}) \approx \delta^\sigma,$$

such that

$$\#\mathbb{T}^\rho \gg \rho^{-2} \qquad \textit{for all } \rho \in (\delta, 1) \textit{ strictly.}$$

The proof uses the lossy decomposition. If there was a scale ρ where $\#\mathbb{T}^\rho \sim \rho^{-2}$, we could apply the Lossy Decomposition (Proposition 1.3) to split into two sub-problems. Both those sub-problems have to also be worst-case-scenario. If one if them is nowhere sticky, we are done. Otherwise, we can keep splitting into sub-problems. Eventually, we have to either find a nowhere sticky sub-problem, or find so many sticky scales that we can apply Sticky Kakeya to get a gain.

- The plank-y improvement (Lemma 4.3) says that if the maximum density convex set is $a \times b \times 1$, then

$$\mathrm{Vol}(\cup\mathbb{T}) \gtrapprox \frac{1}{C_F(\mathbb{T})}(\delta\frac{b}{a})^\sigma.$$

- In the proof we will need to study δ-tubes inside of a w-tube. Let $\psi : T^w \to [0,1]^3$ be a rescaling map, and consider the rescaled set $\psi(\mathbb{T}[T^w])$. The inverse Frostman constant is estimated by

$$\begin{aligned}\frac{1}{C_F(\psi(\mathbb{T}[T^w]))} &= \inf_{U\subset T^w} \#(U \text{ to cover } \mathbb{T}[T^w])\frac{\mathrm{Vol}(U)}{\mathrm{Vol}(T^w)} \\ &= \inf_{U\subset T^w} \frac{\#(U \text{ to cover all of } \mathbb{T})\,\mathrm{Vol}(U)}{\#\mathbb{T}^w\,\mathrm{Vol}(T^w)} \\ &\geq \frac{1}{C_F(\mathbb{T})}\frac{1}{\#\mathbb{T}^w\,\mathrm{Vol}(T^w)}. \qquad (5.1)\end{aligned}$$

Suppose $\mathbb{T}$ is Frostman. Then $\frac{1}{C_F(\mathbb{T})} \approx 1$, and $\frac{1}{\#\mathbb{T}^w\,\mathrm{Vol}(T^w)} \lessapprox 1$. If the number of w-tubes is the minimal amount, w^{-2}, then the right hand side is ~ 1. In other words, $\psi(\mathbb{T}[T^w])$ is itself Frostman. The more w-tubes there are, the farther $\mathbb{T}[T^w]$ is from Frostman.

We are ready to start the proof. Let $\mathbb{T}$ be a nowhere sticky, worst-case scenario set of tubes, so

$$\mathrm{Vol}(\cup\mathbb{T}) \approx \delta^\sigma.$$

We would like to find a contradiction.

Pick a scale $\rho \in [\delta, 1]$. At the end of the proof we will talk about how to choose ρ.

Let $\mathbb{B}$ denote the collection of ρ-balls active in covering E. For $B \in \mathbb{B}$, let

$$\mathbb{T}_B = \{T \cap B \,:\, T \in \mathbb{T}\}$$

be the set of $\delta \times \delta \times \rho$ tubelets active inside of B, as in Figure 1. A priori, we don't know much about $\mathbb{T}_B$. We don't even know how many tubelets there are in $\mathbb{T}_B$. As a first step, let $U = a \times b \times \rho$ be the extremizing convex set for $\mathbb{T}_B$.

If U is $\rho\times\rho\times\rho$, i.e. $\mathbb{T}_B$ is Frostman, we could finish the proof right now. We split into a problem above scale ρ and a problem below scale ρ,

$$\mathrm{Vol}(E) \gtrapprox \#\mathbb{B}\,\mathrm{Vol}(B)\frac{\mathrm{Vol}(\cup\mathbb{T}_B)}{\mathrm{Vol}(B)}.$$

To estimate $\mathrm{Vol}(\cup\mathbb{T}_B)$, use $\mathbf{A}(\sigma)$. To estimate $\#\mathbb{B}\,\mathrm{Vol}(B) = \mathrm{Vol}(\cup\mathbb{T}^\rho)$, use the nowhere sticky reduction and the high density lemma. Altogether,

$$\mathrm{Vol}(E) \gtrapprox \delta^\sigma(\#\mathbb{T}^\rho\rho^2)^{\sigma/2} \gg \delta^\sigma,$$

contradicting our assumption that $\mathbb{T}$ is worst-case-scenario. To deal with the general case, when U is not necessarily $\rho\times\rho\times\rho$, we need to use our plank-y improvement (Lemma 4.3).

Towards estimating the Frostman constant of $\mathbb{T}_B$, define

$$\begin{aligned}\#(a\times b\times\rho) &= \text{The total number of } a\times b\times\rho \text{ convex sets active in covering } E\\ &= \#\mathbb{B}\,\#(a\times b\times\rho \text{ needed to cover each } \mathbb{T}_B).\end{aligned}$$

The inverse Frostman constant of $\mathbb{T}_B$ is given by

$$\frac{1}{C_F(\mathbb{T}_B)} = \frac{\#(a\times b\times\rho)\,\mathrm{Vol}(a\times b\times\rho)}{\#\mathbb{B}\,\mathrm{Vol}(B)}.$$

By the plank-y improvement (Lemma 4.3) applied to $\mathbb{T}_B$,

$$\frac{\mathrm{Vol}(\cup\mathbb{T}_B)}{\mathrm{Vol}(B)} \gtrapprox \frac{\#(a\times b\times\rho)\,\mathrm{Vol}(a\times b\times\rho)}{\#\mathbb{B}\,\mathrm{Vol}(B)}\Big(\frac{\delta}{\rho}\frac{b}{a}\Big)^\sigma.$$

Summing over all $B\in\mathbb{B}$ gives an estimate for $\mathrm{Vol}(E)$,

$$\mathrm{Vol}(E) = \#\mathbb{B}\,\mathrm{Vol}(B)\frac{\mathrm{Vol}(\cup\mathbb{T}_B)}{\mathrm{Vol}(B)} \gtrapprox \#(a\times b\times\rho)\,\mathrm{Vol}(a\times b\times\rho)\Big(\frac{\delta}{\rho}\frac{b}{a}\Big)^\sigma. \tag{5.2}$$

We've turned the problem of estimating $\mathrm{Vol}(E)$ into the problem of estimating $\#(a\times b\times\rho)$.

Suppose, as an example, $U=\delta\times\delta\times\rho$. We need to estimate the total number of tubelets. If we're lucky and every $\delta\times\delta\times\rho$ tubelet has just one δ-tube through it, then

$$\#(\delta\times\delta\times\rho) = \#(\delta\text{-tubes})\rho^{-1} = \delta^{-2}\rho^{-1},$$

giving the favorable bound $\#(\delta\times\delta\times\rho)\,\mathrm{Vol}(\delta\times\delta\times\rho)\gtrsim 1$.

What if there are several δ-tubes through each $\delta\times\delta\times\rho$ tubelet? All these δ-tubes have to lie in the $\frac{\delta}{\rho}$-tube we get by scaling up our original tubelet, and they all have ρ^{-1} many tubelets along them. If we take a new δ-tube through one of these other

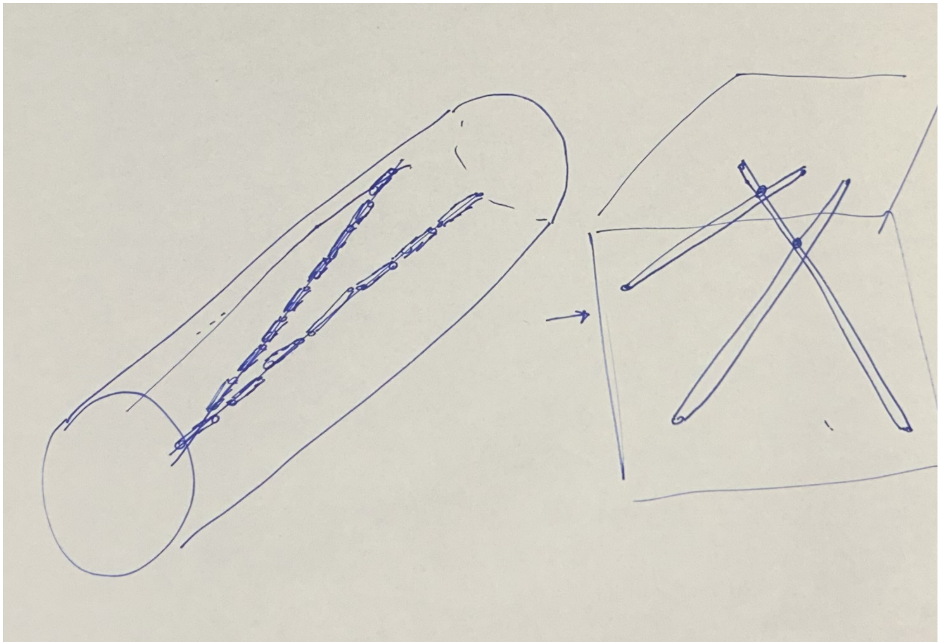

FIGURE 3. To estimate the total number of $\delta \times \delta \times \rho$ tubelets active in covering E, rescale δ/ρ tubes.

tubelets, it still lies in the same $\frac{\delta}{\rho}$-tube. What we are seeing is another Kakeya type problem. If we rescale this $\frac{\delta}{\rho}$-tube to the unit cube,

$$\begin{aligned} \frac{\delta}{\rho}\text{-tube} &\mapsto \text{Unit cube}, \\ \delta\text{-tube} &\mapsto \rho\text{-tube}, \\ \delta \times \delta \times \rho\text{-tubelet} &\mapsto \rho\text{-ball}. \end{aligned}$$

To count the total number of $\delta \times \delta \times \rho$-tubelets, we can apply $\mathbf{A}(\sigma)$ inside of each δ/ρ-tube, and multiply by the number of $\frac{\delta}{\rho}$ tubes. See Figure 3.

A similar idea works if U is $a \times b \times \rho$. The best thing to do is zoom into b/ρ tubes and count copies of U in each of these. Inside each b/ρ-tube, the count of $a \times b \times \rho$ convex sets is related to the volume of the union of a-tubes via

$$\#(a \times b \times \rho \text{ inside } T^{b/\rho})\frac{\mathrm{Vol}(a \times b \times \rho)}{\mathrm{Vol}(T^{b/\rho})} \gtrsim \frac{\mathrm{Vol}(\cup \mathbb{T}^a[T^{b/\rho}])}{\mathrm{Vol}(T^{b/\rho})}.$$

We apply $\mathbf{A}(\sigma)$ to estimate the volume of the union of a-tubes and get

$$\frac{\mathrm{Vol}(\cup \mathbb{T}^a[T^{b/\rho}])}{\mathrm{Vol}(T^{b/\rho})} \gtrapprox (a\frac{\rho}{b})^\sigma \frac{1}{C_F(a\text{-tubes inside } b/\rho\text{-tube})}.$$

To estimate the Frostman constant, we use (5.1),

$$\frac{1}{C_F(a\text{-tubes inside } b/\rho\text{-tube})} \gtrapprox (a\frac{\rho}{b})^\sigma \frac{1}{\#(\mathbb{T}^{b/\rho})\,\mathrm{Vol}(T^{b/\rho})}.$$

When we combine these estimates to count the total number of $a \times b \times \rho$, the factor $\#(\mathbb{T}^{b/\rho})\,\mathrm{Vol}(T^{b/\rho})$ cancels and we are left with

$$\#(a \times b \times \rho)\,\mathrm{Vol}(a \times b \times \rho) \gtrapprox (a\frac{\rho}{b})^\sigma.$$

Plugging this estimate into (5.2) gives

$$\mathrm{Vol}(E) \gtrapprox \delta^\sigma.$$

We have recovered $\mathbf{A}(\sigma)$ from itself by splitting into various sub-problems. This is the *lossless decomposition*.

- The first sub-problem was applying Lemma 4.3 to $\mathbb{T}_B$. Under the hood, $\mathbf{A}(\sigma)$ is applied to various collections of tubes related to $\mathbb{T}_B$.
- The second sub-problem was applying $\mathbf{A}(\sigma)$—more precisely, the general version Lemma 4.1—to the collection of a-tubes inside a b/ρ-tube.

We hope to get an improvement on the second sub-problem, to contradict the hypothesis that $\mathbb{T}$ is worst-case-scenario. If $a \gg \delta$, then by the nowhere sticky reduction, the number of a-tubes inside a b/ρ-tube is greater than the easy lower bound

$$\frac{a^{-2}}{\#\mathbb{T}^{b/\rho}},$$

so by the refined high density lemma (Lemma 5.1) provides an improvement.

If $a = \delta$ and $b \ll \rho$, then the low density lemma Lemma 5.2 provides an improvement. There's one remaining case we haven't dealt with: If

$$U = \delta \times \rho \times \rho,$$

then the second sub-problem is the same one we started with, so the high density lemma does not give an improvement.

To deal with this last case, we will choose ρ very close to 1, in a way that depends on σ. If $U = \delta \times \rho \times \rho$, the L^2 method implies

$$\mathrm{Vol}(E) \gtrsim \#(\delta \times \rho \times \rho)\,\mathrm{Vol}(\delta \times \rho \times \rho),$$

and the Convex Frostman hypothesis implies

$$\#(\delta \times \rho \times \rho)\,\mathrm{Vol}(\delta \times \rho \times \rho) \gtrsim \rho^{10}.$$

We choose $\rho = \delta^{\sigma/100}$, so that these estimates give a gain over $\mathbf{A}(\sigma)$. Remember that the gain from the high density lemma requires ρ to be strictly in the range $(\delta, 1)$, which is why we can't choose $\rho = 1$.

References

[1] Larry Guth, *Outline of the Wang–Zahl proof of the Kakeya conjecture in* $\mathbb{R}^3$, arXiv preprint arXiv:2508.05475 (2025). https://arxiv.org/abs/2508.05475.

[2] ———, *Introduction to the proof of the Kakeya conjecture*, arXiv preprint arXiv:2505.07695 (2025). https://arxiv.org/abs/2505.07695.

[3] Nets Katz and Joshua Zahl, *An improved bound on the Hausdorff dimension of Besicovitch sets in* $\mathbb{R}^3$, J. Amer. Math. Soc. **32** (2019), no. 1, 195–259, DOI 10.1090/jams/907.

[4] Tamás Keleti and Pablo Shmerkin, *New bounds on the dimensions of planar distance sets*, Geom. Funct. Anal. **29** (2019), 1886–1948, DOI 10.1007/s00039-019-00500-9.

[5] Hong Wang and Joshua Zahl, *Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions*, arXiv preprint arXiv:2502.17655 (2025). https://arxiv.org/abs/2502.17655.

[6] ———, *Sticky Kakeya sets and the sticky Kakeya conjecture*, arXiv preprint arXiv:2210.09581 (2022). https://arxiv.org/abs/2210.09581.