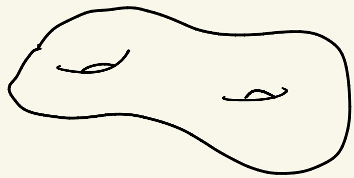
Higher dimensional fractal uncertainty

Quantum chaos



M a cpct hyperbolic mfld

$\Delta u_j = h_j^{-2} u_j$

Q What does $|u_j|^2$ look like?

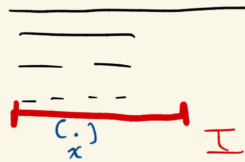
Conj (QUE) $|u_j|^2 \to d_{Leb}$

Fourier uncertainty principle	Info about $|u_j|^2$
1 Volume bound $\mathrm{supp}\,\hat{f} \subset Y \Rightarrow \|f 1_X\|_2 \le |X|^{1/2} |Y|^{1/2} \|f\|_2$	$\to$ $|u_j|^2$ is not concentrated on a closed geodesic (Anantharaman)
2 Entropic uncertainty principle	$\to$ Entropy lower bound (A. + Nonnenmacher)
3 Fractal uncertainty principle	$\to$ $\int_U |u_j|^2 dx \geq c(U) > 0$ all open sets U (Dyatlov + Jin)

1 and 2 uses $L^1 \to L^\infty$. 3 uses complex analysis

Def $X \subset \mathbb{R}$ is porous if on scales $\alpha_0 < R < \alpha_1$ if

$$\forall\ |I| = R,\ \exists\, x \in I \ \text{ s.t. } \ B_{\nu R}(x) \cap X = \emptyset$$



Thm (Bourgain-Dyatlov)

- $X \subset [0,1]$ porous from h to 1
- $Y \subset [0,1]$ porous from 1 to h^{-1}

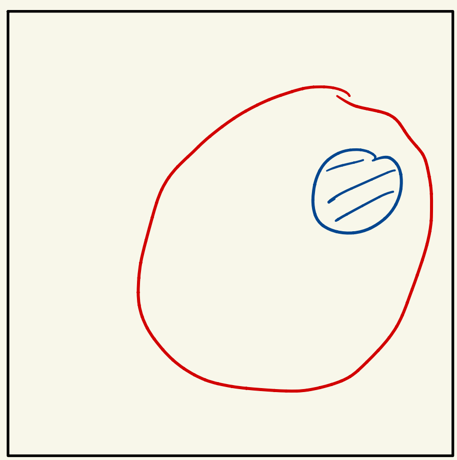
$$\operatorname{supp} \hat{f} \subset Y \Rightarrow \|f 1_X\|_2 \leq h^{\beta} \|f\|_2 \qquad \beta = \beta(\nu) > 0$$

[Modify porous $\longrightarrow$ porous on balls]

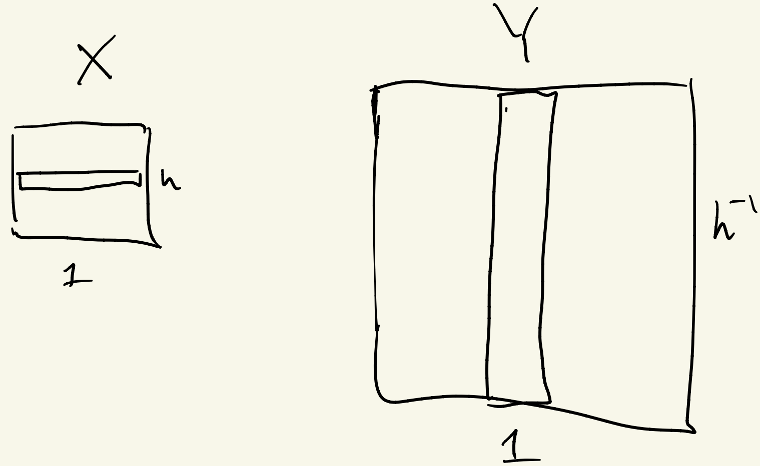
Def $X \subset \mathbb{R}$ is porous on balls if

$\forall B \subset \mathbb{R}^d$ an R-ball,

$\exists\, x \in B$ s.t. $B_{\nu R}(x) \cap X = \emptyset$



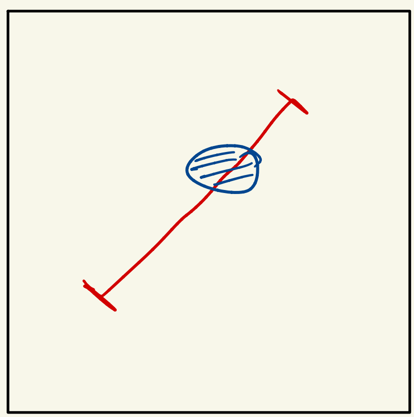
Ex



$$\widehat{1_X} \approx 1_Y$$

X and Y are porous on balls but do <u>not</u> satisfy FUP.

Def $X \subset \mathbb{R}^d$ is porous <u>on lines</u>



For all line segments τ of length

$\alpha_0 < R < \alpha_1$,

$\exists\, x \in \tau$ s.t. $B_{\nu R}(x) \cap X = \emptyset$.

Main thm

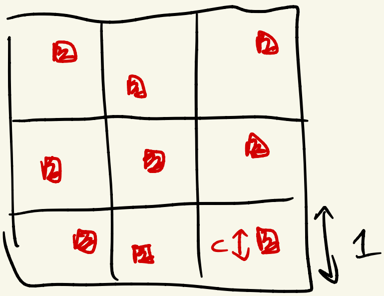
- $X \subset [0,1]^d$ porous on balls from h to 1

- $Y \subset [0,h^{-1}]^d$ porous on lines from 1 to h^{-1}

$$\operatorname{supp} \hat{f} \subset Y \implies \|f 1_X\|_2 \leq h^{\beta} \|f\|_2$$

Quantitative unique continuation

$\mathcal{U} =$



Goal: $\operatorname{Supp} \hat{f} \subset Y \Rightarrow \|f 1_{\mathcal{U}}\|_2 \geq \gamma \|f\|_2$

Def: $\psi \in L^2(\mathbb{R}^d)$ is a damping function for Y if

- $\operatorname{Supp} \psi \subset B_1$ $\quad \psi \neq 0$
- $|\hat{\psi}(\xi)|$ decays very quickly for $\xi \in Y$

$$\text{Damping function} \xrightarrow{f \to f * \psi} \text{QUC}$$

Thm (Beurling – Malliavin)

Let $\omega : \mathbb{R} \to \mathbb{R}_{\leq 0}$ have

1. Regularity $|\omega(x) - \omega(y)| \leq K|x-y|$
2. Growth $\displaystyle\int_{-\infty}^{\infty} \frac{|\omega(t)|}{1+t^2}\,dt \leq K$

Then $\exists f \in L^2(\mathbb{R})$, $\operatorname{supp} \hat{f} \subset [-1,1]$, $|f(x)| \leq e^{\omega(x)}$

Thm

Let $\omega : \mathbb{R}^d \to \mathbb{R}_{\leq 0}$ satisfy

1. Regularity
2. Growth $\longleftarrow$ mass of ω on lines through origin

Then $\exists f \in L^2(\mathbb{R})$ $\operatorname{supp} \hat{f} \subset B_1$ $f \neq 0$

$$|f(x)| \leq e^{\omega(x)}$$

Ex

1. $\omega(x) = -|x|$ ✗
2. $\omega(x) = \dfrac{-|x|}{\log(2+|x|)}$ ✗
3. $\omega(x) = \dfrac{-|x|}{(\log(2+|x|))^2}$ ✓

Thm (Paley–Wiener)

TFAE $f \in L^2(\mathbb{R})$

- $\operatorname{supp} \hat{f} \subset B_\sigma$ ~ polynomial coeffs
- $f(z)$ analytic and $f(\underline{x}+iy) \leq Ae^{\sigma|y|}$ ~ polynomial roots

Main idea

$f: \mathbb{C}^d \to \mathbb{C}^d$ analytic then $\log|f|$ is plurisubharmonic

And this is the most important info about f.

Ex $f = (z-\alpha_1)\cdots(z-\alpha_N)$

$$\Delta(\log|f|) = 2\pi \sum \delta_{\alpha_j}$$

Fourier analysis $\operatorname{supp}\hat{f} \subset B_1$, $|f(x)| \leq e^{\omega(x)}$

$$\Downarrow$$

Complex analysis $|f(x+iy)| \leq Ae^{\sigma|y|}$

$$\Downarrow$$

Potential theory $(*)$
$$\begin{cases} u = \log|f| \text{ is plurisubharmonic} \\ u(x+iy) \leq A + \sigma|y| \\ u(x) \leq \omega(x) \qquad x \in \mathbb{R} \end{cases}$$

Step 1 Solve (✱)

Step 2 Use u to construct f

Use Hörmander $\bar\partial$ following Bourgain

Psh construction

$$u = E\omega + C|y|$$

bdd

make u more psh

$$E\omega(x+iy) = \int \frac{\omega(x+ty)}{1+t^2}\frac{dt}{\pi}$$

= separate harmonic extension to every real-linear complex line

Lem If ω satisfies two props

① and ②

then $E\omega + C|y|$ is psh

In general, take

$$\omega \to \tilde{\omega} \le \omega$$

and satisfies ① and ②

$u = E\tilde{\omega} + C|y|$ solves step 1.

Modification of the weight

Observation: If you zoom out far enough, all lines look like they pass through the origin

Choose $g \le 0$ constant in radial direction and s.t. for all $\hat{v} \in S^{d-1}$,

$$\int_0^{\infty} \frac{(w+g)(t\hat{v})}{t^2}\, dt = \text{const. indep of } \hat{v}$$

$$\tilde{w} = w + g$$

Construction of analytic function from weight

Thm (Hörmander $\bar\partial$)

Let $\cdot\ \varphi : \mathbb{C}^d \to \mathbb{R}$ be strictly psh

$\cdot\ \eta$ a $(0,1)$ form on $\mathbb{C}^d$ w/ $\bar\partial \eta = 0$

Then

$$\bar\partial g = \eta$$

has a soln $g : \mathbb{C}^d \to \mathbb{C}$ satisfying

$$(*) \quad \int_{\mathbb{C}^d} |g|^2 e^{-\varphi(z)} \le \int |\eta|^2 \frac{e^{-\varphi(z)}}{K(z)} \leftarrow \text{psh lower bd}$$

$$\varphi \approx u + 10d \log|z|_\infty + \cdots$$

h = Bump function const. on B_1, vanish outside B_2

$\eta = \bar\partial h$

$\bar\partial g = \bar\partial h$

$f = g - h$