# FRACTAL UNCERTAINTY PRINCIPLE OVER THE $p$-ADICS

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0.1. **Summary.** Bourgain and Dyatlov [1] proved a fractal uncertainty principle (FUP) for porous subsets of $\mathbb{R}$. The proof breaks down over the $p$-adics $\mathbb{Q}_p$. I would be very interested to know if FUP still holds.

0.2. **The fractal uncertainty principle over $\mathbb{R}$.** Let $(\Omega, d)$ be a metric space. We say a set $\mathbf{X} \subset \Omega$ is *$\nu$-porous* from scales $\alpha_0$ to $\alpha_1$ if for every ball $\mathsf{B}$ of diameter $R \in (\alpha_0, \alpha_1)$, there exists a point $\mathbf{x} \in \mathsf{B}$ such that the ball $\mathsf{B}_{\nu R}(\mathbf{x})$ (with center $\mathbf{x}$ and radius $\nu R$) is disjoint from $\mathbf{X}$. Porous sets are an example of fractals. Bourgain and Dyatlov proved an uncertainty principle for porous subsets of $\mathbb{R}$.

**Theorem 0.1** (Bourgain–Dyatlov [1, Theorem 4])**.** *Let $\nu > 0$ and suppose that*

- $\mathbf{X} \subset [-1, 1] \subset \mathbb{R}$ *is $\nu$-porous from scales $h$ to $1$, and*
- $\mathbf{Y} \subset [-h^{-1}, h^{-1}] \subset \mathbb{R}$ *is $\nu$-porous from scales $1$ to $h^{-1}$.*

*Then there exist constants $\beta, C > 0$, depending only on $\nu$, such that for all $f \in L^2(\mathbb{R})$*

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \quad \Longrightarrow \quad \|f \mathbf{1}_{\mathbf{X}}\|_2 \leq C\, h^{\beta}\, \|f\|_2. \tag{0.1}$$

We use the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x}\, f(x)\, dx, \tag{0.2}$$

and the inverse Fourier transform

$$f^{\vee}(x) = \int_{\mathbb{R}} e^{2\pi i \xi x}\, \hat{f}(\xi)\, d\xi. \tag{0.3}$$

The proof of Theorem 0.1 is based on analyzing decay in the tails. To give an example of this analysis, suppose $\hat{f}$ has compact Fourier support. We can use the Fourier extension formula (0.3) to extend $f$ to an entire function on $\mathbb{C}$. If you plug in a complex number $x + iy$, the exponential term $e^{2\pi i \xi (x+iy)}$ may grow exponentially in $\xi$, but because we assumed $\hat{f}$ is compactly supported, the integral is still defined. As entire functions have isolated zeros, $f$ has full support. This is called a unique continuation principle, because the values of $f$ everywhere are determined by its values on any interval $I \subset \mathbb{R}$.

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Bourgain–Dyatlov quantitatively strengthened this result. If $E \subset \mathbb{R}$ satisfies

$$E \cap [n, n+1] \text{ contains a } \nu\text{-interval for every } n \in \mathbb{Z},$$

then there is a mass lower bound

$$\hat{f} \text{ decays very rapidly} \implies \|f 1_E\|_2 \geq c(\nu)\|f\|_2. \tag{0.4}$$

Bourgain–Dyatlov then upgraded this theorem to prove

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \implies \|f 1_E\|_2 \geq c(\nu)\|f\|_2. \tag{0.5}$$

To prove (0.5) from (0.4), Bourgain and Dyatlov construct compactly supported functions $\psi$ whose Fourier transforms decay very rapidly on $\mathbf{Y}$. If $\operatorname{supp} \hat{f} \subset \mathbf{Y}$, then $f * \psi$ decays very rapidly, and (0.4) can be applied to $f * \psi$.

0.3. **Fractal uncertainty conjecture over the $p$-adics.** For lots of problems in harmonic analysis, the $p$-adics $\mathbb{Q}_p$ behave much like the real numbers. For this problem, they behave differently.

Elements of the $p$-adics are formal sums

$$x = \sum_{n \geq n_0} a_n p^n \qquad \text{where } a_n \in \{0, \ldots, p-1\} \text{ and } n_0 \in \mathbb{Z}.$$

Addition and multiplication are defined by carrying from left to right. We define the valuation of $x$ as

$$\operatorname{val}_p(x) := \text{The smallest } n_0 \text{ such that } a_{n_0} \neq 0.$$

Using the valuation, we place a norm (and hence a metric) on $\mathbb{Q}_p$,

$$|x|_p = p^{-\operatorname{val}_p(x)}.$$

A key property of the metric is that $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. This implies that balls around the origin are subgroups. The ball of radius $p^j$ around the origin is the subgroup

$$B_{p^j}(\mathbb{Q}_p) = \{\sum_{n \geq j} a_n p^n\},$$

and the unit ball is the group of $p$-adic integers,

$$B_1(\mathbb{Q}_p) = \mathbb{Z}_p = \{\sum_{n \geq 0} a_n p^n\}.$$

The $p$-adics come equipped with a natural Haar measure under which the unit ball has measure 1. Fix the character

$$\chi(x) = \exp(2\pi i \sum_{n \leq 0} a_n p^n).$$

Use this character to define the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{Q}_p} \chi(-x\xi) f(x) dx$$

and the inverse Fourier transform

$$f^\vee(x) = \int_{\mathbb{Q}_p} \chi(x\xi) f(\xi) dx.$$

From the perspective of unique continuation, the $p$-adics behave very differently from the real numbers. The indicator function of the unit ball is mapped to itself under the Fourier transform,

$$\widehat{1}_{B_1(\mathbb{Q}_p)} = 1_{B_1(\mathbb{Q}_p)}. \tag{0.6}$$

This is very different from the real numbers, where functions with compactly supported Fourier transform must have full support. Because of (0.6), the unique continuation principle (0.4) is not true over the $p$-adics, so the proof of Theorem 0.1 breaks down in a fundamental way.

I tried and failed to construct a counterexample to FUP over the $p$-adics. I'll optimistically conjecture that FUP still holds.

**Conjecture 0.2** (FUP over $p$-adics)**.** *Let $\nu > 0$ and $h = p^{-n}$. Suppose*

- $\mathbf{X} \subset B_1 \subset \mathbb{Q}_p$ *is $\nu$-porous from scales $h$ to 1, and*
- $\mathbf{Y} \subset B_{h^{-1}} \subset \mathbb{Q}_p$ *is $\nu$-porous from scales 1 to $h^{-1}$.*

*Then there exist constants $\beta, C > 0$, depending only on $\nu$, such that for all $f \in L^2(\mathbb{Q}_p)$*

$$\operatorname{supp} \hat{f} \subset \mathbf{Y} \quad \Longrightarrow \quad \|f \mathbf{1}_{\mathbf{X}}\|_2 \le C\, h^\beta\, \|f\|_2. \tag{0.7}$$

In this theorem statement, the fact that $\operatorname{supp} \hat{f} \subset B_{h^{-1}}$ implies that $f$ is constant on balls of radius $h$. What does this mean? In the inverse Fourier transform formula

$$f^\vee(x) = \int_{\mathbb{Q}_p} \chi(x\xi) f(\xi) dx,$$

the frequency $\xi$ may have negative powers of $p$ going down to $h = p^{-n}$, but no more than that:

$$\xi = \sum_{j \ge -n} b_j p^{-j}.$$

So, if we take

$$x = \sum a_j p^j$$

and change the coefficients $a_j$ for $j \ge n$, that changes $x\xi$ by an element of $\mathbb{Z}_p$, and the value of $\chi(x\xi)$ stays the same.

In (0.7) we may assume $\operatorname{supp} f \subset B_1(\mathbb{Q}_p)$. First, we may assume $\mathbf{Y}$ is a union of $h$-balls. Then, if we take $f$ and replace it with $f 1_{B_1(\mathbb{Q}_p)}$, that has the effect of convolving $f$ with $1_{B_1(\mathbb{Q}_p)}$, which does not impact the Fourier support.

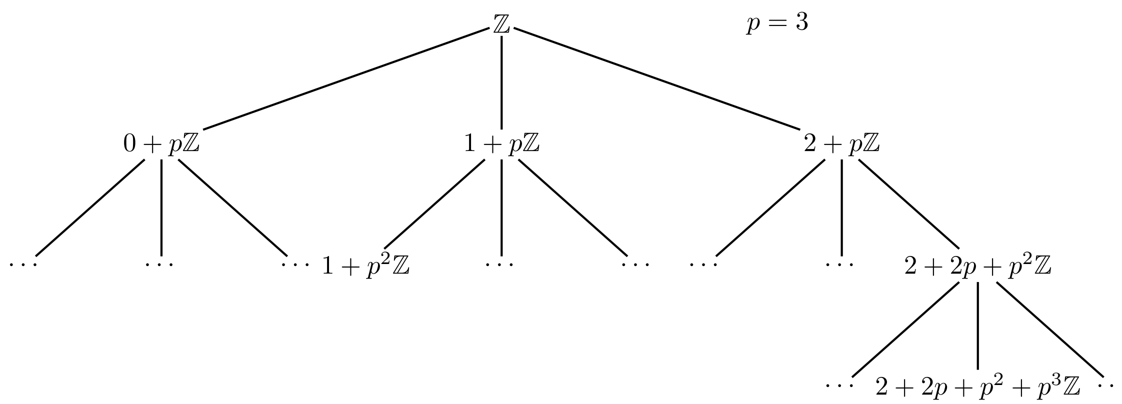


FIGURE 1. Tree structure on $\mathbb{Z}/p^k\mathbb{Z}$ with $p = 3$.

What have we learned? From the perspective of (0.7), it is sufficient to consider functions $f$ with

$$\operatorname{supp} f \subset B_1(\mathbb{Q}_p) \qquad \text{and} \qquad f \text{ const. on balls of radius } h,$$

which is equivalent to the class of functions

$$\operatorname{supp} \hat{f} \subset B_{h^{-1}}(\mathbb{Q}_p) \qquad \text{and} \qquad \hat{f} \text{ const. on balls of radius } 1.$$

The first class is the functions on the group $B_1(\mathbb{Q}_p)/B_h(\mathbb{Q}_p)$, and the second is the functions on the group $B_{h^{-1}}(\mathbb{Q}_p)/B_1(\mathbb{Q}_p)$. Both of these groups are isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$.

We can restate Conjecture 0.2 in terms of the group $\mathbb{Z}/p^n\mathbb{Z}$. One can define the Fourier transform on $\mathbb{Z}/p^n\mathbb{Z}$ by taking a non-degenerate character $\chi : \mathbb{Z}/p^n\mathbb{Z} \to S^1$ and setting

$$\hat{f}(\xi) = |G|^{1/2}\mathbb{E}_G \chi(-x\xi) f(x), \qquad g^{\vee}(x) = |G|^{1/2}\mathbb{E}_G \chi(x\xi) g(\xi),$$

and state an FUP in this context. The valuation on $\mathbb{Z}/p^n\mathbb{Z}$ is

$$\operatorname{val}(x) = \text{The smallest } j \text{ such that } x \in p^j\mathbb{Z}/p^n\mathbb{Z},$$

and the metric is $|x|_{\mathbb{Z}/p^n\mathbb{Z}} = p^{-\operatorname{val}(x)}$. Under this metric,

$$B_{p^{-j}}(\mathbb{Z}/p^n\mathbb{Z}) = p^j\mathbb{Z}/p^n\mathbb{Z}.$$

This metric puts a tree structure on $\mathbb{Z}/p^n\mathbb{Z}$. There are $p$-many balls of radius $p^{-1}$, corresponding to the $p$-many residue classes mod $p$. Each of these $p$-many balls contain $p$ many balls of radius $p^{-2}$, corresponding to the residue mod $p^2$. This continues down to $p^n$. See Figure 1.

In the special case $\nu = \frac{1}{p}$, a porous set is constructed iteratively. To construct a porous set $\mathbf{X}$, first, select all but 1 of the balls of radius $p^{-1}$. Then, in each of these, select all but 1 of the balls of radius $p^{-2}$. Continue down to scale $p^{-n}$. You will have constructed a porous set with $(p-1)^n$ elements.

0.4. **Self-similar Cantor sets.** Dyatlov and Jin [2] proved an FUP when $\mathbf{X}$ and $\mathbf{Y}$ are self-similar Cantor sets. Fix alphabets $\mathcal{A} \subsetneq \{0, \dots, p-1\}$ and $\mathcal{B} \subsetneq \{0, \dots, p-1\}$. Let

$$\mathbf{X} = \{\sum_{j=0}^{n-1} a_j p^j \,:\, a_j \in \mathcal{A}\} \subset \mathbb{Z}/p^n\mathbb{Z},$$

and

$$\mathbf{Y} = \{\sum_{j=0}^{n-1} a_j p^j \,:\, a_j \in \mathcal{B}\} \subset \mathbb{Z}/p^n\mathbb{Z}.$$

For these self-similar Cantor sets, $\mathbf{X}$ and $\mathbf{Y}$ are also porous in the Euclidean metric on $\mathbb{Z}$, so FUP holds for the same reason as Theorem 0.1. But in general, porous sets in $\mathbb{Q}_p$ need not be porous in the Euclidean metric, and a new idea is needed.

0.5. **A comment on the Fourier transform on $\mathbb{Z}/p^n\mathbb{Z}$.** From the $p$-adic perspective, it is helpful to distinguish the two copies $B_1(\mathbb{Q}_p)/B_h(\mathbb{Q}_p)$ and $B_{h^{-1}}(\mathbb{Q}_p)/B_1(\mathbb{Q}_p)$. There is a canonical isomorphism

$$B_1(\mathbb{Q}_p)/B_{p^{-n}}(\mathbb{Q}_p) \longrightarrow \mathbb{Z}/p^n\mathbb{Z},$$

and the character $\chi$ on $\mathbb{Q}_p$ gives an isomorphism

$$B_{p^n}(\mathbb{Q}_p)/B_1(\mathbb{Q}_p) \longrightarrow \mathbb{T}_{p^n} =: \{y \in \mathbb{R}/\mathbb{Z} \,:\, p^n y \equiv 0\}.$$

Let $x \in \mathbb{Z}/p^n\mathbb{Z}$ and $y \in \mathbb{T}_{p^n} \subset \mathbb{R}/\mathbb{Z}$. Because $p^n y \in \mathbb{Z}$, there is a natural pairing $xy \in \mathbb{R}/\mathbb{Z}$. Taking $\chi$ to be the standard character on $\mathbb{R}/\mathbb{Z}$ gives rise to a Fourier transform

$$L^2(\mathbb{Z}/p^n\mathbb{Z}) \to L^2(\mathbb{T}_{p^n}).$$

One should think of $\mathbb{Z}/p^n\mathbb{Z}$ as living inside the unit ball down to scale $p^{-n}$, and $\mathbb{T}_{p^n}$ as living inside the ball of radius $p^n$ down to scale 1.

From this perspective, the Fourier transform between $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{T}_{p^n}$ is analogous to the usual Fourier transform on $\mathbb{R}$, in which scale $h$ is paired to scale $h^{-1}$, and the Fourier transform between $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Z}/p^n\mathbb{Z}$ is analogous to the semiclassical Fourier transform on $\mathbb{R}$, in which scale $h$ is paired to scale 1.

I think one could tell a similar story for $\mathbb{Z}/M^k\mathbb{Z}$ rather than $\mathbb{Z}/p^k\mathbb{Z}$, where $M$ is not necessarily prime, and not much would change.

## References

[1] Jean Bourgain and Semyon Dyatlov, *Spectral gaps without the pressure condition*, Ann. of Math. (2) **187** (2018), no. 3, 825–867.

[2] Semyon Dyatlov and Long Jin, *Resonances for open quantum maps and a fractal uncertainty principle*, Comm. Math. Phys. **354** (2017), no. 1, 269–316.