A CIRCLE OF IDEAS AROUND THE GAUSS-BONNET THEOREM

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Contents

1. Introduction

The Gauss-Bonnet theorem is a fundamental result which establishes a connection between geometry and topology.

Theorem 1 (Gauss-Bonnet, ∼1848)**.** *For M a compact oriented Riemann surface, $K : M \to \mathbb{R}$ the Gaussian curvature, $\chi(M)$ the Euler characteristic, we have*

$$\int_M K dA = 2\pi\chi(M).$$

A two dimensional oriented manifold is classified by its Euler characteristic, which is related to the genus by

$$\chi(M) = 2 - 2g.$$

The Gauss-Bonnet theorem says that the average curvature on a Riemannian surface does not depend on the choice of metric, but only on the genus. Qualitatively: any Riemannian metric on a sphere must have positive total curvature, any metric on a torus must have zero total curvature, and any metric on a genus > 1 surface has negative total curvature. See Figure

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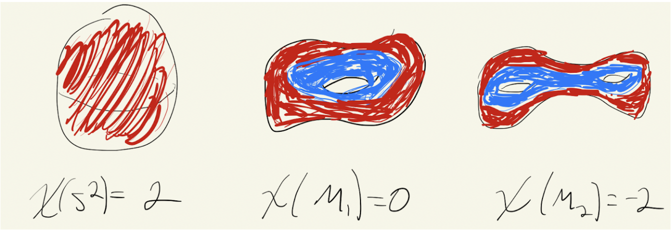


Figure 1. The larger the genus, the smaller the total curvature. Red is positive curvature and blue is negative.

1. Imagine, for instance, embedding a sphere in $\mathbb{R}^3$. The unit sphere has uniform Gaussian curvature equal to one, so $\int_{S^2} K dA = 4\pi$, the area of the sphere. This comports with the theorem, as $\chi(S^2) = 2$. The sphere can be deformed to induce areas of higher and lower curvature, but all such variations must cancel out. It took many years for the proper higher dimensional analogue to be discovered.

Theorem 2 (Chern, 1945)**.** *Let M be a $2n$-dimensional Riemannian manifold with curvature matrix Ω (each entry of which is a 2-form) relative to a frame. Then the Pfaffian polynomial evaluated on this matrix $\mathrm{Pf}(\Omega)$ is a $2n$-form which does not depend on the frame, and satisfies*

$$\frac{1}{(2\pi)^n} \int_M \mathrm{Pf}(\Omega) = \chi(M).$$

Chern's original proof of this theorem was extremely elegant, especially for the time: an alternative route towards these ideas by Weil and collaborators involved much more complex calculations.

Chern's theorem led great development of this connection between geometry and topology. One piece of that story is Chern-Weil theory, which places Chern's theorem in the larger context of characteristic classes for vector bundles and principal bundles. This point of view explains why the Pfaffian polynomial appears in Chern's theorem: an analagous version holds when the Pfaffian is replaced by any polynomial in the entries of a matrix that is invariant under a change of basis. It also leads to a nice proof Chern's theorem, which concludes the general result from the rank two version using naturalness properties for vector bundles. The Gauss-Bonnet-Chern theorem for vector bundles is due to Bell, its statement is a straightforward generalization of the original.

Theorem 3 (Bell, 2000)**.** *Let E denote a real orientable Riemannian vector bundle of rank $2n$ over a compact manifold M of dimension $2n$. Then*

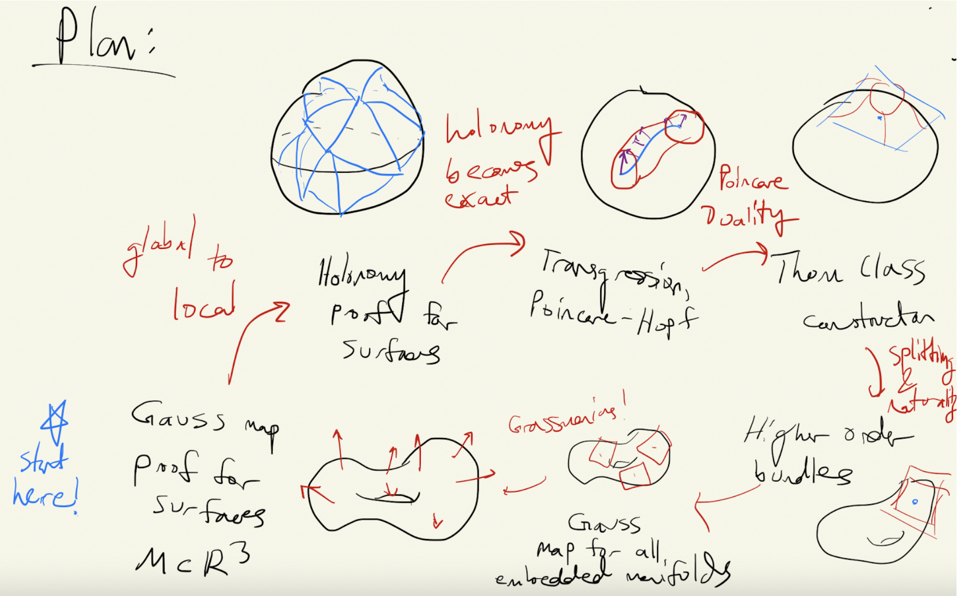
$$\frac{1}{(2\pi)^n} \int_M \mathrm{Pf}(\Omega) = \chi(E).$$

As before, Ω is the curvature matrix of an arbitrary metric connection on E, Pf is the Pfaffian polynomial, and $\chi(E)$ is the Euler characteristic of E.

We will attempt to draw out some of the ideas in this story by discussing a number of proofs of the Gauss-Bonnet theorem, eventually moving towards Bell's theorem for vector bundles. The plan is as follows:

1. We begin by proving the Gauss-Bonnet theorem for hypersurfaces in $\mathbb{R}^3$
2. We generalize to all Riemannian surfaces by introducing holonomy and the local Gauss-Bonnet theorem
3. We introduce the transgression phenomenon as a more elegant way of understanding the connection between holonomy and curvature, and give Chern's original proof of Theorem 2 specialized to surfaces
4. We use a connection to build the Thom class for surfaces. We illustrate the connection between the Thom class and Euler class.
5. We give Bell's proof of Theorem 3 using the splitting principle.
6. We give a proof of the Gauss-Bonnet-Chern theorem for any embedded manifold using a higher dimensional Gauss map

All manifolds and vector bundles will be smooth and oriented. We assume some topological prerequisites: in the sections on the Gauss-Bonnet theorem, we use the Poincare-Hopf index theorem relating the Euler characteristic to the index of a vector field, and we use Euler's formula $\chi(M) = V - E + F$. In the latter sections we use some basic facts about the Euler class: its definition in terms of the Thom class for a vector bundle, and naturalness under vector bundle operations.



2. Extrinsic proof for surfaces embedded in $\mathbb{R}^3$

Let $M \subset \mathbb{R}^3$ be a compact surface. For $x \in M$, let $\mathbf{n}(x)$ be the outward pointing normal vector to M at x. The shape operator $S : T_pM \to T_pM$

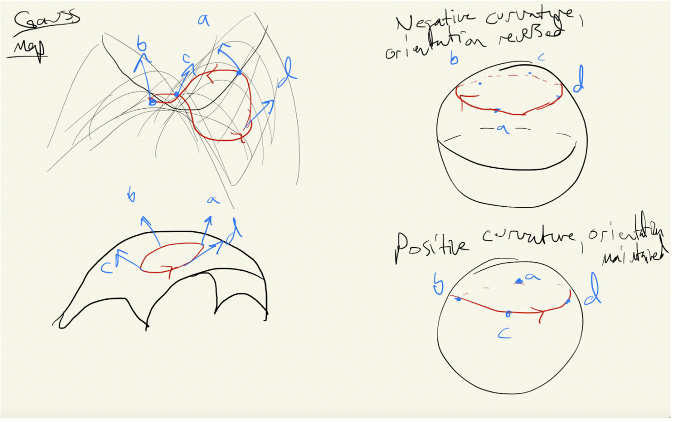


FIGURE 2. The determinant of the Gauss map gives the curvature.

takes $v \mapsto \nabla_v \mathbf{n}$—in other words, the shape operator measures the rate of change of the normal vector. Gauss's theorem egregium says that

$$\det S_p = K_p,$$

that is, the Gaussian curvature (an intrinsic quantity) is equal to the determinant of the shape operator (an extrinsic quantity).

Taking normal vectors gives a map $\mathbf{n} : M \to S^2$, called the Gauss map. We consider S^2 as embedded in $\mathbb{R}^3$ and give M and S^2 the outward pointing orientation. We can view the shape operator as the derivative of this map, by identifying T_pM with $T_{\mathbf{n}(p)}S^2$. Then picking an orthonormal basis for T_pM and $T_{\mathbf{n}(p)}S^2$, the determinant of $d\mathbf{n}$ is well defined, and $K_p = \det d\mathbf{n}$. See Figure 2.

The Gauss map has some topological degree $\deg \mathbf{n}$. If we smoothly vary the embedded manifold M, then the Gauss map smoothly varies as well, giving a homotopy, so the new Gauss map has the same topological degree. It follows, then, that the degree of the Gauss map depends only on the topology of an embedded manifold, not on the particular embedding. In fact, $\deg \mathbf{n} = \frac{1}{2}\chi(M)$. For a proof, take $v \in S^2$ such that $\pm v$ are both regular values. Then if $\mathbf{n}^{-1}(v) = \{x_1, \ldots, x_n\}$ and $\mathbf{n}^{-1}(-v) = \{y_1, \ldots, y_m\}$, we have

$$\deg \mathbf{n} = \sum_{i=1}^{n} \operatorname{sgn} d\mathbf{n}|_{x_i} = \sum_{i=1}^{m} \operatorname{sgn} d\mathbf{n}|_{y_i}.$$

On the other hand, consider the vector field V on M with V_p the orthogonal projection of v onto T_pM. The critical points of this vector field are precisely $x_1, \ldots, x_n, y_1, \ldots, y_m$. Moreover, the index of V at x_i, is equal to $d\mathbf{n}|_{x_i}$ and similarly for y_i. By the Poincare-Hopf theorem,

$$\sum_{i=1}^{n} \operatorname{sgn} d\mathbf{n}|_{x_i} + \sum_{i=1}^{m} \operatorname{sgn} d\mathbf{n}|_{y_i} = \chi(M)$$

so $\deg \mathbf{n} = \frac{1}{2}\chi(M)$.

We can also calculate the degree of $\mathbf{n}$ by integrating. Indeed, let dA be the area form (on M or on S^2). Then using the general fact that for $f : M \to N$ a smooth map we have $\int_M f^*\omega = \deg f \int_N \omega$, it follows that

$$\int_M \mathbf{n}^* dA_{S^2} = \deg \mathbf{n} \int_{S^2} dA_{S^2} = 4\pi \deg \mathbf{n}$$

But $\mathbf{n}^* dA_{S^2} = (\det d\mathbf{n}) dA_M = K dA_M$, so

$$\int_M K dA = 2\pi\chi(M)$$

which is the Gauss-Bonnet theorem.

This proof does not cover the Gauss-Bonnet theorem in full generality, but it does help explain why the integral of the curvature does not depend on the metric.

Example 1 (S^2). In the case of the unit sphere, the Gauss map is simply the identity map, leading to a topological degree of one which is indeed half of $\chi(S^2) = 2$.

3. Local Gauss-Bonnet

3.1. **Holonomy.** While the above proof is very clean, it only works for embedded manifolds in $\mathbb{R}^3$, which isn't general enough—for instance, it does not apply to a flat torus. We will give another proof based on the notion of holonomy.

Let M be a surface with Riemannian metric, and $\gamma : [0, T] \to M$ a closed unit speed curve with $\gamma(0) = \gamma(T) = p$. Let $v_0 = \gamma'(0) \in T_pM$, and imagine parallel translating v_0 about the curve γ. We will end up with some other vector $v_1 \in T_pM$, which in general will be a rotated copy of v_0. The holonomy of the curve refers to the amount that v_1 is rotated away from v_0.

Measuring holonomy is a little subtle. One approach might be to write $T(t) = \gamma'(t)$ and $N(t) = \frac{\gamma''(t)}{\|\gamma''(t)\|}$, then write the parallel translated tangent vector $v(t)$ as $v(t) = \cos\theta\, T(t) + \sin\theta\, N(t)$, choosing θ to vary continuously and have $\theta(0) = 0$. Then $\theta(T)$ measures the total amount that $v(t)$ has rotated away from the tangent vector to the curve. The problem is that if we measure this quantity for a circle $S^1 \subset \mathbb{R}^2$, for instance, we will get that the holonomy is -2π. But really we should get zero for any curve in $\mathbb{R}^2$: holonomy is supposed to measure the extra rotation a parallel translated vector feels due to the curvature of the surface, not just due to the rotation of the curve.

In fact this problem is fundamental. That is because the holonomy phenomenon we are trying to measure is not defined for a not closed curve $\gamma : [0, T] \to M$: that is, it isn't defined locally. Indeed, parallel translation gives us a way to map from the tangent space at $\gamma(0)$ to the tangent space at $\gamma(T)$, but there is no other natural map to compare this to. The holonomy phenomenon only arises because once we have rotated around a whole curve, we can compare the parallel translated copy of T_pM to the stationary copy of T_pM.

To solve this issue, we measure holonomy relative to a frame. Let e_1, e_2 give a frame in an open set containing γ. Also suppose this frame is positively oriented, so $dA(e_1, e_2) = 1$. Let $v(t) = \cos\varphi\, e_1 + \sin\varphi\, e_2$, and choose φ to vary continuously and have $\varphi(0) = 0$. Then $\varphi(T)$ is the holonomy of γ *relative to our framing*, and different framings will give holonomies that differ by a multiple of 2π.

It turns out that $\varphi'(t) = -\langle \nabla_{\gamma'(t)} e_1, e_2\rangle$, where ∇_{e_1} is the covariant derivative. Indeed, because $v(t)$ is by definition a parallel translation,

$$\begin{aligned} 0 = \nabla_{\gamma'} v &= \cos\varphi \nabla_{\gamma'} e_1 + \sin\varphi \nabla_{\gamma'} e_2 - \varphi' \sin\varphi e_1 + \varphi' \cos\varphi e_2 \\ &= \cos\varphi \langle \nabla_{\gamma'(t)} e_1, e_2\rangle e_2 - \sin\varphi \langle \nabla_{\gamma'(t)} e_1, e_2\rangle e_1 - \varphi' \sin\varphi e_1 + \varphi' \cos\varphi e_2 \end{aligned}$$

so by matching up like terms, we see $\varphi'(t) = -\langle \nabla_{\gamma'(t)} e_1, e_2\rangle$.

Let $\gamma'(t) = \cos\theta\, e_1 + \sin\theta\, e_2$. So φ measures the angle of the parallel translated vector relative to our framing, and θ measures the angle of the tangent vector to γ relative to our framing. Now let $\kappa(t) = dA(\gamma'(t), \gamma''(t))$. So $|\kappa(t)| = \|\gamma''(t)\|$, and the sign depends on which way γ bends (relative to the orientation on M). The quantity $\kappa(t)$ is the *geodesic curvature*, and we have

$$\kappa(t) = \theta'(t) - \varphi'(t).$$

Indeed,

$$\begin{aligned} \gamma'' &= -\theta' \sin\theta e_1 + \theta' \cos\theta e_2 + \cos\theta \nabla_{\gamma'} e_1 + \sin\theta \nabla_{\gamma'} e_2 \\ &= -\theta' \sin\theta\, e_1 + \theta' \cos\theta\, e_2 - \varphi' \cos\theta e_2 + \varphi' \sin\theta e_1 \\ &= (\varphi' - \theta') \sin\theta\, e_1 - (\varphi' - \theta') \cos\theta\, e_2 \\ \kappa(t) &= dA(\gamma', \gamma'') = \theta' - \varphi' \end{aligned}$$

as desired. So the reason holonomy is tricky to measure is that there are two factors contributing the the geodesic curvature: θ', which measures how much the tangent vector γ' rotates relative to a frame, and φ', which measures how much parallel translated vectors rotate relative to a frame. A choice of frame gives us a way to decompose the total geodesic curvature into these two parts, even locally. After integrating about a simple closed curve,

$$\int_\gamma \kappa(t)\, dt = \theta(T) - \varphi(T). \tag{1}$$

The left hand side does not depend on the choice of framing, so neither does the right hand side. For a different choice of framing, $\theta(T)$ (the change in angle) and $\varphi(T)$ (the holonomy) can change by the same multiple of 2π.

Suppose $P \subset M$ is a submanifold homeomorphic to a disk, and ∂P is parametrized by $\gamma(t)$. Suppose $\{e_1, e_2\}$ is a framing on an open neighborhood of P. Then the Hopf Umlaufsatz says that $\theta(T) = 2\pi$. An interpretation is that when γ is a simply closed curve bounding a region, we can canonically define the holonomy by fixing $\theta(T)$ to be 2π. We have

$$2\pi = \int_0^1 \kappa(t)\,dt + \varphi(T)$$
$$\varphi(T) = -\int_0^T \langle \nabla_{\gamma'(t)} e_1, e_2\rangle = 2\pi - \int_0^T \kappa(t)\,dt.$$

This is a very interesting formula. The geodesic curvature measures, in some sense, the second order deviation of γ from a geodesic. On the other hand, letting $\omega(v) = \langle \nabla_v e_1, e_2\rangle$, the holonomy $\gamma(T) - \gamma(0)$ equals $-\int_\gamma \omega$, the integral of a one-form. It is not at all obvious that this "first order quantity" $\int \omega$ and this second order quantity $\int_0^T \kappa(t)\,dt$ should have such a relation. We can apply Stoke's theorem to obtain

$$\int_P d\omega = \int_0^T \kappa(t)\,dt - 2\pi.$$

But $d\omega = -K dA$, where K is the Gaussian curvature. Indeed,

$$\begin{aligned} d\omega(e_1, e_2) &= e_1(\omega(e_2)) - e_2(\omega(e_1)) - \omega([e_1, e_2]) \\ &= e_1\langle \nabla_{e_2} e_1, e_2\rangle - e_2\langle \nabla_{e_1} e_1, e_2\rangle - \langle \nabla_{[e_1,e_2]} e_1, e_2\rangle \\ &= \langle ([\nabla_{e_1}, \nabla_{e_2}] - \nabla_{[e_1,e_2]}) e_1, e_2\rangle \\ &= \langle R_{e_1,e_2} e_1, e_2\rangle = -K \end{aligned}$$

as needed. As a consequence,

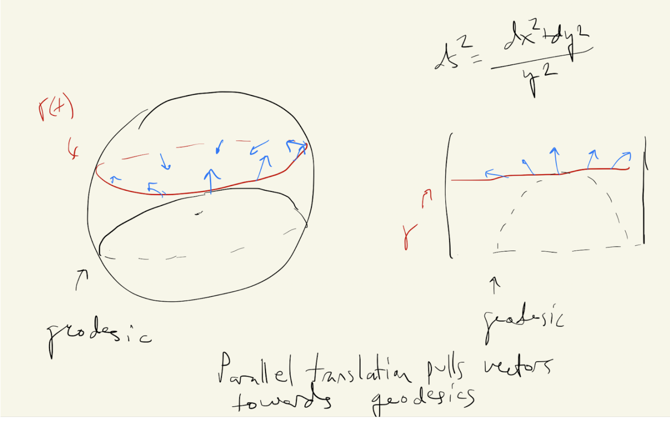
$$\int_P K dA + \int_0^T \kappa(t)\,dt = 2\pi. \tag{2}$$

This formula is remarkable: it shows how holonomy is related to curvature, and from one perspective, it is the essense of the Gauss-Bonnet theorem.

Remember that in order to derive Equation (2), we assumed that a neighborhood of P had a framing. But the framing never appears in (2), because the holonomy does not appear, it was absorbed by Stoke's theorem into $\int_P K dA$. In fact it is no restriction to demand a framing on a neighborhood of P, because we can always construct two independent vector fields in a neighborhood of P and apply the Gram-Schmidt process.

We will need a mild generalization of Equation (2) from simple closed curves to polygons. Let γ be a piecewise smooth curve with corners $0 < t_1 < t_2 < \cdots < t_n < 1$. Then the modified Hopf Umlaufsatz says that $\theta(1) = 2\pi - \sum_{i=1}^n (\pi - \alpha_i)$, where α_i is the angle at the ith corner. It follows from the same calculation that

$$\int_P K\,dA + \int_0^T \kappa(t)\,dt = (2-n)\pi + \sum_{i=1}^n \alpha_i. \tag{3}$$



This equation is sometimes called the local Gauss-Bonnet theorem or the Gauss-Bonnet theorem for polygons.

Example 2 (Geodesic triangles). In the special case where γ is a geodesic triangle,

$$\int_T K\, dA = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

For instance, on the unit sphere $K = 1$, so $\int_T K dA$ is just the area of the triangle T, and the right hand side is $\alpha_1+\alpha_2+\alpha_3-\pi$. This gives the familiar formula for the area of a spherical triangle, and implies in particular that the sum of the angles is at least π. Similarly, on a surface of constant curvature $K = -1$, the area of a geodesic triangle is given by $\pi - \sum_{i=1}^n \alpha_i$. It follows that the sum of the angles is at most π, and the maximum area of a triangle is π.

Example 3 (The sphere S^2). Consider the unit sphere S^2. We can smoothly parametrize it minus the north pole by $(\theta, \varphi) \mapsto (\cos\theta \sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$.

The metric is given by

$$\begin{aligned}
g(\partial_\theta, \partial_\theta) &= \langle -\sin\theta\sin\varphi\partial_x + \cos\theta\sin\varphi\partial_y, -\sin\theta\sin\varphi\partial_x + \cos\theta\sin\varphi\partial_y\rangle \\
&= \sin^2\varphi \\
g(\partial_\theta, \partial_\varphi) &= \langle -\sin\theta\sin\varphi\partial_x + \cos\theta\sin\varphi\partial_y, \cos\theta\cos\varphi\partial_x + \sin\theta\cos\varphi\partial_y - \sin\varphi\partial_z\rangle \\
&= 0 \\
g(\partial_\varphi, \partial_\varphi) &= \langle \cos\theta\cos\varphi\partial_x + \sin\theta\cos\varphi\partial_y - \sin\varphi\partial_z, \cos\theta\cos\varphi\partial_x + \sin\theta\cos\varphi\partial_y - \sin\varphi\partial_z\rangle \\
&= 1
\end{aligned}$$

Using

$$\Gamma_{ij}^k = \frac{1}{2}\sum_m g^{mk}\left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

The covariant derivatives of the fundamental vector fields are given by

$$\begin{aligned}
\nabla_{\partial_\theta}\partial_\theta &= -\frac{1}{2}\sin(2\varphi)\partial_\varphi\\
\nabla_{\partial_\theta}\partial_\varphi &= \frac{2\sin\varphi\cos\varphi}{2\sin^2\varphi}\partial_\theta = \operatorname{cotan}\varphi\,\partial_\theta\\
\nabla_{\partial_\varphi}\partial_\theta &= \operatorname{cotan}\varphi\,\partial_\theta\\
\nabla_{\partial_\varphi}\partial_\varphi &= 0
\end{aligned}$$

We have

$$\begin{aligned}
\langle R_{\partial_\theta,\partial_\varphi}\partial_\varphi,\partial_\theta\rangle &= \langle[\nabla_{\partial_\theta},\nabla_{\partial_\varphi}]\partial_\varphi,\partial_\theta\rangle\\
&= -\langle\nabla_{\partial_\varphi}\operatorname{cotan}\varphi\,\partial_\theta,\partial_\theta\rangle\\
&= \left\langle\frac{1}{\sin^2\varphi}\partial_\theta + \operatorname{cotan}^2\varphi\partial_\theta,\partial_\theta\right\rangle\\
&= 1-\cos^2\varphi = \sin^2\varphi
\end{aligned}$$

So

$$\begin{aligned}
K &= \frac{\sin^2\varphi}{\langle\partial_\theta,\partial_\theta\rangle\langle\partial_\varphi,\partial_\varphi\rangle - \langle\partial_\theta,\partial_\varphi\rangle^2}\\
&= 1.
\end{aligned}$$

Now, for $\gamma = (\gamma^1,\gamma^2)$ a curve in (θ,φ) coordinates, $Z = Z^1\partial_\theta + Z^2\partial_\varphi$ a vector field on γ, we have

$$\begin{aligned}
Z' = \frac{dZ^1}{dt}\partial_\theta &+ \frac{dZ^2}{dt}\partial_\varphi +\\
&Z^1\left(\frac{d\gamma^1}{dt}\nabla_{\partial_\theta}\partial_\theta + \frac{d\gamma^2}{dt}\nabla_{\partial_\varphi}\partial_\theta\right) + Z^2\left(\frac{d\gamma^1}{dt}\nabla_{\partial_\theta}\partial_\varphi + \frac{d\gamma^2}{dt}\nabla_{\partial_\varphi}\partial_\varphi\right)\\
= \frac{dZ^1}{dt}\partial_\theta &+ \frac{dZ^2}{dt}\partial_\varphi - \frac{1}{2}\sin(2\varphi)Z^1\frac{d\gamma^1}{dt}\partial_\varphi + \operatorname{cotan}\varphi Z^1\frac{d\gamma^2}{dt}\partial_\theta + \operatorname{cotan}\varphi Z^2\frac{d\gamma^1}{dt}\partial_\theta
\end{aligned}$$

So the parallel translate equations become

$$\begin{aligned}
\frac{dZ^1}{dt} + \operatorname{cotan}\varphi\left(Z^1\frac{d\gamma^2}{dt} + Z^2\frac{d\gamma^1}{dt}\right) &= 0\\
\frac{dZ^2}{dt} - \sin(\varphi)\cos(\varphi)Z^1\frac{d\gamma^1}{dt} &= 0.
\end{aligned}$$

Suppose γ traces out the circle $\theta(t) = t$, $\varphi(t) = \varphi_0$. Then the parallel translate equations become

$$\begin{aligned}
\frac{dZ^1}{dt} &= -\operatorname{cotan}\varphi_0 Z^2\\
\frac{dZ^2}{dt} &= \sin(\varphi_0)\cos(\varphi_0)Z^1
\end{aligned}$$

or

$$\begin{pmatrix} \dot{Z}^1 \\ \dot{Z}^2 \end{pmatrix} = \cos\varphi_0 \begin{pmatrix} 0 & -1/\sin\varphi_0 \\ \sin\varphi_0 & 0 \end{pmatrix} \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix}$$

Solving this,

$$\begin{pmatrix} Z^1(t) \\ Z^2(t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t) & -\frac{1}{\sin\varphi_0}\sin(\alpha t) \\ \sin\varphi_0 \sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix}$$

where $\alpha = \cos\varphi_0$. Thus we see that parallel translation rotates at a constant rate $\cos\varphi_0 = \alpha$. At $t = 2\pi$, corresponding to a full loop, the holonomy is $2\pi\cos\varphi_0$.

To calculate the holonomy we chose to measure the angle of the parallel translated vector relative to a tangent vector. But in this notion, holonomy agrees with geodesic curvature, so the holonomy equation (2) predicts that the holonomy is equal to $2\pi - \int_P dA$, where P is the region bounded by γ. But that region has area equal to $2\pi(1-\cos\varphi_0)$ because it is a spherical cap. This gives the correct answer of $2\pi\cos\varphi_0$.

Example 4 (The half plane $\mathbb{H}$). The hyperbolic metric on the upper half plane is $ds^2 = \frac{dx^2+dy^2}{y^2}$. Using the formula for Christoffel symbols in terms of the metric, as before, we obtain

$$\Gamma^1_{ij} = \begin{pmatrix} 0 & -\frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}, \qquad \Gamma^2_{ij} = \begin{pmatrix} \frac{1}{y} & 0 \\ 0 & -\frac{1}{y} \end{pmatrix}$$

so

$$\nabla_{\partial x}\partial x = \frac{1}{y}\partial_y, \quad \nabla_{\partial x}\partial y = -\frac{1}{y}\partial_x, \quad \nabla_{\partial y}\partial x = -\frac{1}{y}\partial_x, \quad \nabla_{\partial y}\partial y = -\frac{1}{y}\partial_y$$

and

$$\begin{aligned} R_{\partial_x,\partial_y}\partial_x &= [\nabla_{\partial_x}, \nabla_{\partial_y}]\partial_y \\ &= -\nabla_{\partial_x}\frac{1}{y}\partial_y + \nabla_{\partial_y}\frac{1}{y}\partial_x \\ &= \frac{1}{y^2}\partial_x - \frac{1}{y^2}\partial_x - \frac{1}{y^2}\partial_x = -\frac{1}{y^2}\partial_x \end{aligned}$$

so

$$K = \frac{\langle R_{\partial_x,\partial_y}\partial_x, \partial_y\rangle}{\langle\partial_x,\partial_x\rangle\langle\partial_y,\partial_y\rangle - \langle\partial_x,\partial_y\rangle^2} = -1.$$

For $\gamma = (\gamma^1, \gamma^2)$ a curve, $Z = Z^1\partial_x + Z^2\partial_y$ a vector field on γ, we have

$$\begin{aligned}
Z' &= \frac{dZ^1}{dt}\partial_x + \frac{dZ^2}{dt}\partial_y + \\
&\quad Z^1\left(\frac{d\gamma^1}{dt}\nabla_{\partial_x}\partial_x + \frac{d\gamma^2}{dt}\nabla_{\partial_y}\partial_x\right) + Z^2\left(\frac{d\gamma^1}{dt}\nabla_{\partial_x}\partial_y + \frac{d\gamma^2}{dt}\nabla_{\partial_y}\partial_y\right) \\
&= \frac{dZ^1}{dt}\partial_x + \frac{dZ^2}{dt}\partial_y + \\
&\quad Z^1\frac{d\gamma^1}{dt}\frac{1}{y}\partial_y - Z^1\frac{d\gamma^2}{dt}\frac{1}{y}\partial_x - Z^2\frac{d\gamma^1}{dt}\frac{1}{y}\partial_x - Z^2\frac{d\gamma^2}{dt}\frac{1}{y}\partial_y \\
&= \frac{1}{y}\left(y\frac{dZ^1}{dt} - Z^1\frac{d\gamma^2}{dt} - Z^2\frac{d\gamma^1}{dt}\right)\partial_x + \frac{1}{y}\left(y\frac{dZ^2}{dt} + Z^1\frac{d\gamma^1}{dt} - Z^2\frac{d\gamma^2}{dt}\right)\partial_y.
\end{aligned}$$

Suppose that $\gamma = (t, y_0)$. Then the parallel transport equations are

$$(Z^1)' = Z^2/y, \quad (Z^2)' = -Z^1/y.$$

Yielding that if $(Z^1, Z^2) = \cos\theta\partial_x + \sin\theta\partial_y$, $\theta' = -1/y$, so $\theta(t) = -t/y$. Holonomy is more mild the farther we are from the x-axis. This makes sense, as holonomy measures the second order difference between a curve and the geodesic through a point, and geodesics become closer and closer to horizontal lines the farther we are from the x-axis.

The geodesic equations are

$$\begin{aligned}
y\frac{d^2\gamma^1}{dt^2} &= 2\frac{d\gamma^1}{dt}\frac{d\gamma^2}{dt} \\
y\frac{d^2\gamma^2}{dt^2} &= \left(\frac{d\gamma^2}{dt}\right)^2 - \left(\frac{d\gamma^1}{dt}\right)^2
\end{aligned}$$

We verify that geodesics are arcs of circles that meet the x-axis at right angles. Indeed, letting

$$\gamma = (a + r\cos\theta(t), r\sin\theta(t))$$

we obtain the equations

$$\begin{aligned}
-r^2\theta''\sin^2\theta - r^2\theta'^2\cos\theta\sin\theta &= -2r^2\theta'^2\sin\theta\cos\theta \\
r^2\theta''\cos\theta\sin\theta - r^2\theta'^2\sin^2\theta &= r^2\theta'^2(\cos^2\theta - \sin^2\theta)
\end{aligned}$$

which reduces to

$$\begin{aligned}
-\theta''\sin^2\theta &= -\theta'^2\sin\theta\cos\theta \\
\theta''\cos\theta\sin\theta &= \theta'^2\cos^2\theta
\end{aligned}$$

which is solved when $\theta'' = \theta'\operatorname{cotan}\theta$.

3.2. **Triangulation proof of Gauss-Bonnet.** Let $P_1 \cup \cdots \cup P_n$ be a triangulation for a Riemannian surface M. Applying the local Gauss-Bonnet formula (3) to each triangle, the integration of the geodesic curvature cancels, and we are left with

$$\int_M KdA = -n\pi + \sum_v \alpha_v$$

where v ranges over the vertices of any triangle in the triangulation. We have $\sum_v \alpha_v = 2\pi V$, where V is the total number of vertices, because the sum of all angles at a vertex is equal to 2π. Thus letting $F = n$ be the number of triangles, E the number of edges,

$$\int_M KdA = 2\pi(V - F/2) = 2\pi(V - E + F) = 2\pi\chi(M)$$

yielding the Gauss-Bonnet formula. Notice that even if we did not already know $V - E + F$ does not depend on the triangulation, it would follow from this work, as that quantity is equal to the integral of the curvature. But $V - E + F$ is invariant under homeomorphism, so just from the equation $\int_M KdA = 2\pi(V - E + F)$ it follows that there is a topological invariant which is equal to $\frac{1}{2\pi}\int_M KdA$ for any Riemannian metric on M, and is equal to $V - E + F$ for any triangulation.

4. Chern's proof

Here we give Chern's original proof of his Theorem 2, specialized to the case of surfaces.

4.1. **Some perspectives on a connection.** We put here some background material on connections that will be needed later. Let $E \xrightarrow{\pi} M$ be a vector bundle with a Riemannian metric. A metric connection on M is a function $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$ such that for any three vector fields X, Y, Z,

$$\begin{aligned} \nabla_{fX} Y &= f\nabla_X Y \\ \nabla_X fY &= (Xf)Y + f\nabla_X Y \\ \nabla_{X+X'} Y &= \nabla_X Y + \nabla_{X'} Y, \quad \nabla_X(Y + Y') = \nabla_X Y + \nabla_X Y'. \end{aligned}$$

We can also view the connection as taking Y to the map $X \mapsto \nabla_X Y$. From this point of view, the connection is a function $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*(M))$ such that for vector fields $X, Y \in \Gamma(E)$,

$$\nabla(fX) = X \otimes df + f\nabla X, \tag{4}$$

$$d\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle. \tag{5}$$

Importantly, a connection on a vector bundle gives a way to parallel transport vectors along a curve. To do so, we must define differentiation along a curve. For γ a curve on M, Z a vector field on γ, the covariant derivative Z' satisfies

- If X is a vector field on M and $Z = X|_\gamma$, then $Z' = D_{\gamma'} X$

- $(fZ)' = \frac{df}{dt}Z + fZ'$
- $(Z+W)' = Z' + W$.

To be explicit, let $e_1, \ldots, e_n$ be a local frame, γ a curve, $Z(t) = a_1(t)e_1 + \cdots + a_n(t)e_n$ a vector field on γ. Then

$$Z' = a_1'e_1 + \cdots + a_n'e_n + a_1\nabla_{\gamma'}e_1 + \cdots + a_n\nabla_{\gamma'}e_n.$$

We then say that a vector field on a curve is parallel if it's covariant derivative is zero, and we can parallel translate vectors along curves by solving the equation $Z(0) = v, Z' = 0$.

Finally, we can view a connection as a function from $TE \to E$. An element of TE is a tangent vector to a curve γ in E, which can be described by $(Z(t), \gamma(t))$ with $\gamma(t) = \pi(Z(t))$. Then $Z(t)$ is a vector field over the curve γ, so the covariant derivative $Z' \in E$ is defined. The connection then takes

$$\nabla\left(\frac{d}{dt}(Z(t), \gamma(t))\right) = Z'(0).$$

This perspective has topological significance. Consider π^*TM, the pullback of the tangent bundle over M to E, and π^*E, the pullback of E over itself. Then there is an exact sequence of vector bundles

$$0 \to \pi^*E \to TE \to \pi^*TM \to 0 \tag{6}$$

The inclusion map takes $(p, v) \mapsto \frac{d}{dt}(p + tv)|_{t=0}$, where $\pi(p) = \pi(v) \in M$. The projection map takes $\frac{d}{dt}(Z(t), \gamma(t)) \mapsto \gamma'(t)$. Another perspective is that an element of π^*E is a pair $(v, w) \in E$ with $\pi_*v = \pi^*w = x$, with the projection map to E given by $(v, w) \mapsto w$. But then $v \in \pi^{-1}(x)$ can also be viewed as an element of $T_w\pi^{-1}(x)$, which by inclusion is naturally an element of TE. The map $TE \to \pi^*TM$ is the pullback of the derivative map $\pi_* : TE \to TM$. A natural question is: does this exact sequence split? Is $TE \cong \pi^*E \oplus \pi^*TM$? The answer turns out to be yes, and the data of a splitting map is exactly a connection. Indeed, the function $\nabla : TE \to E$ we is linear, so the pullback gives rise to a map $TE \to \pi^*E$, which it is easy to verify is a section map using the formula for covariant differentiation (in this case, $\gamma' = 0$).

This is a very useful point of view, a connection is a continuous way of splitting TE into a *vertical* component π^*E, and a *horizontal* component π^*TM. It also illuminates the topological significance of a connection. The bundle TE is always isomorphic to $\pi^*E \oplus \pi^*TM$, but not canonically so, and an isomorphism is exactly a connection.

Of course, there are many other perspectives on a connection, some of which may come up later.

4.2. **Transgression.** A peculiar aspect of the local Gauss-Bonnet formula is that Equation (3) requires a choice of frame in the proof, but the result does not depend on the frame. The reason is that once we've chosen a frame e_1, e_2, we can define a one-form $\omega(v) = \langle \nabla_v e_1, e_2 \rangle$ which satisfies

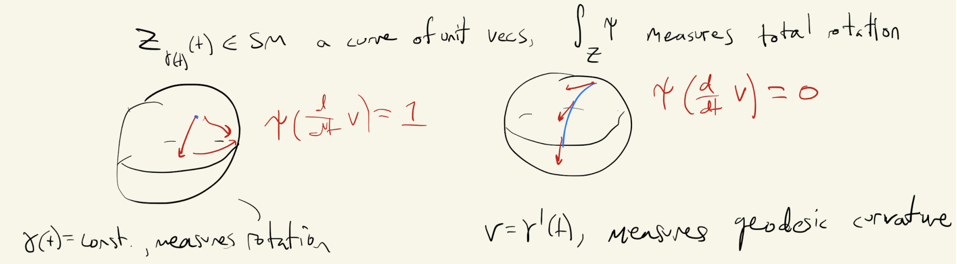


Figure 3. A connection is exactly the data needed to build a global angular form, from which we can build the Thom class.

$\Omega = -d\omega$. There is no one best choice of the one-form ω, there is a different and perfectly valid choice for each frame. But we must choose some frame in order to make Ω exact and apply Stoke's theorem.

Chern's way past this pecularity is to pass to a larger manifold on which Ω is canonically exact. To be precise, define SM as the manifold of all unit tangent vectors to M, it should be viewed as a submanifold of TM. There is a projection map $SM \to M$ given by $v_p \to p$. The *transgression* phenomenon is that $\pi^*\Omega = d\psi$ for a one-form ψ.

As discussed in the prior section, a connection ∇ gives a map $TSM \to TM$. Define ψ by

$$\psi(w_p) = dA(p, \nabla w)$$

where $p \in SM$, $w_p \in TSM|_p$, and dA is the area 2-form on M. We can understand this one form in terms of the isomorphism $TE \cong \pi^*E \oplus \pi^*TM$. Any $v \in TE$ can be written as $v = x + y$ where $\pi_* x = 0$ and $\nabla y = 0$. Then $\psi(y) = 0$, so $\psi(v) = \psi(x)$. Another way of saying this is that $(TE)^* = (\pi^*E)^* \oplus (\pi^*TM)^*$, and ψ lives in the first component.

The form ψ is called the *global angular form.* If $Z_{\gamma(t)}(t) \in SM$ is a curve of unit vectors, then $\int_Z \psi$ measures the total rotation of Z. For instance, if $\gamma(t)$ is constant and $Z_p(t) = \cos\theta\, e_1 + \sin\theta e_2$, then $\int_Z \psi = \theta_1 - \theta_0$—it is the total change in angle. On the other hand, if $\gamma(t) \in M$ is a curve on a Riemannian surface and $Z(t) = \gamma'(t)$, then $\int_Z \psi$ measures the geodesic curvature of γ. In particular, if γ is a geodesic, $\int_Z \psi = 0$. See Figure 3.

Let us prove that $\pi^*\Omega = d\psi$. Let e_1, e_2 be a framing, $\omega_1^2(X) = \langle \nabla_X e_1, e_2 \rangle$ the holonomy one-form. Then $\Omega = d\omega_1^2$. The one-forms e_1^*, e_2^*, ω_1^2 on M give functions in $C^\infty(SM)$. Let $a = e_1^*$, $b = e_2^*$ viewed as functions on SM instead of one-forms on M. Then $a^2 + b^2 = 1$ so $ada + bdb = 0$, the gradients are linearly dependent. It turns out that

$$\psi = adb - bda + \pi^*\omega_1^2.$$

This is verified by explicit computation:

$$\begin{aligned}\psi(\frac{d}{dt}(\gamma, \alpha e_1 + \beta e_2)) &= -\langle \alpha' e_1 + \beta' e_2 + \alpha \nabla_{\gamma'} e_1 + \beta \nabla_{\gamma'} e_2,\ \alpha e_2 - \beta e_1 \rangle \\ &= -(\alpha'\beta - \beta'\alpha) - \alpha^2 \omega_1^2(\gamma') - \beta^2 \omega_1^2(\gamma') \\ &= (-bda + adb - \pi^* \omega_1^2)\left(\frac{d}{dt}(\gamma, \alpha e_1 + \beta e_2)\right).\end{aligned}$$

Then

$$d\psi = 2dadb - d\pi^*\omega_1^2 = -\pi^* d\omega_1^2 = -\pi^*\Omega.$$

Here, because da and db are linearly dependent, $dadb = 0$ yielding the result. The amazing thing is that even though ω_1^2 is only locally defined, ψ is globally defined.

Using the formula $d\psi = -\pi^*\Omega$ we derive the Gauss-Bonnet theorem using the Poincaré Hopf theorem. Let s be a vector field of unit vectors on M with finitely many singularities. Let $x_1, \ldots, x_n$ be the singularities, and $B_1, \ldots, B_n$ disjoint balls around those points, of radius r. Then s gives a section of the bundle $SM \to M$ in the set $U = M - (B_1 \cup \cdots \cup B_n)$. Now, $s(U) \subset SM$, and by injectivity and choice of orientations,

$$\int_U \Omega = \int_U s^*\pi^*\Omega = \int_{s(U)} \pi^*\Omega.$$

By Stoke's theorem,

$$\begin{aligned}\int_{s(U)} \pi^*\Omega &= -\int_{s(U)} d\psi \\ &= -\int_{s(\partial U)} \psi \\ &= -\sum_{j=1}^n \int_{s(\partial B_j)} \psi.\end{aligned}$$

We let $r \to 0$. We show that

$$\lim_{r\to 0} \int_{s(\partial B_j)} \psi = -2\pi \operatorname{ind}_{x_j} s. \tag{7}$$

It follows that

$$\begin{aligned}\lim_{r\to 0} \int_U \Omega &= -\lim_{r\to 0} \sum_{j=1}^n \int_{s(\partial B_j)} \psi \\ \int_M K dA &= 2\pi\chi(M)\end{aligned}$$

as desired.

Now let us prove Equation (7). Recall that $\operatorname{ind}_{x_j} s = \deg \alpha$, where $\alpha : \partial B_j \to S^1$ is given by composing $s : M \to SM$ with a projection $SM \to S^1$ in a neighborhood of B_j. As a realization of α, we choose $\pi^{-1}(x_j)$ as our

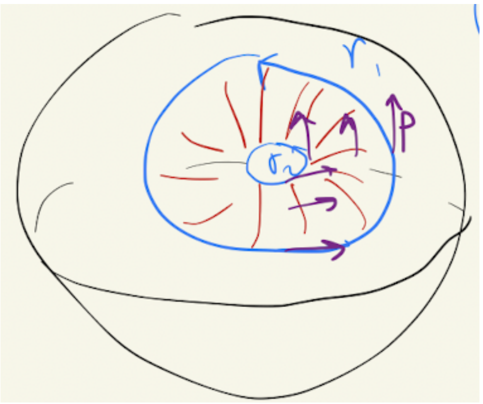


Figure 4. The global angular form can be used to relate geodesic curvature to curvature.

copy of S^1. Choose a frame in a neighborhood of B_j, so we can realize $\pi^{-1}(B_j)$ as $B_j \times S^1$, and then let α be s composed with the projection from $\pi^{-1}(B_j) \to \pi^{-1}(x_j)$. The degree of this projection map is precisely $\operatorname{ind}_{x_j} s$. So

$$\int_{s(\partial B_j)} \pi^* \psi = (\operatorname{ind}_{x_j} s) \int_{\pi^{-1} x_j} \psi = -2\pi \operatorname{ind}_{x_j} s.$$

Now,

$$\lim_{r \to 0} \int_{s(\partial B_j)} \pi^* \psi = \lim_{r \to 0} \int_{s(\partial B_j)} \psi$$

because $\pi^* \psi \to \psi$. This yields the desired formula.

Remark 1. We can also recover the local Gauss-Bonnet formula using the global angular form ψ. In particular, let P be a region with a field of unit vectors V on P that is tangent to the boundary. Then $V : P \to SM$, and the total geodesic curvature is given by

$$\begin{aligned}
\int_{\partial P} \kappa dt &= \int_{V(\partial P)} \psi \\
&= \int_{V(P)} d\psi \\
&= -\int_{V(P)} \pi^* \Omega \\
&= -\int_P \Omega.
\end{aligned}$$

Taking P to be a region homeomorphic to a disk, $x \in P$ a point, $B_r(x)$ a ball about x of radius r, we can construct a unit vector field on P which is tangent to the boundary, and we have

$$\int_P \Omega - \int_{B_r} \Omega = -\int_{\partial P} \kappa dt + \int_{\partial B_r(x)} \kappa dt$$

$$\lim_{r\to 0} \int_{\partial B_r(x)} \kappa dt = 2\pi, \quad \lim_{r\to 0} \int_{B_r} \Omega = 0$$

together yielding the local Gauss-Bonnet formula

$$\int_P K\, dA + \int_{\partial P} \kappa\, dt = 2\pi.$$

See Figure 4.

5. Thom class perspective

5.1. **Building the Thom class.** The point of view that a connection is a splitting of TTM into $\pi^*TM \oplus \pi^*TM$ leads naturally to yet another proof of the Gauss Bonnet theorem, the Thom class approach.

For $\pi : E \to M$ a smooth rank-n vector bundle over an m-dimensional compact base, the Thom class Φ is the unique element of $H^n_{cv}(E)$ which integrates to 1 along fibers.[1] That is,

$$\int_{\pi^{-1}(x)} \Phi = 1 \qquad \text{for all } x.$$

Constructing the Thom class may at first seem trivial: simply define it to be some localized copy of the area form in the vertical direction of each fiber. The trickiness is that we must choose some smoothly varying splitting of TTM along fibers into a vertical and a horizontal part. A connection provides exactly this data.

Let us first consider a rank 2 bundle. We follow the ideas in Bott & Tu, the section on the global angular form (around page 71) (but using the data of a connection rather than a trivialization).

First, on $M \times \mathbb{R}^2$, we could do the following. Let ρ be a bump function with $\rho(x) = -1$ for $|x| < 1/2$ and $\rho(x) = 0$ for $|x| > 3/2$. Now let $d\theta$ be the angular form on $\mathbb{R}^2 - \{0\}$. Then $\frac{1}{2\pi} d\rho \wedge d\theta$ integrates to one along each fiber. Because $d\theta$ is closed, this form is equal to $\frac{1}{2\pi} d(\rho d\theta)$. We will denote the form $d\theta$ by ψ so it does not get confused for an exact form. Now $\frac{1}{2\pi}(d\rho\psi)$ integrates to one along each fiber. Here ψ is a global one form on $M \times \mathbb{R}^2$, coming from the one form on the factor $\mathbb{R}^2$. Notice that the common kernel of the two 1-forms $d\rho$ and ψ at a point $(x, y) \in M \times \mathbb{R}^2$ is the subspace of horizontal vectors of the form $(v, 0)$, $v \in T_xM$.

In general, let $E \xrightarrow{\pi} M$ be a rank two vector bundle, and fix an inner product $\langle \cdot, \cdot \rangle$ and metric connection ∇ on E. We will construct a *global*

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[1] $H^n_{cv}(E)$ is the quotient space of degree n closed forms compactly supported in the vertical direction, modulo exact forms of the same type

angular form ψ on $E^0 = E - M$ which gives the angular form $d\theta$ when restricted to each fiber. Notice that the Euclidean structure specifies an angular form on each fiber. Let ρ be a bump function as before, we will denote by ρ the smooth function $\rho(r)$ on E. We will set $\Phi = d(\rho\psi) = d\rho \wedge \psi + \rho d\psi$. At first it seems that Φ should only be defined on E^0, but it turns out that it can sensibly be interpreted as a two-form on all of E. We note also that the common vanishing set of ψ and dr is a two dimensional subspace of TTM, consisting of the "horizontal" vectors with covariant derivative zero: a curve γ in TM is parallel translated if and only if γ' lies in the common kernel of ψ and dr. In this way, specifying a global angular form is really the same as specifying a connection. Another way to see this is that a global angular form tells us how to calculate the amount a curve of unit vectors is "rotating", which is also what a connection tells us. See Figure 3.

Let $S \subset E$ consist of the unit vectors, so S is an S^1 bundle over M. As in the prior section, define the angular form $\tilde{\psi}$ on S by

$$\tilde{\psi}(v_p) = dA(p, \nabla v_p)$$

where dA is the area two form coming from the Euclidean structure, $p \in S$ is a unit vector in E lying over $x \in M$, and $\nabla v_p \in T_x M$. This is really the same one-form as ψ from the prior section. Define ψ on all of E^0 by $\psi = P^*\tilde{\psi}$, where $P : E^0 \to S$ is the map $v \mapsto v/\|v\|$.

Lemma 1. *The global angular form ψ agrees with the angular form on each fiber coming from the metric.*

Proof. It suffices to show that $\tilde{\psi}$ agrees with the unit length form on $\pi^{-1}(x) \cong S^1$s, as the angular form for the metric is the pullback of this. But indeed, this is true because ∇ is a section map for (6):

$$\tilde{\psi}(\frac{d}{d\theta}(\cos\theta\, e_1 + \sin\theta\, e_2)) = dA(\cos\theta\, e_1 + \sin\theta\, e_2, -\sin\theta\, e_1 + \cos\theta\, e_2) = 1$$

(Note: it is slightly curious that we do not seem to rely on the fact that ∇ is a metric connection, only that it is a connection. I am not sure what to make of this.) □

Now set

$$\Phi = d(\rho\psi) = d\rho \wedge \psi + \rho d\psi.$$

It is not at first clear that Φ extends to a smooth form on all of E, rather than just E^0. But $d\rho$ is supported away from $M \subset E$ so $d\rho \wedge \psi$ is smooth on all of E, and it turns out that $d\psi = P^* d\tilde{\psi}$ is of the form $\pi^* e$, for e a two form on M, so in fact $d\psi$ extends to the two-form $\pi^* e$ on all of E.

We show $d\tilde{\psi} = \pi^* e$ for an appropriate form e on M. Actually, we already know this is true: we showed in the prior section that $d\tilde{\psi} = \pi^*\Omega$. Nevertheless, it's instructive to show that $d\tilde{\psi}$ arises in this way directly.

Lemma 2. *There exists a form Ω, called the geometric Euler class, such that $d\tilde{\psi} = \pi^*\Omega$.*

Proof. We prove this in the following steps.

- If $p \in S$ lies over $x \in M$ and $v_p \in T_pS$, and $\pi_* v_p = 0$, then $d\tilde{\psi}(v_p, w_p) = 0$ for any $w_p \in T_pS$
- Let R_θ be the map which rotates the unit vectors in S by θ degrees counterclockwise. Then $R_\theta^* \tilde{\psi} = \tilde{\psi}$

The second point is obvious by the definition of $\tilde{\psi}$. For the first, consider the vector field $R(p) = \frac{d}{d\theta} R_\theta(p)\Big|_{\theta=0}$ on SM. Explicitly, the vector at $p = \cos\theta e_1 + \sin\theta e_2$ is $R(p) = \cos\theta e_2 - \sin\theta e_1$. The vector $R(p)$ spans the one dimensional subspaces $\ker \pi_*$ at p. Thus we must show that $d\tilde{\psi}(R, X) = 0$ for any X commuting with R. Indeed,

$$\begin{aligned} d\tilde{\psi}(R, X) &= R(\tilde{\psi}(X)) - X(\tilde{\psi}(R)) \\ &= R(\tilde{\psi}(X)) = \frac{d}{d\theta}\tilde{\psi}(X) \circ R_\theta = 0 \end{aligned}$$

by property two.

Altogether then we may define $\Omega(v, w)$ by $d\tilde{\psi}(\tilde{v}, \tilde{w})$ for any lift by π, and the value will not depend on our choice. □

Now, $e = s_0^* \Phi = \frac{1}{2\pi}\Omega$. The left hand side is by definition the Euler class of a vector bundle. Thus, we have proved

$$\chi(M) = \int_M e = \frac{1}{2\pi}\int_M \Omega$$

where Ω is the curvature of any metric connection.

The Euler class can be interpreted by Poincare duality as the self intersection number of M inside E. So Poincare duality gives a way to pass between Chern's original proof and the Thom class proof: integrating the zero section of the Thom class calculates the self intersection number of M in TM which is equal to the total index of a vector field. By Poincare-Hopf, that in turn is equal to $\chi(M)$.

The Chern-Weil theory explains that the Euler class measures curvature of the global angular form. One can show directly that the integral of the Euler class does not depend on the connection, but it already follows from the fact that the Thom class is unique in De Rham cohomology. Note that the quantization of the Euler class is nontrivial, and requires some topological interpretation—e.g., that it counts the self intersection number of M inside E.

6. The Euler class, Gauss-Bonnet-Chern for vector bundles

6.1. **Plan.** Here is out strategy for proving Theorem 3, Gauss-Bonnet-Chern for vector bundles. First, we will show that the form $e_g = \frac{1}{(2\pi)^n}\operatorname{Pf}(\Omega)$ is closed and is a diffeomorphism invariant (does not depend on the connection)—this is a specialization of the Chern-Weil theorem. Then we show e_g behaves well under direct sums and pullbacks of vector bundles, so it is in fact a characteristic class. As a base case, we have already shown in the prior section that the geometric Euler class e_g agrees with the topological Euler class e_t for rank two bundles, and we use the splitting principle to show that e_g must agree with e_t for all vector bundles as desired. We note that Mathai & Quillen give an explicit construction of the Thom class from a connection, not relying on the splitting principle, but this is quite involved.

6.2. **The Chern-Weil theorem.** Let $E \xrightarrow{\pi} M$ be a rank $2n$ vector bundle with Riemannian metric g and metric connection ∇. Relative to a fixed frame let $\omega^i_j = \langle \nabla e_j, e_i \rangle$ be a connection matrix of one-forms and $\Omega = d\omega + \omega \wedge \omega$ be the corresponding curvature matrix of two-forms. The Pfaffian polynomial in the curvature matrix is the $2n$-form

$$\operatorname{Pf}(\Omega) = \sum_{\alpha} \operatorname{sgn}\alpha \, \Omega_{i_1,j_1} \wedge \cdots \wedge \Omega_{i_n,j_n}$$

where α ranges over all unordered matchings $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ of $\{1, \ldots, 2n\}$. Under a change of frame the curvature matrix changes by conjugation of an element of $\mathrm{SO}(2n)$, which leaves $\operatorname{Pf}(\Omega)$ unchanged because Ω is a skew symmetric matrix. Thus $\operatorname{Pf}(\Omega)$ is a degree $2n$ form on M which does not depend on the frame, only on the connection.

Theorem 4 (Chern-Weil)**.**

(1) *The form* $\operatorname{Pf}(\Omega)$ *is closed.*

(2) *The De Rham cohomology class* $[\operatorname{Pf}(\Omega)]$ *does not depend on the connection* ∇*.*

Proof.

(1) We follow the proof from Milnor-Stasheff Appendix C. Viewing Pf as a polynomial in the variables x_{ij}, $i < j$, let Pf' be the matrix with polynomial entries

$$\operatorname{Pf}' = \begin{pmatrix} 0 & \frac{\partial \operatorname{Pf}}{\partial x_{12}} & \cdots & \frac{\partial \operatorname{Pf}}{\partial x_{1m}} \\ -\frac{\partial \operatorname{Pf}}{\partial x_{12}} & 0 & \cdots & \frac{\partial \operatorname{Pf}}{\partial x_{2m}} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{\partial \operatorname{Pf}}{\partial x_{1m}} & -\frac{\partial \operatorname{Pf}}{\partial x_{2m}} & \cdots & 0 \end{pmatrix}$$

We have

$$d \operatorname{Pf}(\Omega) = \operatorname{Tr}(\operatorname{Pf}'(\Omega) \wedge d\Omega).$$

Now, for A the skew symmetric matrix of indeterminates x_{ij}, $\operatorname{Pf}'(A)A = A \operatorname{Pf}'(A)$. For a proof, it suffices to show this identity holds for any particular values of x_{ij}. But because $\operatorname{Pf}'(BAB^t) = B \operatorname{Pf}'(A) B^t$ for any $B \in SO(2n)$, it suffices to consider the case where A is a direct sum of skew symmetric 2×2 matrices, for which it is easy. It follows then at the level of matrices of forms that $\operatorname{Pf}'(\Omega) \wedge \Omega = \Omega \wedge \operatorname{Pf}'(\Omega)$

We use the second Bianchi identity

$$d\Omega = d(d\omega + \omega \wedge \omega) = \Omega \wedge \omega - \omega \wedge \Omega$$

to evaluate

$$\begin{aligned} d\,\mathrm{Pf}(\Omega) &= \mathrm{Tr}(\mathrm{Pf}'(\Omega) \wedge d\Omega) \\ &= \mathrm{Tr}(\mathrm{Pf}'(\Omega) \wedge \Omega \wedge \omega - \mathrm{Pf}'(\Omega) \wedge \omega \wedge \Omega) \\ &= 0 \end{aligned}$$

combining the fact that $\mathrm{Pf}'(\Omega)$ and Ω commute with the cyclicity of the trace.

(2) Let ∇^0 and ∇^1 be two metric connections. Consider $f : M \times [0,1] \to M$, and consider the pullback bundle f^*E. We can put two connections on f^*E—$f^*\nabla^0$ and $f^*\nabla^1$. Consider the connection $\tilde{\nabla} = (1-t)f^*\nabla^0 + tf^*\nabla^1$. Let i_0, i_1 be the inclusions $M \to M \times [0,1]$ at times zero and one. Then $\nabla^0 = i_0^*\tilde{\nabla}$ and $\nabla^1 = i_1^*\tilde{\nabla}$. We have $\mathrm{Pf}(\Omega_0) = i_0^*\,\mathrm{Pf}(\tilde{\Omega})$ and $\mathrm{Pf}(\Omega_1) = i_1^*\,\mathrm{Pf}(\tilde{\Omega})$. By the first part, $\mathrm{Pf}(\tilde{\Omega})$ is closed, and because i_0 and i_1 are homotopic, $[\mathrm{Pf}(\Omega_0)] = [\mathrm{Pf}(\Omega_1)]$ as desired. □

One can replace the Pfaffian polynomial with any invariant polynomial for $\mathrm{SO}(n)$, and in this way one can obtain all the Chern classes through Chern-Weil theory.

Because $\mathrm{Pf}(\Omega)$ is closed and does not depend on the connection, it is a well defined element of De Rham cohomology, and we can define the *geometric Euler class.*

Definition. Define the *geometric Euler class* e_g of E to be

$$e_g = \frac{1}{(2\pi)^n}\,\mathrm{Pf}(\Omega)$$

and the *topological euler class* to be

$$e_t = s_0^*\Phi,$$

the pullback of the Thom class by the zero section.

Theorem. *For M a rank $2n$ vector bundle, we have $e_g = e_t$ as De Rham cohomology classes. In other words,*

$$\frac{1}{(2\pi)^n}\int_M \mathrm{Pf}(\Omega) = \int_M s_0^*\Phi = \chi(E)$$

In fact, $e_g = e_t = 0$ for any odd rank vector bundle, so the Theorem is true for any vector bundle. Now we establish that e_g behaves like a characteristic class.

Lemma 3. *Both e_g and e_t satisfy:*

(1) *Let E, F be even dimensional vector bundles over M. Then*

$$e(E \oplus F) = e(E) \wedge e(F).$$

(2) *For $f : N \to M$ be smooth map, $E \xrightarrow{\pi} M$ an even rank vector bundle, f^*E the pullback bundle, we have $e(f^*E) = f^*(e(E))$.*

Proof. First for e_t: the first property holds because $\Phi_{E\oplus F} = \pi_E^*\Phi_E \wedge \phi_F^*\Phi_F$ at the level of forms, so

$$e(E \oplus F) = s_0^*(\Phi_{E\oplus F}) = s_0^*(\pi_E^*\Phi_E) \wedge s_0^*(\pi_F^*\Phi_F) = e(E) \wedge e(F).$$

Similarly, for the second property, $\Phi_{f^*E} = f^*\Phi_E$ at the level of forms (this follows from the fact that bundle maps are isomorphisms of fibers).

Now for e_g. Because e_g does not depend on the choice of connection, we are free to choose a convenient connection on N. We will choose the direct sum connection on $E_1 \oplus E_2$ and the pullback connection on f^*E.

With this choice, the connection matrices satisfy $\omega_{E\oplus F} = \omega_E \oplus \omega_F$, so the curvature matrices do as well, $\Omega_{E\oplus F} = \Omega_E \oplus \Omega_F$. Then the fact follows from the fact about Pfaffians of any skew symmetric matrix that

$$\mathrm{Pf}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{Pf}(A)\,\mathrm{Pf}(B).$$

For the second fact, $\Omega_{f^*E} = f^*\Omega_E$ gives the result. □

Next we state without proof the splitting principle.

Lemma 4. *For E an even dimensional vector bundle over a manifold M, there exists a manifold N and a map $g : N \to M$ so that*

- *$g^* : H^*(M) \to H^*(N)$ is injective*
- *g^*E is a direct sum of plane bundles.*

Now, note that we already proved Theorem 3 in the rank two case by explicitly constructing the Thom class from the connection. In this case, the Pfaffian is just the $1, 2$ entry of a 2×2 skew symmetric matrix. The general case follows easily from what we've proven. For E a rank $2n$ bundle over M, choose $g : N \to M$ according to the splitting principle so that $g^*E = E_1 \oplus \cdots \oplus E_n$; then

$$\begin{aligned} g^*e_g(E) = e_g(g^*E) = e_g(E_1 \oplus \cdots \oplus E_n) &= e_g(E_1) \wedge \cdots \wedge e_g(E_n) \\ &= e_t(E_1) \wedge \cdots \wedge e_t(E_n) \\ &= e_t(E_1 \oplus \cdots \oplus E_n) \\ &= e_t(g^*E) = g^*e_t(E) \end{aligned}$$

and by injectivity, $e_g(E) = e_t(E)$ as desired. Note that as long as we can split in the smooth category, we can after the fact put the Riemannian metric on g^*E associated to E in order to make the proof work.

The Gauss-Bonnet-Chern theorem follows by specializing to the case where $E = TM$ and choosing the Levi-Civita connection. The use of vector bundles and the splitting principle puts the theorem in useful wider context: the connection coming with E will be the Levi-Civita connection, but in order to establish the Gauss-Bonnet-Chern theorem we prove the formula for a different connection coming from a direct sum of rank two bundles, and use the Chern-Weil theorem to conclude that the two forms have the same integral.

7. Extrinsic proof for any embedded Riemannian manifold

We give a sketch of another proof of the Gauss-Bonnet-Chern theorem based on an embedding.

Let M be an $n = 2k$ dimensional Riemannian manifold embedded in $\mathbb{R}^N$ (this is always possible due to the Nash embedding theorem). Let $P = \mathrm{Gr}(n, N)$ be the Grassmannian. The total space E naturally forms a Riemannian bundle over P by taking the inner product in $\mathbb{R}^N$. Then the map $f : M \to P$ taking $x \in M$ to $T_xM \subset \mathbb{R}^N$ gives rise to a bundle map $F : TM \to E$ which is an isometry on fibers. Now, consider the connection on E so that for $\gamma(t) \in P$ a curve of planes, $v(t) \in \gamma(t)$ vectors, covariant differentiation yields

$$\nabla_{\gamma'} v = \pi_{\gamma(0)} \frac{dv}{dt}.$$

That is, we view $v(t)$ as an element of $\mathbb{R}^N$, take $\frac{dv}{dt}$ in $\mathbb{R}^N$, and then orthogonally project onto the plane $\gamma(0)$. Another way to see this connection is from the principal bundle perspective. The bundle of orthonormal frames over P is the Stiefel manifold $\mathrm{St}(n, N)$ consisting of n orthogonal unit vectors. This is an $SO(n)$ bundle. Represent $\mathrm{St}(n, N)$ by $N \times n$ matrices A such that $A^t A = I$. The connection is then the $\mathfrak{so}(n)$-valued one form ω such that $\omega(\frac{d}{dt} A(t)) = A^t \dot{A}$. Indeed,

$$\frac{d}{dt}(A^t A) = A^t \dot{A} + \dot{A}^t A = 0$$

so $A^t \dot{A}$ is skew symmetric. Also, for $B(t) \in \mathrm{SO}(N)$,

$$\omega\left(\frac{d}{dt} AB(t)\right) = A^t A \dot{B} = \dot{B}$$

as needed for ω to give a connection. This Ehresmann connection gives rise to the differentiation along a curve that we explicitly defined earlier.

We start with a Lemma.

Lemma 5. *We have* $e_g(P) = e_t(P)$

Proof idea. I have not worked this out and hope to add it later... but it should be an explicit construction, similar to the construction of the Thom class for a 2-bundle. We need to show that the Pfaffian of the curvature matrix of this particular connection for $\mathrm{Gr}(2n, m)$ agrees with the pullback of the Thom class of the zero section. This should involve giving some natural construction of the Thom class and relating it to the natural connection on $\mathrm{Gr}(2n, m)$. □

The bundle map $F : TM \to E$ is compatible with the Levi-Civita connection on TM and the aforementioned connection ∇ on E. This is because the

Levi-Civita connection on an embedded manifold is given by projecting the Levi-Civita connection on the ambient space onto the embedded manifold. Indeed, for $Z(t)$ a curve of tangent vectors on M, then

$$\begin{aligned} F(Z'(0)) &= F\left(\pi_{T_{\gamma(0)}M}\frac{dZ}{dt}\right) \\ &= (T_{\gamma(0)}M, \pi_{T_{\gamma(0)}M}\frac{dZ}{dt}) \\ &= (F \circ Z)' \end{aligned}$$

as needed.

Now, because the connections on M and P are compatible,

$$f^* e_P = \frac{1}{(2\pi)^n} f^* \operatorname{Pf}(\Omega_P) = \frac{1}{(2\pi)^n} \operatorname{Pf}(\Omega_M)$$

Indeed, If $e_1, \ldots, e_n$ is a local framing of P, these pull back under F to give a framing of M, and the connection matrices satisfy $\omega^i_{jM} = f^*\omega^i_{jP}$. Thus the curvature matrices satisfy the same relation $\Omega^i_j|_M = f^*\Omega^i_{jP}$, and of course the Pfaffian polynomial in the curvature matrices also pull back. Thus it follows from almost no real work that $e_t = e_g$ for the manifold M. This proof works for all Riemannian manifolds: for instance, if we want to prove the Gauss-Bonnet-Chern theorem for a flat torus, we could embed it in $\mathbb{R}^4$ and apply this proof.

Notice that the map $f : M \to P = \operatorname{Gr}(n, N)$ is a well defined smooth map independent of the bundles we put on top of them. In the case where $n = 2$, f is essentially a realization of the Gauss map, and this proof reduces to our first proof of the Gauss-Bonnet theorem for Riemannian surfaces embedded in $\mathbb{R}^3$. Notice that $f^* dV_P = \det df\, dV_M$, so

$$\det df = \operatorname{Pf}(\Omega).$$

Recall that the determinant $\det df$ has nothing to do with the Riemmanian bundle E we put on P, but rather with the Riemannian metric on $\operatorname{Gr}(n, N)$, so this equation is really saying something interesting. This equation can be read in two ways. First, the quantity $\det df$ can be computed in terms of the curvature of the Riemannian manifold M, leading to the topological significance of Ω, and second, the we can interpret the form $\operatorname{Pf}(\Omega)$ as measuring the amount of distortion in the higher dimensional Gauss map $f : M \to P$. This is in direct analogy to Gauss's theorem egregium, which lets us interpret curvature of a manifold as distortion of the Gauss map for surfaces in $\mathbb{R}^3$. This extrinsic proof further illustrates of power of viewing the Gauss-Bonnet-Chern theorem from the broader viewpoint of vector bundles.

8. Acknowledgements

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Appendix A. Analyzing the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$

A.1. $\mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}$ **realization of the Hopf fibration.** Realize $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ as $\{(z,w) : |z|^2 + |w|^2 = 1\}$. Realize $S^2 \subset \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ as $\{(\xi, y) : |\xi|^2 + |y|^2 = 1\}$. Then the Hopf fibration is given by the map

$$\varphi : (z,w) \mapsto (2z\bar{w}, |z|^2 - |w|^2).$$

Indeed,

$$|2z\bar{w}|^2 + (|z|^2 - |w|^2)^2 = 4|z|^2|w|^2 + |z|^4 + |w|^4 - 2|z|^2|w|^2 = (|z|^2 + |w|^2)^2 = 1$$

We have $\varphi(e^{i\theta}z, e^{i\theta}w) = \varphi(z,w)$, this is the action of $U(1) \cong S^1$ on S^3. The fiber of $(\xi, y) \in S^2$ is precisely this copy of $U(1)$. Suppose $2z\bar{w} = \xi$ and $|z|^2 - |w|^2 = y$. Multiplying z and w by a phase, suppose w is real and ≥ 0. Then $|z|^2 - y = w^2$ and $z = \xi/2w$. So $z = \frac{\xi}{2\sqrt{|z|^2 - y}}$. Because the choice of z and w is unique in this case, it follows that the preimage of (ξ, y) is precisly the $U(1)$ orbit of this choice fo z and w.

View $S^3 \to S^2$ as a $U(1)$ principal bundle with the action $e^{i\theta}(z,w) = (e^{i\theta}z, e^{i\theta}w)$. We would like to find a connection on this bundle. Now, the tangent space to S^3 at (z,w) is given by

$$T_{(z,w)}S^3 = \{a, b \in \mathbb{C}^2 : \operatorname{Re}(a\bar{z} + b\bar{w}) = 0\}$$

as $\operatorname{Re}(a\bar{z} + b\bar{w})$ gives the Euclidean inner product in $\mathbb{R}^4$. The connection should be a 1-form valued in $\mathfrak{u}(1) = i\mathbb{R}$. This motivates us defining

$$\omega(a,b) = a\bar{z} + b\bar{w}.$$

Indeed, by the tangent space condition, $\omega(a,b) \in i\mathbb{R}$. We must verify that this is $\mathfrak{u}(1)$ equivariant. Indeed, the vertical vector field associated to $i \in \mathfrak{u}(1)$ is $\frac{d}{d\theta}(e^{i\theta}z, e^{i\theta}w) = (iz, iw)$. Then

$$\omega(iz, iw) = i|z|^2 + i|w|^2 = i$$

as needed.

Now, the curvature is given by

$$\begin{aligned}
\Omega &= d\omega + \frac{1}{2}[\omega, \omega] \\
\Omega((a,b),(c,d)) &= d\omega((a,b),(c,d)) \\
&= \partial_{(a,b)}\omega(c,d) - \partial_{(c,d)}\omega(a,b) \\
&= c\bar{a} + d\bar{b} - a\bar{c} - b\bar{d} \\
&= -2i\operatorname{Im}(a\bar{c} + b\bar{d}).
\end{aligned}$$

It should be that Ω vanishes when either one of its inputs is a vertical vector. Indeed,

$$\Omega((iz, iw), (c,d)) = -2i\operatorname{Im}(iz\bar{c} + iw\bar{d}) = -2i\operatorname{Re}(z\bar{c} + w\bar{d}) = 0.$$

It should be that $\Omega(v,w) = C(\xi,y)dA(\pi_* v, \pi_* w)$ where dA is the area two-form on the sphere, which is given by

$$dA = (u_1 du_2 du_3 - u_2 du_1 du_3 + u_3 du_1 du_2) = \frac{du_1 du_2}{u_3}$$

in $\mathbb{R}^3$ coordinates. In $\mathbb{C}\times\mathbb{R}$ coordinates, at (ξ,y), we have

$$dA((v_1,t_1),(v_2,t_2)) = -\frac{\operatorname{Im}(v_1\bar{v}_2)}{y}.$$

A basis for $T_{(z,w)}S^3$ is given by $\{(iz,iw),(-\bar{w},\bar{z}),(-i\bar{w},i\bar{z})\}$.[2] This is a particularly nice basis: (iz,iw) is a vertical vector, and $(-w,z)$ and $(-iw,iz)$ are both horizontal in the sense that the vanish on ω. Thus

$$C(\xi,y) = \frac{\Omega((-w,z),(-iw,iz))}{idA(\pi_*(-w,z),\pi_*(-iw,iz))}$$

where $z,w \mapsto \xi, y$. We have

$$\Omega((-\bar{w},\bar{z}),(-i\bar{w},i\bar{z})) = -2i\operatorname{Im}(-i|w|^2 - i|z|^2) = 2i.$$

In general,

$$\pi_*(a,b) = (2a\bar{w} + 2z\bar{b}, 2\operatorname{Re}(a\bar{z} - b\bar{w}))$$

so

$$\begin{aligned} dA(\pi_*(a,b),\pi_*(c,d)) &= -\frac{1}{(|z|^2-|w|^2)}\operatorname{Im}((2a\bar{w}+2\bar{b}z)(2\bar{c}w+2d\bar{z})) \\ &= -\frac{4}{(|z|^2-|w|^2)}\operatorname{Im}(a\bar{c}|w|^2 + \bar{b}d|z|^2 + (ad-bc)\bar{z}\bar{w}) \end{aligned}$$

In particular,

$$\begin{aligned} dA(\pi_*(-\bar{w},\bar{z}),\pi_*(-i\bar{w},i\bar{z})) &= \frac{4}{|z|^2-|w|^2}\operatorname{Im}(-i|w|^4 + i|z|^4) \\ &= \frac{4(|z|^4-|w|^4)}{|z|^2-|w|^2} \\ &= 4. \end{aligned}$$

It follows that $C = \frac{1}{2}$. Does this make sense?

Note that while the above calculations fail at $|z|^2 - |w|^2 = 0$, they extend there by continuity.

A.2. $\mathbb{H}\to\mathbb{H}$ **realization of the Hopf fibration.** The Hopf fibration can be realized more symmetrically in the following way. Let $\mathbb{H}$ be the quaternions, $\{a+bi+cj+dk : a,b,c,d\in\mathbb{R}\}$. Realize $S^3\subset\mathbb{H}$ as the unit quaternions in $\mathbb{H}$. Realize $S^2\subset\mathbb{H}$ as the imaginary unit quaternions, i.e., $S^2 = \{bi+cj+dk : b^2+c^2+d^2=1\}$. Then let $\varphi : S^3\to S^2$ be given by

$$q\mapsto q\cdot i\cdot q^{-1}.$$

---

[2] Could we instead take $\{(iz,iw),(z,-w),(iz,-iw)\}$?

Certainly $|\varphi(q)| = 1$. Also,

$$\overline{q \cdot i \cdot q^{-1}} = (q \cdot i \cdot q^{-1})^{-1} = -q \cdot i \cdot q^{-1}$$

so $\varphi(q) \in S^2$. The action of $U(1)$ is $q \mapsto qe^{i\theta}$. Indeed,

$$qe^{i\theta} \cdot i \cdot e^{-i\theta} q^{-1} = q \cdot i \cdot q^{-1}$$

so this action preserves fibers. In fact, fibers are precisely the orbit of this action. Suppose that for $q, u \in S^3$, $qiq^{-1} = uiu^{-1}$. The $(u^{-1}q)i = i(u^{-1}q)$. So $u^{-1}q$ commutes with i. But the centralizer of i is precisely $\mathbb{R} + i\mathbb{R}$, so $u^{-1}q = e^{i\theta}$ for some θ as needed.

The tangent space to S^3 in $\mathbb{H}$ is

$$T_q S^3 = \{u \in \mathbb{H} : \operatorname{Re}(u\bar{q}) = \operatorname{Re}(uq^{-1}) = 0\}.$$

A very nice basis for $T_q S^3$ is given by $\{qi, qj, qk\}$. Indeed, by the same argument we gave for why φ maps to S^2, we have that

$$\operatorname{Re}(qiq^{-1}) = \operatorname{Re}(qjq^{-1}) = \operatorname{Re}(qkq^{-1}) = 0.$$

We have that $\frac{d}{d\theta} qe^{i\theta} = qi$ spans the vertical tangent space, so we will take our connection to satisfy $\omega(qi) = i$, $\omega(qj) = \omega(qk) = 0$. From the Cartan perspective, we have chosen the span of qj, qk for our horizontal tangent space. This is certainly $\mathfrak{u}(1)$ equivariant because $je^{i\theta} = j\cos\theta - k\sin\theta$ and $ke^{i\theta} = k\cos\theta + j\sin\theta$.

Consider the vector fields E_i, E_j, E_k on S^3 given by $E_i(q) = qi$, similarly for the others. We have

$$\Omega(E_i, E_j) = \partial_{qi}\omega(qj) - \partial_{qj}\omega(qi) - \omega([E_i, E_j]) = -\omega([E_i, E_j])$$

as the other terms are zero. The flow lines along these vector fields are quite simple—they are given by

$$\Phi_i(q, t) = qe^{it}, \qquad \Phi_j(q, t) = qe^{jt}.$$

Thus

$$\begin{aligned}
[E_i, E_j] &= \lim_{t \to 0} \frac{\Phi_i(\Phi_j(q, t), t) - \Phi_j(\Phi_i(q, t))}{t^2} \\
&= \lim_{t \to 0} \frac{qe^{jt}e^{it} - qe^{it}e^{jt}}{t^2} \\
&= q \lim_{t \to 0} \frac{e^{jt}e^{it} - e^{it}e^{jt}}{t^2} \\
&= -2qk.
\end{aligned}$$

Similarly,

$$[E_i, E_j] = -2qk, \quad [E_j, E_k] = -2qi, \quad [E_k, E_i] = -2qj.$$

Thus plugging in our formula for the curvature form,

$$\Omega(E_i, E_j) = 0, \quad \Omega(E_j, E_k) = 2i, \quad \Omega(E_k, E_i) = 0.$$

There is a function $C(x)$ on S^2 such that at $p \in S^3$,

$$\Omega(u, v) = iC(\varphi(p))dA(\pi_*(u), \pi_*(v)).$$

To find $C(\varphi(p))$, we must have

$$2i = \Omega(E_j, E_k) = iC(\varphi(p))dA(\pi_*E_j, \pi_*E_k).$$

Now,

$$\begin{aligned}\pi_*E_j &= \lim_{t\to 0} \frac{1}{t}(qe^{jt}i(qe^{jt})^{-1} - qiq^{-1})\\ &= \lim_{t\to 0} \frac{1}{t} qe(^{jt}ie^{-jt} - i)q^{-1}\\ &= -2qkq^{-1}.\end{aligned}$$

Similarly, $\pi_*(E_k) = 2qjq^{-1}$. Now, $\pi_*(E_j)$ and $\pi_*(E_k)$ are orthogonal as tangent vectors to S^2, because $\operatorname{Re} \pi_*(E_j)\pi_*(E_k)^{-1} = 0$. Thus $dA(\pi_*E_j, \pi_*E_j) = \pm|\pi_*(E_j)||\pi_*(E_k)| = 4$. Choosing signs appropriately, we find $C = \frac{1}{2}$ as before.

APPENDIX B. ANALYZING THE HOPF FIBRATION $S^3 \hookrightarrow S^7 \to S^4$

Realize $S^7 \subset \mathbb{H}^2$ as $\{(q_1, q_2) : |q_1|^2 + |q_2|^2 = 1\}$. Realize $S^4 \subset \mathbb{H} \times \mathbb{R}$ as $\{(\xi, t) : |\xi|^2 + t^2 = 1\}$. We map $S^7 \to S^4$ by

$$\varphi : (q_1, q_2) \mapsto (2q_1\bar{q}_2, |q_1|^2 - |q_2|^2).$$

The action of $\{q \in \mathbb{H} : |q| = 1\} \cong SU(2) \cong S^3$ is given by $(q_1, q_2) \mapsto (q_1v, q_2v)$ where v is a unit quaternion.

The tangent space to S^7 is given by

$$T_{(q_1,q_2)}S^7 = \{(a, b) : \operatorname{Re}(\bar{q}_1a + \bar{q}_2b) = 0\}.$$

Motivated by this, we define an Ehresmann connection by

$$\omega(a, b) = \bar{q}_1a + \bar{q}_2b.$$

For $(a, b) \in T_{(q_1,q_2)}S^7$, $\omega(a, b) \in \mathfrak{su}(2)$, which we identify with the quaternions that have real part zero.

Three vertical tangent vectors, which span the vertical tangent space, are

$$\begin{aligned}\frac{d}{d\theta}(q_1, q_2)e^{i\theta} &= (q_1i, q_2i)\\ \frac{d}{d\theta}(q_1, q_2)e^{j\theta} &= (q_1j, q_2j)\\ \frac{d}{d\theta}(q_1, q_2)e^{k\theta} &= (q_1k, q_2k)\end{aligned}$$

Indeed,

$$\omega(q_1i, q_2i) = i, \quad \omega(q_1j, q_2j) = j, \quad \omega(q_1k, q_2k) = k$$

satisfying one of the properties of a connection. More generally, for $y \in \mathfrak{su}(2)$ a quaternion with real part zero, $\omega(q_1 y, q_2 y) = y$.

On the other hand, a horizontal vector is given by $(\bar{q}_1^{-1}, -\bar{q}_2^{-1}) = (q_1/|q_1|^2, -q_2/|q_2|^2)$. A basis for the horizontal space is given by

$$(q_1/|q_1|^2, -q_2/|q_2|^2), \quad (q_1/|q_1|^2, -q_2/|q_2|^2)i, \quad (q_1/|q_1|^2, -q_2/|q_2|^2)j, \quad (q_1/|q_1|^2, -q_2/|q_2|^2)k$$

In total, we have three vector fields, the first three of which span the vertical space and the last four span the horizontal space:

$$V_1 = (q_1 i, q_2 i), \quad V_2 = (q_1 j, q_2 j), \quad V_3 = (q_1 k, q_2 k)$$
$$H_1 = (q_1/|q_1|^2, -q_2/|q_2|^2), \quad H_2 = (q_1/|q_1|^2, -q_2/|q_2|^2)i$$
$$H_3 = (q_1/|q_1|^2, -q_2/|q_2|^2)j, \quad H_4 = (q_1/|q_1|^2, -q_2/|q_2|^2)k$$

We would like to determine the curvature form. In order to do so, we must evaluate the commutators of the H_i's. We have the following flow lines through these vector fields:

$$\begin{aligned}
\Phi_1(q_1, q_2, t) &= (q_1\sqrt{2t/|q_1|^2 + 1}, q_2\sqrt{-2t/|q_2|^2 + 1}) \\
\Phi_2(q_1, q_2, t) &= (q_1 e^{it/|q_1|^2}, q_2 e^{-it/|q_2|^2}) \\
\Phi_3(q_1, q_2, t) &= (q_1 e^{jt/|q_1|^2}, q_2 e^{-jt/|q_2|^2}) \\
\Phi_4(q_1, q_2, t) &= (q_1 e^{kt/|q_1|^2}, q_2 e^{-kt/|q_2|^2})
\end{aligned}$$

Using these, we compute the commutators:

$$[H_i, H_j] = \begin{pmatrix} 0 & 2(-q_1/|q_1|^4, -q_2/|q_2|^4)i & 2(-q_1/|q_1|^4, -q_2/|q_2|^4)j & 2(-q_1/|q_1|^4, -q_2/|q_2|^4)k \\ 2(q_1/|q_1|^4, q_2/|q_2|^4)i & 0 & 2(q_1/|q_1|^4, q_2/|q_2|^4)k & -2(q_1/|q_1|^4, q_2/|q_2|^4)j \\ 2(q_1/|q_1|^4, q_2/|q_2|^4)i & -2(q_1/|q_1|^4, q_2/|q_2|^4)k & 0 & 2(q_1/|q_1|^4, q_2/|q_2|^4)i \\ 2(q_1/|q_1|^4, q_2/|q_2|^4)i & 2(q_1/|q_1|^4, q_2/|q_2|^4)j & -2(q_1/|q_1|^4, q_2/|q_2|^4)i & 0 \end{pmatrix}.$$

Then

$$\omega([H_i, H_j]) = \frac{1}{|q_1|^2|q_2|^2}\begin{pmatrix} 0 & -2i & -2j & -2k \\ 2i & 0 & 2k & -2j \\ 2j & -2k & 0 & 2i \\ 2k & 2j & -2i & 0 \end{pmatrix}$$

We have $\Omega(A, B) = -\omega([A, B])$ for A, B any of the vector fields given above, because Ω is zero on vertical vectors, and all the connection values are constant.

Thus $\Omega = \Omega^1 i + \Omega^2 j + \Omega^3 k$, where

$$\Omega^1(H_1, H_2) = \frac{2}{|q_1|^2|q_2|^2}, \quad \Omega^1(H_3, H_4) = -\frac{2}{|q_1|^2|q_2|^2}$$
$$\Omega^2(H_1, H_3) = \frac{2}{|q_1|^2|q_2|^2}, \quad \Omega^2(H_2, H_4) = \frac{2}{|q_1|^2|q_2|^2}$$
$$\Omega^3 s(H_1, H_4) = \frac{2}{|q_1|^2|q_2|^2}, \quad \Omega^3(H_2, H_3) = -\frac{2}{|q_1|^2|q_2|^2}$$

and all other combinations from our vector fields give zero.

There is essentially only one invariant polynomial on $\mathfrak{su}(2)$. Consider the action $q \mapsto qiq^{-1}$, $q \in SU(2)$ a unit quaternion. Then this maps i onto $S^2 \subset \mathfrak{su}(2)$. Similarly, for any $x \in \mathfrak{su}(2)$, the image of the adjoint representation brings x to any other vector with the same norm. The invariant polynomials are generated by the polynomial $ai+bj+ck \mapsto a^2+b^2+c^2$. The corresponding polynomial of the curvature form is

$$f(\Omega) = \Omega^1 \wedge \Omega^1 + \Omega^2 \wedge \Omega^2 + \Omega^3 \wedge \Omega^3$$

giving a four-form on S^7. The four-form is given by

$$f(\Omega)(H_1, H_2, H_3, H_4) = -\frac{24}{|q_1|^4|q_2|^4}$$

Letting dV be the volume form in S^4, we would like to find that function $C(p)$ such that

$$f(\Omega)(H_1, H_2, H_3, H_4)_{(q_1,q_2)} = C(\varphi(q_1, q_2))dV(\pi_*H_1, \pi_*H_2, \pi_*H_3, \pi_*H_4)$$

Let $\varphi(q_1, q_2) = (2q_1\bar{q}_2, |q_1|^2 - |q_2|^2) = (z, s)$. Then

$$f(\Omega)(H_1, H_2, H_3, H_4)_{(q_1,q_2)} = -\frac{24}{|z/2|^4}$$

and

$$\begin{aligned}
\pi_*H_1 &= (2q_1\bar{q}_2(1/|q_1|^2 - 1/|q_2|^2), 4) \\
\pi_*H_2 &= (2q_1i\bar{q}_2(1/|q_1|^2 + 1/|q_2|^2), 0) \\
\pi_*H_3 &= (2q_1j\bar{q}_2(1/|q_1|^2 + 1/|q_2|^2), 0) \\
\pi_*H_4 &= (2q_1k\bar{q}_2(1/|q_1|^2 + 1/|q_2|^2), 0)
\end{aligned}$$

Now we must compute $dV(\pi_*H_1, \dots, \pi_*H_4)$. Notice that any two of these vectors are orthogonal. So

$$\begin{aligned}
|dV(\pi_*H_1, \dots, \pi_*H_4)| &= |\pi_*H_1| \cdots |\pi_*H_4| \\
&= 2^3|q_1|^{-3}|q_2|^{-3}\sqrt{4|q_1|^{-2}|q_2|^{-2}(|q_2|^2 - |q_1|^2)^2 + 16} \\
&= 2^4|q_1|^{-3}|q_2|^{-3}\sqrt{|q_1|^{-2}|q_2|^{-2}(|q_2|^4 + |q_1|^4) + 2} \\
&= 2^4|q_1|^{-3}|q_2|^{-3}\sqrt{|q_1|^{-2}|q_2|^{-2}(1 - 2|q_1|^2|q_2|^2) + 2} \\
&= 2^4|q_1|^{-4}|q_2|^{-4} \\
&= 2^4|z/2|^{-4}
\end{aligned}$$

Thus

$$C(z, s) = 24|z/2|^{-4}|z/2|^4 2^{-4} = \frac{3}{2}$$

and

$$\int_{S^4} C(z, s)\, dV = \frac{3}{2}\int_{-1}^{1}(1 - y^2)\sigma_3\, dy = 2\sigma_3$$

where σ_3 is the surface area of S^3. This is equal to $2\pi^2$, so $\int_{S^4} C(z,s)dV = 4\pi^2$. This is the characteristic class of the Hopf fibration. After normalizing by $\frac{1}{4\pi^2}$ we get that the Chern class is 1.

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