Branching functions in phase space

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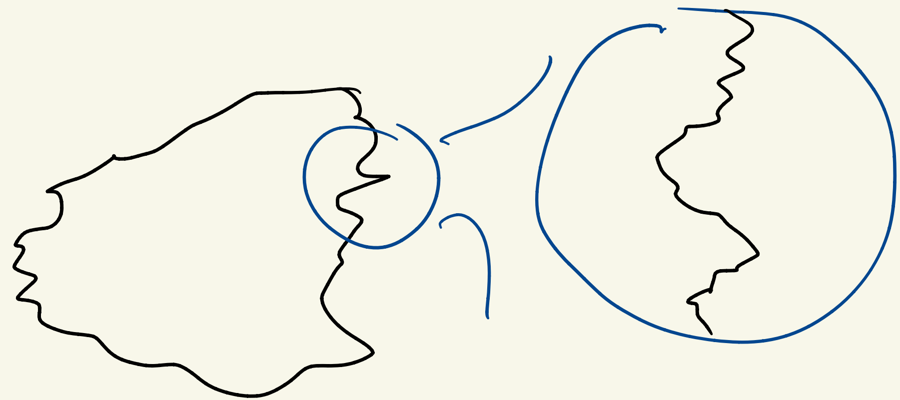
Riddle: How long is the coastline of Britain?



You get a larger and larger number the smaller your measuring stick is.

How to know the right measuring stick length to choose?

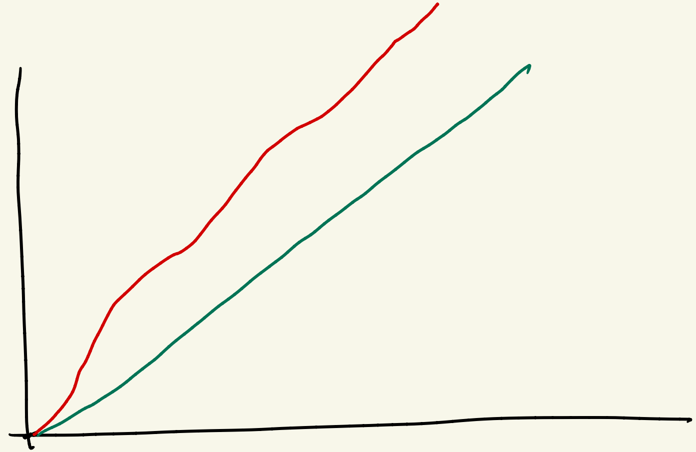
Fractal geometry's insight: DON'T just pick one measuring stick.

INSTEAD, think about all of them!

$|\text{Coastline}|_\omega$ = # of ω-balls to cover the coastline

covering number

$|\text{Coastline}|_w = 2^{-y}$

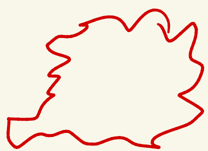


$w = 2^{-x}$

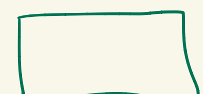
Branching function = log-log plot of covering number

<u>Fractal Geometry</u>:

pts in a set $\longrightarrow$ branching func



Britain

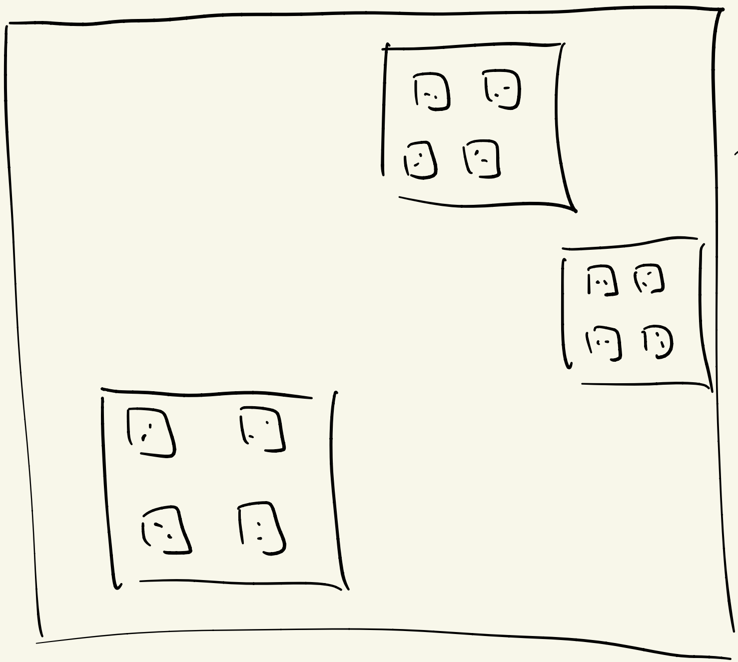


Colorado

$|\text{Britain coastline}|_w \gg w^{-1}$

$|\text{Colorado coastline}|_w = \text{Length} \cdot w^{-1}$

When faced with a set of pts, it's useful to assume it is built like a tree.



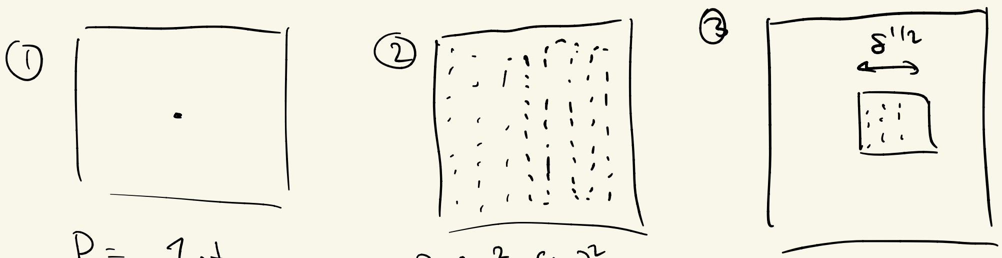
Def P is uniform on a set of scales $\mathcal{S}$ if for $u \in \mathcal{S}$,

$$P \subset \bigcup_{Q \in \mathcal{Q}_u} Q \leftarrow \text{collection of } u \times u \text{ square}$$

$$|P \cap Q| \sim \text{const.} \qquad \forall Q \in \mathcal{Q}_u$$

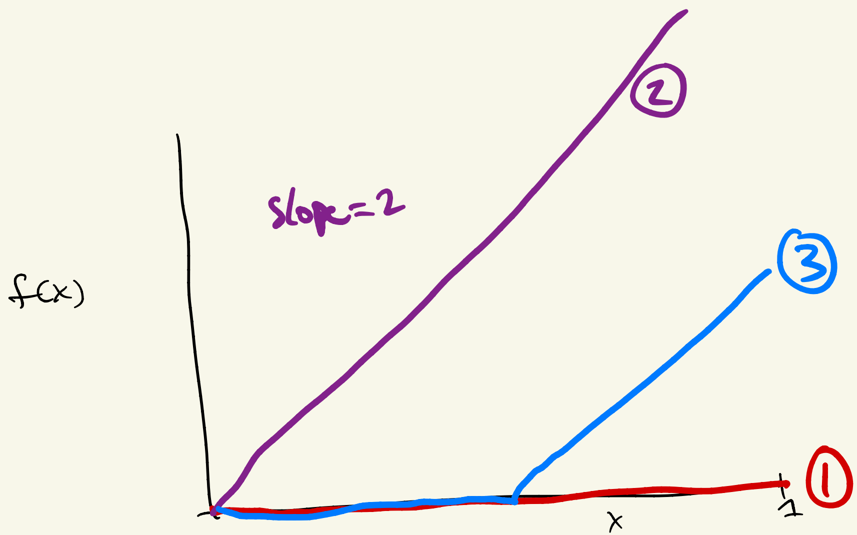
Branching function $f(x)$: $\quad 2^{-f(x)} = |\mathcal{Q}_{2^{-x}}|$

← covering number



$P = 1$ pt

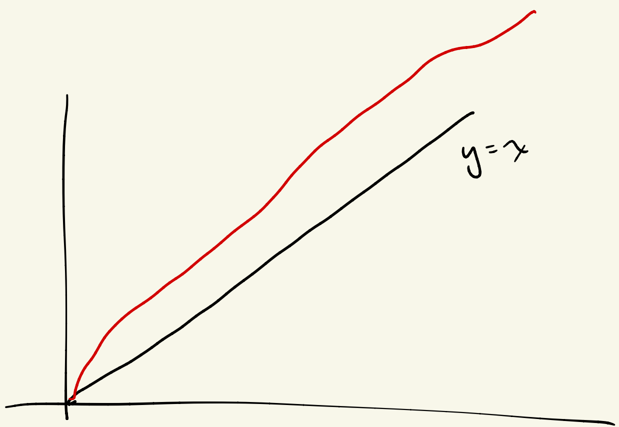
$P = \delta\mathbb{Z}^2 \cap [0,1]^2$

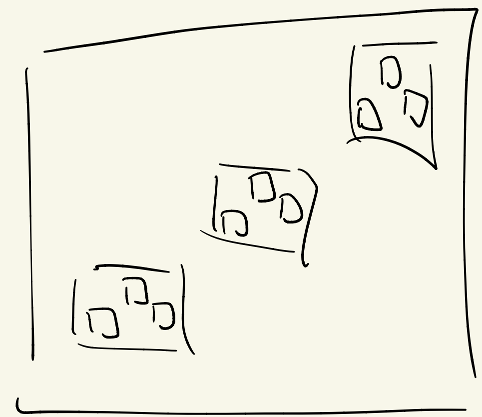


We say P is s-Frostman if

$$|P \cap Q| \leq C\, w^s |P|, \quad Q \text{ a } w \times w \text{ square.}$$

Using branching fncs:
$$f(x) \geq sx - \varepsilon$$



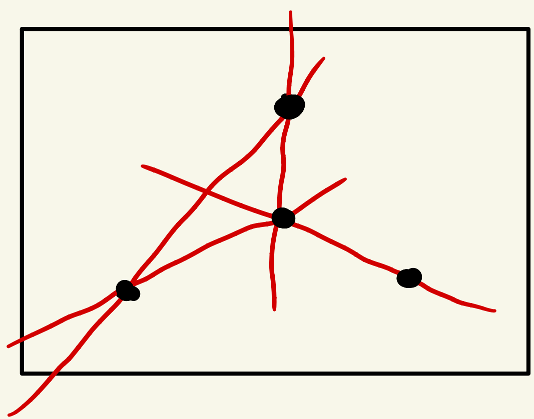


Incidence geometry in the plane:

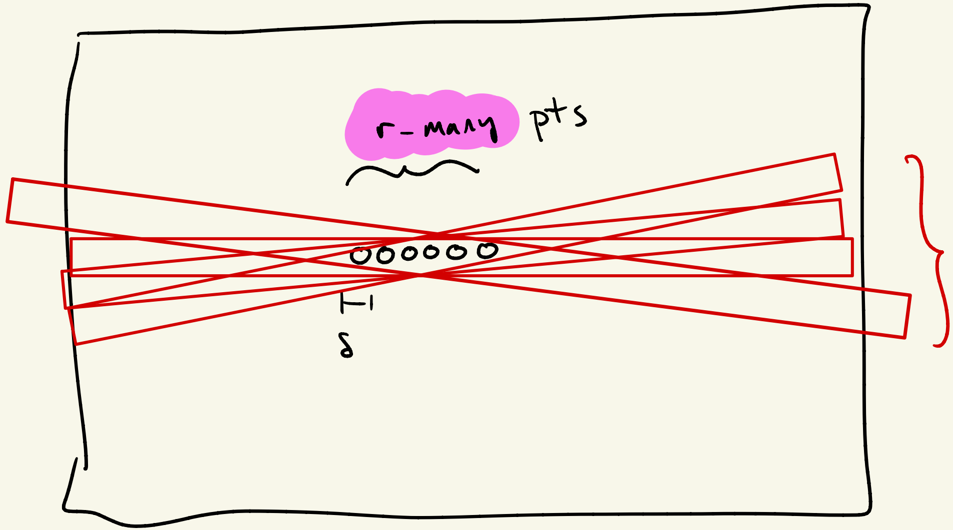
Let $P = \{p_1, \ldots, p_m\}$ be a set of m-many pts

For each $p \in P$, $L_p = \{$ set of r-many lines through $p\}$, $L = \bigcup_p L_p$

Thm (Szemerédi - Trotter) $|L| \geq \min\{r^{3/2} m^{1/2}, \text{other term}\}$



The Szemerédi-Trotter theorem <u>fails</u> if we try to apply it to points and tubes.



$m = \frac{\delta^{-1}}{r}$ many tubes through each pt

Line sets L_p highly concentrated in one direction

In order to prove incidence estimates for points & tubes, need to turn Cardinality $\rightsquigarrow$ size at different scales

Thm (Furstenberg sets, Orponen–Shmerkin, Ren–Wang)

Let $P \subset [0,1]^2$ be t-Frostman (down to scale δ)

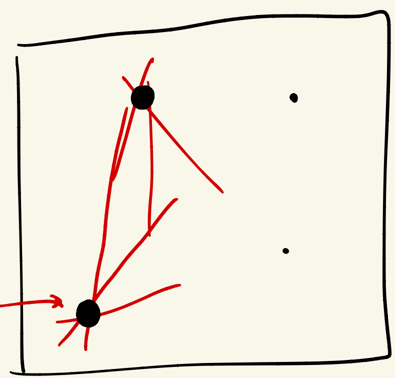
For each $p \in P$, L_p is an s-Frostman set of lines (down to scale δ)

Then

$$|L_P|_\delta \gtrsim \min\left\{ \delta^{-\left(\frac{3}{2}s + \frac{1}{2}t\right)}, \text{ other terms} \right\}$$

Line sets L_p

not highly concentrated

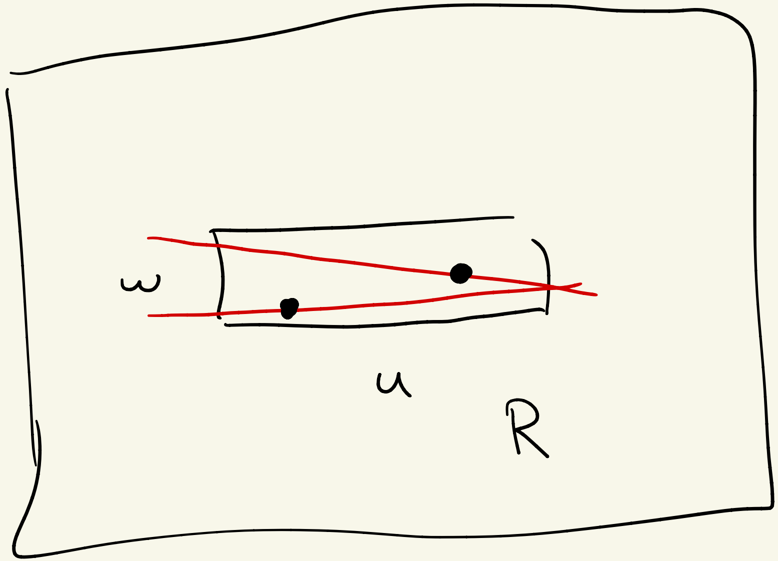


Q How to describe a configuration of point-line incidences with a branching function?

$(P, L) \longrightarrow f(x, y, z)$ a branching function of 3 variables

Uniformity Find a set of restricted incidences $\{(p_j, \ell_k)\}$

s.t.



Every rectangle R covers ~ the same number of incidences

(+ a bit more)

$2^{-f(x,y,z)}$ = Covering # by rectangles

Simple facts about rectangles $\to$ simple facts about f

- Monotonic
- Lipschitz
- Submodular

f also contains info about branching patterns of

L_p (= Lines through a chosen pt)

Normal version

- Let $P \subset [0,1]^2$ be t-Frostman
- For each $p \in P$, L_p is an s-Frostman set of lines

Then

$$|L_p|_\delta \gtrsim \min\left\{ \delta^{-(\frac{3}{2}s + \frac{1}{2}t)}, \text{ other terms} \right\}$$

Branching version: Let f be a branching func.

- suppose $f(x,0,x) \geq tx$
- suppose $f(1,y,1) - f(1,0,1) \geq sy$

Then

$$f(0,1,1) \geq \min\left\{ \frac{3}{2}s + \frac{1}{2}t, \text{ other terms} \right\}$$

Incidence thm $\rightsquigarrow$ If f is big in some places, it is also big other places.

Branching functions are a location to State questions about incidences, and to synthesize information.