Spin(7)-Manifolds and Multisymplectic Geometry

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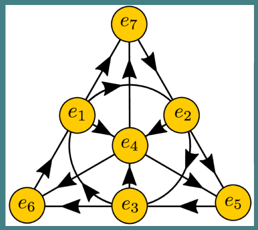

Spin(7)-Manifolds

Motivation: Berger Classification[1], Ricci-Flat, Parallel Spinor.

Definition (Admissible 4-Form). Pointwise identified with model 4-Form on $\mathbb{R}^8$

Theorem[2]. Admissible 4-Form in 1:1 Correspondence with Spin(7)-Structure. 8-Manifold admits Admissible 4-Form iff orientable, spin, and satisfies particular characteristic class condition.

Remark. May decompose spaces of k-forms into irreducible representations of Spin(7).

Theorem[2]. Manifold has holonomy contained in Spin(7) iff admits torsion-free Spin(7)-Structure iff admissible 4-Form is closed.

Theorem[2]. On Compact Manifold, Torsion-Free Spin(7)-Structure has Spin(7) Holonomy iff $\hat{A}$ genus is 1. Such manifolds are simply-connected and satisfy Betti number relation.

$$25 = b_3 - b_2 + b_4^+ - 2b_4^-$$

Key Questions: Which manifolds admit Spin(7)-Holonomy Metrics and what properties do these manifolds have?

Compact Examples by Joyce[2]: Resolutions of Toroidal Orbifolds and Calabi-Yau Four-Orbifolds with antiholomorphic involutions

Complete Noncompact Examples: Bryant/Salamon[3], Cvectič/Gibbons/Lu/Pope[4], Foscolo[5].

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Key Point. Admissible 4-Form associated to Torsion-Free Spin(7)-Structure is a Multisymplectic Form.

Goal: Treat Spin(7)-Geometry as a Special Type of Multisymplectic Geometry. Constrain tensors on manifolds with torsion-free Spin(7)-Structures.

Remark. Multisymplectic Interior Product works very nicely with decomposition of differential forms into irreducible representations of Spin(7).

Lemma. Maps given by contracting a vector field, 2-multivector field, and 3-multivector field into the admissible 4-form are injective, bijective, and surjective linear maps, respectively.

Implication. Hamiltonian 2-multivector fields exist on any manifold with torsion-free Spin(7)-Structure and Hamiltonian 3-multivector fields exist in abundance.

Theorem. There do not exist any nontrivial Hamiltonian vector fields associated to a Torsion-Free Spin(7)-Structure on a compact manifold.

Results on Hamiltonian 2-Multivector Fields. Expressions for components in terms of irreducible representations of Spin(7). Cannot be globally decomposable on compact manifold, may not be closed on compact manifold, relationships given between pointwise norms of components.

Results on Hamiltonian 3-Multivector Fields. Expressions for components in terms of irreducible representations of Spin(7). May not be closed on compact manifold. Relationships given to Spin(7) triple cross product.

Future Goals: Further push the (tentative) analogy between symplectic geometry and exceptional geometries to fully clarify the relationships between these disciplines.

[1]Berger, M., Sur les groupes d'holonomie homogénes de variétés a connexion ane et des variétés riemanniennes. Bulletin de la Société Mathématique de France, Tome 83 (1955) , pp. 279-330. doi : 10.24033/bsmf.1464. [2]Joyce, D., Compact manifolds with special holonomy, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. [3]Bryant, R. and Salamon, S, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 , 829-850 (1989). [4]Cvetič, M., Gibbons, G.W., Lü, H., Pope, C.N., New complete noncompact Spin(7) manifolds, Nuclear Physics B, 620, 29-54, (2002). [5]Foscolo, L.,. Complete non-compact Spin(7) manifolds from self-dual Einstein 4-orbifolds Geom. Topol. 25(1): 339-408 (2021). [6]Ryvkin L., and Wurzbacher, T., An invitation to multisymplectic geometry, Journal of Geometry and Physics, 142, 9-36, (2019). [7] Cantrijn, F., Ibort, A. and de León, M., Hamiltonian structures on multisymplectic manifolds, Rend. Sem. Mat. Univ. Politec. Torino 54(3), 22523 (1999).

Multisymplectic Geometry

Motivation: Provides a natural generalization of symplectic geometry and models phase space in the Hamiltonian formalism for classical field theory. Has implications for quantization of such theories[7].

Definition (Multisymplectic k-form). A closed k-form is Multisymplectic if the interior product of a nontrivial vector field with the k-form does not vanish.

Theorem[6]. On an 8-Manifold every cohomology class of four-forms features a Multisymplectic form. Space of Multisymplectic forms is dense in closed forms.

Theorem[6]. Number of linear types of Multisymplectic 4-forms on 8 manifolds is 21

Key Questions. What are properties of specific types of multisymplectic geometries? In what sense precisely do various generalizations of results from symplectic geometry hold?

May define an interior product of a k-form with multivector field by splitting into decomposable pieces and contracting the vector fields into the form (in reverse order).

Definition (Hamiltonian Multivector Fields and Differential Forms). Satisfy the following equation involving multisymplectic form f, multivector X and form a.

$$i_X f = da$$