RELATIONSHIP BETWEEN HOMOGENEITY AND INDEPENDENCE CONSTRAINTS FOR CLASSIFICATION ALGORITHMS

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We describe a set of functional universal constraints for classification algorithms that correspond to particular systems of symmetric universal constraints.

We consider some aspect of the relationship between symmetric and functional constraints for classification algorithms. These constraints are one of the objects of study in the theory of universal and local constraints [1-3], which are a component of the algebraic approach to the design of correct classification algorithms [4, 5].

Formal constructions utilizing information about independence and/or homogeneity of various objects and classes for the solution of classification problems are described in [3, 6]. This information is expressed by appropriate constraints which are imposed on the form of the mappings realized by the sought algorithms. The formal equivalent of homogeneity constraints are so-called symmetric categories, which are defined by subgroups of the symmetric group σ_0 acting on the set of pairs $S = \{(1, 1), \ldots, (q, \ell)\}$, where q is the number of classes, ℓ is the number of objects in a control sample. Simultaneous homogeneity and independence constraints correspond to sets of mappings defined by functional signatures φ. A functional signature φ is a collection $(S_{(1,1)}, \ldots, S_{(q,\ell)})$ of linearly ordered subsets of the set S, called tuples, with the function λ, where λ takes S to the set $\{1, \ldots, t\}$, $t \geq q\ell$. For any pairs (i_1, j_1) and (i_2, j_2) we have

$$(\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow (|S_{(i_1, j_1)}| = |S_{(i_2, j_2)}|).$$

It is shown in [3] that only so-called admissible functional signatures define categories. A functional category is called admissible if the following conditions are satisfied:

(1) $(i, j) \in S_{(i,j)}$ for all $(i, j) \in S$;

(2) $(\lambda(i_1, j_1) = \lambda(i_2, j_2))$ & $((i_1, j_1) = s(i_1, j_1, k)) \Rightarrow ((i_2, j_2) = s(i_2, j_2, k))$ for all (i_1, j_1), (i_2, j_2) from S and all k from $\{1, \ldots, |S_{(i_1, j_1)}|\}$;

(3) $(\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow (\lambda(s(i_1, j_1, k)) = \lambda(s(i_2, j_2, k)))$ for all (i_1, j_1), (i_2, j_2) from S and all k from $\{1, \ldots, |S_{(i_1, j_1)}|\}$;

(4) $((i_1, j_1) \in S_{(i_2, j_2)}) \Rightarrow (S_{(i_1, j_1)} \subseteq S_{(i_2, j_2)})$ for all (i_1, j_1) and (i_2, j_2) from S;

(5) $(\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow ((s(s(i_1, j_1, k), r) = s(i_1, j_1, n)) \equiv (s(s(i_2, j_2, k), r) = s(i_2, j_2, n)))$ for all (i_1, j_1), (i_2, j_2) from S, all k, n from $\{1, \ldots, |S_{(i_1, j_1)}|\}$, and all r from $\{1, \ldots, |S_{s(i_1, j_1, k)}|\}$.

The relationship between symmetric and functional constraints which is relevant for classification problems is manifested in that sometimes functional categories are Γ-complete in symmetric categories. Informally, this means that problems posed using symmetric constraints may be solved using more constructively defined functional constraints. The condition ensuring this completeness was determined in [3]. It has the form

(6) $\forall (i_1, j_1) \in S\, ((\exists (i_2, j_2) : S_{(i_1, j_1)} \subset S_{(i_2, j_2)}) \Rightarrow (|\lambda(\lambda(i_1, j_1))| = 1))$.

It is also shown in [3] that the functional category Φ defined by the functional signature φ is a subcategory of the symmetric category Σ defined by the group σ if and only if σ is a subgroup of the group σ_φ, where σ_φ is defined as follows: the permutation g is in σ_φ if and only if it satisfies the following conditions:

(7) $\lambda(g(i, j)) = \lambda(i, j)$ for all (i, j) in S;

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(8) $g(s(i, j, k)) = s(g(i, j), k)$ for all (i, j) in S and all k in $\{1,\dots,|S_{(i,j)}|\}$.

Thus, if a feasible functional signature φ satisfies condition (6), then we can construct a symmetric category Σ_φ corresponding to the group σ_φ such that the category Φ is a Γ-complete subcategory of the category Σ_φ. However, the same group σ_φ may correspond to different functional signatures. Therefore below we consider the description of the set of functional signatures to which the same group σ_φ corresponds. We consider only admissible functional signatures that satisfy condition (6). The symbol "=" in application to ordered sets is understood in the sense of sets of identical composition and the symbol "~" denotes equality of ordered sets.

Definition 1. Admissible functional signatures φ and φ' are called σ-equivalent if $\sigma_\varphi = \sigma_{\varphi'}$.

Let a functional signature φ be given. Consider the orbits of two different elements of the set S by the group σ_φ. We know that the orbits of two different elements are either disjoint or identical. Orbits will be called classes of functional similarity. We denote the class containing the pair (i_1, j_1) by the symbol $\gamma_{(i_1,j_1)}$. The following lemma was proved in [6].

LEMMA 1. Let φ be an admissible functional signature, $(i_1, j_1) \neq (i_2, j_2)$, and $\lambda(i_1, j_1) = \lambda(i_2, j_2)$. Under these conditions,

1) if $S_{(i_1,j_1)} \neq S_{(i_2,j_2)}$, then there exists a permutation $g \in \sigma_\varphi$ such that

$$(g(i_1, j_1) = (i_2, j_2)) \,\&\, (g(i_2, j_2) = (i_1, j_1)) \,\&$$
$$\&\, (\forall (i_3, j_3)\, ((i_3, j_3) \notin S_{(i_1,j_1)} \cup S_{(i_2,j_2)} \Rightarrow$$
$$\Rightarrow g(i_3, j_3) = (i_3, j_3)));$$

2) if $S_{(i_1,j_1)} = S_{(i_2,j_2)}$, then there exists a permutation $g \in \sigma_\varphi$ such that

$$(g(i_1, j_1) = (i_2, j_2)) \,\&\, (\forall (i_3, j_3)\, ((i_3, j_3) \notin S_{(i_1,j_1)}$$
$$\cup\, S_{(i_2,j_2)} \Rightarrow g(i_3, j_3) = (i_3, j_3))).$$

From this lemma and Definition 1 of the group σ_φ it follows that

(9) $(\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Leftrightarrow (\gamma_{(i_1,j_1)} = \gamma_{(i_2,j_2)})$.

Thus, there exists a one-to-one correspondence between the set of values λ and the set of functional similarity classes. We denote the class corresponding to the value $\lambda = \ell$ by γ^ℓ.

Assume that for some pair (i_1, j_1) we have the inclusion $S_{(i_1,j_1)} \subset S_{(i_2,j_2)}$. Since we are considering functional signatures that satisfy condition (6), we have $|\lambda^{-1}(\lambda(i_1, j_1))| = 1$, i.e., the corresponding functional similarity class consists of a single element. The pairs (i, j) entering such classes will be called stationary, because any permutation from σ_φ leaves them fixed. In what follows, we consider signatures that do not contain stationary pairs, because the presence of such pairs does not affect the composition of the group σ_φ. Note that under these assumptions we also have

(10) $\forall (i_1, j_1), (i_2, j_2) \in S$

$$((S_{(i_1,j_1)} = S_{(i_2,j_2)}) \vee (S_{(i_1,j_1)} \cap S_{(i_2,j_2)} = \varnothing)).$$

For each $(i_1, j_1) \in S$, we use the symbol $St_{(i_1,j_1)}$ to denote the stabilizer of this element in the group σ_φ, i.e., the subgroup of the group σ_φ such that any permutation from this subgroup leaves the pair (i_1, j_1) unchanged.

Definition 2. The block $w_{(i_1,j_1)}$ is the set of pairs (i_2, j_2) such that

$$((i_2, j_2) \in w_{(i_1,j_1)}) \Leftrightarrow (St_{(i_1,j_1)} = St_{(i_2,j_2)}).$$

LEMMA 2. $\forall (i_1, j_1) \in S\ (S_{(i_1,j_1)} \in w_{(i_1,j_1)})$.

Proof. We will show that for any (i_2, j_2) in S we have $St_{(i_1,j_1)} = St_{(i_2,j_2)}$.

1. $St_{(i_1,j_1)} \subseteq St_{(i_2,j_2)}$. Indeed, if $g(i_1, j_1) = (i_1, j_1)$, then $S_{g(i_1,j_1)} \sim S_{(i_1,j_1)}$ by definition of σ_φ. Thus, for any (i_2, j_2) in S we have $g(i_1, j_1) = (i_2, j_2)$.

2. $St_{(i_2,j_2)} \subseteq St_{(i_1,j_1)}$. Assume that $g(i_2, j_2) = (i_2, j_2)$, $g(i_1, j_1) = (i_3, j_3) \neq (i_1, j_1)$, and $S_{(i_3,j_3)} \sim g(S_{(i_1,j_1)})$. Let $(i_2, j_2) = s(i_1, j_1, k)$. Then $s(i_3, j_3, k) = s(i_1, j_1, k)$. Since tuples are either disjoint or identical (10), we have $S_{(i_3,j_3)} = S_{(i_1,j_1)} =$

$S_{(i_2,j_2)}$. For $(i_1, j_1) = s(i_1, j_1, k)$ we have $g(S_{(i_2,j_2)}) \sim S_{g(i_2,j_2)} \sim S_{(i_2,j_2)}$. At the same time, $g(s(i_2, j_2, m)) = (i_3, j_3)$. We obtain that $S_{(i_2,j_2)} \neq S_{(i_2,j_2)}$. A contradiction. Q.E.D.

LEMMA 3. The tuple $S_{(i_1,j_1)}$ is either identical with the block $w_{(i_1,j_1)}$ or the block is representable as the union of two disjoint tuples $S_{(i_1,j_1)}$ and $S_{(i_2,j_2)}$, where (i_1, j_1) and (i_2, j_2) form a two-element functional similarity class:

$$\forall (i_1, j_1) \in S \, (S_{(i_1,j_1)} = w_{(i_1,j_1)}) \vee (\exists (i_2, j_2) \in S:$$
$$: (w_{(i_1,j_1)} = S_{(i_1,j_1)} \cup S_{(i_2,j_2)}) \,\&\, (S_{(i_1,j_1)} \cap S_{(i_2,j_2)} = \varnothing)$$
$$\&\, (\forall g \in \sigma_\varphi \, g(i_1, j_1) \in \{(i_1, j_1), (i_2, j_2)\})).$$

Proof. 1. We will show that any permutation g from σ_φ satisfies the inclusion $g(i_1, j_1) \in w_{(i_1,j_1)}$ if $S_{(i_1,j_1)} \neq w_{(i_1,j_1)}$. Assume that $g(i_1, j_1)$ is not included in $w_{(i_1,j_1)}$. Suppose that (i_2, j_2) is included in $w_{(i_1,j_1)}$ and is not included in $S_{(i_1,j_1)}$. Then by Lemma 1 there exists a permutation h in σ_φ such that $h(i_1, j_1) = g(i_1, j_1)$ and $h(i_2, j_2) = (i_2, j_2)$ or $(i_2, j_2) \in S_{g(i_1,j_1)}$. But if condition 1 of Lemma 1 is satisfied, then (i_2, j_2) is not included in $w_{(i_1,j_1)}$, a contradiction. In case of condition 2 of Lemma 1, we obtain that $g(i_1, j_1)$ is not included in $w_{(i_1,j_1)}$. Therefore $S_{(i_2,j_2)} \neq S_{g(i_1,j_1)}$. But the intersection of the tuples $S_{(i_2,j_2)}$ and $S_{g(i_1,j_1)}$ is nonempty, a contradiction with (10).

2. We will show that no permutation g from σ_φ is included in $S_{(i_1,j_1)}$. Assume that $g(i_1, j_1) \in S_{(i_1,j_1)}$. Then $S_{g(i_1,j_1)} = S_{(i_1,j_1)}$ and by Lemma 1 there exists a permutation h in σ_φ such that $h(i_1, j_1) = g(i_1, j_1)$ and $\forall (i_2, j_2) \in w_{(i_1,j_1)} \backslash S_{(i_1,j_1)} \, (h(i_2, j_2) = (i_2, j_2))$. But then (i_2, j_2) is not included in $w_{(i_1,j_1)}$, a contradiction.

3. Assume that there exist permutations h and g in σ_φ such that $h(i_1, j_1) \neq (i_1, j_1)$, $g(i_1, j_1) \neq (i_1, j_1)$, and $h(i_1, j_1) \neq g(i_1, j_1)$. We have shown that $h(i_1, j_1)$ and $g(i_1, j_1)$ are not included in $S_{(i_1,j_1)}$ and are included in $w_{(i_1,j_1)}$. Since $g(i_1, j_1) = g(h^{-1}(h(i_1, j_1))$, $g(i_1, j_1)$ is not included in $S_{h(i_1,j_1)}$. But then there exists a permutation f such that $f(i_1, j_1) = h(i_1, j_1)$ and $f(g(i_1, j_1)) = h(i_1, j_1)$, i.e., $g(i_1, j_1)$ is included in $w_{(i_1,j_1)}$, a contradiction. We thus have the following assertion:

$$\forall g \in \sigma_\varphi (g(i_1, j_1) \in \{(i_1, j_1), (i_2, j_2)\}),$$
$$(i_1, j_1) \in w_{(i_1,j_1)}.$$

Assume that there exists an element (i_3, j_3) of the block $w_{(i_1,j_1)}$ which is included neither in $S_{(i_1,j_1)}$ nor in $S_{(i_2,j_2)}$. Then by Lemma 1 there exists a permutation h in σ_φ such that $h(i_3, j_3) = (i_3, j_3)$ and $h(i_1, j_1) = (i_2, j_2)$, i.e., (i_3, j_3) is not included in $w_{(i_1,j_1)}$. Therefore $w_{(i_1,j_1)} = S_{(i_1,j_1)} \cup S_{(i_2,j_2)}$. Q.E.D.

Lemmas 2 and 3 link the notions of tuple and block. A block is identical with a tuple considered as an unordered set, with the exception of the case of two-element functional similarity classes. Let us now describe the set of σ-equivalent functional signatures.

We introduce the following transformations of functional signatures.

1. For some functional similarity class γ^ℓ define the permutation ψ^ℓ acting on the set $\{1,\ldots,v_\ell\}$, where v_ℓ is the cardinality of the tuples corresponding to the elements of this class.

Construct a new signature φ'. Let $\lambda' \equiv \lambda$. For all pairs (i_1, j_1) contained in γ^ℓ, define new tuples $s'(i_1, j_1, k) = s(i_1, j_1, \psi^\ell(k))$; for other pairs, leave the tuples unchanged.

This transformation reduces to a compatible change of order in tuples corresponding to the elements of one of the functional similarity classes.

Example 1.

$$\varphi: \{((1,1), (1, 2), (1, 3)), ((1, 2), (1, 3), (1, 1)), ((1, 3), (1, 1) (1, 2)), ((2, 1), (1, 2), (2, 3)), ((2, 2), (2, 3), (2, 1)), ((2, 3), (2, 1) (2, 2))\};$$
$$\lambda(1, 1) = \lambda(1, 2) = \lambda(1, 3) = 1; \; \lambda(2, 1) = \lambda(2,2) = \lambda(2, 3) = 2;$$
$$\psi^1: \begin{pmatrix} (1,1), & (1,2), & (1,3) \\ (1,2), & (1,1), & (1,3) \end{pmatrix},$$
$$\varphi': \{((1, 2, (1, 1), (1, 3)), ((1, 3), (1, 2), (1, 1)), ((1, 1), (1, 3) (1, 2)), ((2, 1), (1, 2), (2, 3)), ((2, 2), (2, 3), (2, 1)), ((2, 3), (2, 1) (2, 2))\};$$
$$\lambda' \equiv \lambda.$$

Transformations 2a and 2b described below are applied to signatures that contain two-element functional similarity classes and affect only elements of these classes. Consider some two-element class $\gamma^{\ell 1} = \{(i_1, j_1), (i_2, j_2)\}$. By

Lemma 3, we may have two cases:

a) $S_{(i_1, j_1)} = S_{(i_2, j_1)} = w$;

b) $S_{(i_1, j_1)} \cap S_{(i_2, j_2)} = \varnothing,\ S_{(i_1, j_1)} \cup S_{(i_2, j_2)} = w$.

From the definition of a block it follows that any pair (i_1, j_1) in w is contained in some two-element functional similarity classes, and both elements of each class are contained in w. Therefore, the block w consists of an even number of elements. Let the cardinality of the block be $|w| = 2p$. The block w is representable as the union $w = \gamma^{\ell_1} \cup \ldots \cup \gamma^{\ell_p}$, where $\gamma^{\ell_k} = \{(i_k, j_k), (i_k', j_k')\}$.

It follows from (10) that either condition a or condition b is satisfied simultaneously for all pairs of tuples $(S_{(i_k, j_k)}, S_{(i_k', j_k')})$.

2a. Assume that condition a is satisfied. Applying transformation 1 to the signature, we can ensure that the following conditions hold:

(11) all tuples $S_{(i_k, j_k)}$ are identical, all tuples $S_{(i_k', j_k')}$ are identical (all tuples have the same composition and the same order of elements);

(12) $\forall g \in \sigma_\varphi : g(i_1, j_1) \neq (i_1, j_1)\ \forall m, k\, (g(s(i_k, j_k, m)) = s(i_k, j_k, m + p))$.

The action of any permutation from σ_φ reduces to interchanging the first and the last p elements in the tuples.

Let us describe transformation 2a: $\lambda' \equiv \lambda$; the tuples of all pairs not included in w remain unchanged: for all tuples corresponding to the elements of w, the tuple length is halved and the first p elements are retained:

$$\forall n \leqslant p\ \ s'(i_k, j_k, n) = s(i_k, j_k, n);$$
$$\forall n \leqslant p\ \ s'(i_k', j_k', n) = s(i_k', j_k', n).$$

Example 2.

$$\begin{aligned}&\varphi: \{((1, 1), (2, 2), (1, 2), (2, 1)), ((1, 2), (2, 1), (1, 1), (2, 2)),\\ &((1, 1), (2, 2), (1, 2), (2, 1)), ((1, 2), (2, 1), (1, 1), (2, 2))\};\\ &\lambda(1,1) = \lambda(1, 2) = 1; \lambda(2, 1) = \lambda(2, 2) = 2;\\ &\varphi': \{((1, 1), (2,2)), ((1, 2), (2, 1)), ((1, 1), (2, 2)), ((1, 2), (2, 1))\};\\ &\lambda' \equiv \lambda.\end{aligned}$$

2b. Assume that condition b is satisfied. Let us describe transformation 2b: $\lambda' = \lambda$; for all elements S not included in w, the tuples do not change; for all elements of w the tuple length is doubled, and

$$\forall n \leqslant p\ \ s'(i_k, j_k, n) = s(i_k, j_k, n),$$
$$\forall n > p\ \ s'(i_k, j_k, n) = s(i_k', j_k', n - p),$$
$$\forall n \leqslant p\ \ s'(i_k', j_k', n) = s(i_k', j_k', n),$$
$$\forall n > p\ \ s'(i_k', j_k', n) = s(i_k, j_k, n - p).$$

The new tuple is a concatenation of the old tuple and the tuple corresponding to the second element of the two-element functional similarity class.

Example 3. If in the example for transformation 2a we consider the transition from φ' to φ, then we obtain transformation 2b.

LEMMA 4. Application of transformations 1, 2a, 2b to admissible functional signatures φ that satisfy condition (6) and do not contain stationary pairs produces admissible functional signatures φ' that also satisfy condition (6) and do not contain stationary pairs, such that $\sigma_{\varphi'} = \sigma_\varphi$.

Proof. 1. Assume that φ' is obtained from φ by transformation 1. Check the admissibility conditions (1)-(5). Conditions (1) and (4) are obviously satisfied by construction of φ'. We will show that the following conditions are satisfied:

$$\begin{aligned}&(2)\ (\lambda'(i_1, j_1) = \lambda'(i_2, i_2))\ \&\ ((i_1, j_1) = s'(i_1, j_1, k)) \Rightarrow\\ &\Rightarrow (\lambda(i_1, j_1) = \lambda(i_2, j_2))\ \&\ ((i_1, j_1) = s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)} \times\\ &\times (k))) \Rightarrow ((i_2, j_2) = s(i_2, j_2, \psi^{-1}_{\lambda(i_1, j_1)}(k)));\\ &\text{since } \lambda(i_1, j_1) = \lambda(i_2, j_2), \text{ then } (i_2, j_2) =\\ &= s(i_2, j_2, \psi^{-1}_{\lambda(i_2, j_2)}(k)) = s'(i_2, j_2, k);\\ &(3)\ (\lambda'(i_1, j_1) = \lambda'(i_2, j_2)) \Rightarrow (\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow\\ &\Rightarrow (\forall k(\lambda(s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)}(k))) = \lambda(s(i_2, j_2, \psi^{-1}_{\lambda(i_2, j_2)} \times\\ &\times (k))) \Rightarrow (\forall k(\lambda'(s'(i_1, j_1, k)) = \lambda'(s'(i_2, j_2, k)))),\end{aligned}$$

$$(5)\ (\lambda'(i_1, j_1) = \lambda'(i_2, j_2)) \Rightarrow (\lambda(i_1, j_1) = \lambda(i_2, j_2)) \Rightarrow$$
$$\Rightarrow (\forall k \forall m \forall r (s(s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)}(k)),$$
$$\psi^{-1}_{\lambda(s(i_1, j_1), \psi^{-1}_{\lambda(i_1, j_1)}}(k)))(r)) = s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)}(m))) \equiv$$
$$\equiv (s(s(i_2, j_2, \psi^{-1}_{\lambda(i_1, j_1)}(k)), \psi^{-1}_{\lambda(s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)}}(k)))(r)) =$$
$$= s(i_2, j_2, \psi^{-1}_{\lambda(i_1, j_1)}(m)))));$$

since $\lambda(i_1, j_1) = \lambda(i_2, j_2)$, then $\psi^{-1}_{\lambda(i_1, j_1)}(k) \equiv \psi^{-1}_{\lambda(i_2, j_2)}(k)$; by condition (5), $\psi^{-1}_{\lambda(s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)}}(k)))(r) = \psi^{-1}_{\lambda(s(i_2, j_2, \psi^{-1}_{\lambda(i_2, j_2)}} \times$ $(k)))(r)$. Passing to s′, we obtain $(s'(s'(i_1, j_1, k), r) = s'(i_1, j_1, m)) \equiv (s'(s'(i_2, j_2, k), r) = s'(i_2, j_2, m))$.

Condition (6) for φ' is obvious. Let us show that $\sigma_{\varphi'} = \sigma_{\varphi}$. We first prove that $\forall g \in \sigma_{\varphi}$ $(g \in \sigma_{\varphi'})$.

Condition (7) is obvious. Let us check condition (8):

$$g(s'(i_1, j_1, k)) = g(s(i_1, j_1, \psi^{-1}_{\lambda(i_1, j_1)}(k)))$$
$$= s(g(i_1, j_1), \psi^{-1}_{\lambda(i_1, j_1)}(k)) = s'(g(i_1, j_1), k).$$

The converse inclusion is proved similarly.

2. Assume that φ' was obtained from φ by transformation 2a. Condition (1) for φ' is obvious. Since the transformation is applied compatibly to all tuples corresponding to the elements of the block w, we have $((i_1, j_1) \in S'_{(i_2, j_2)}) \Rightarrow (S'_{(i_1, j_1)} = S'_{(i_2, j_2)})$, which implies that (4) is satisfied.

Conditions (2), (3), (5), (6) are obviously satisfied by construction. Let us show that $\sigma_{\varphi} = \sigma_{\varphi'}$. Consider the two-element functional similarity class $\gamma^{k} = \{(i_k, j_k), (i_k', j_k')\}$. We first prove that $\forall g \in \sigma_{\varphi}$ $(g \in \sigma_{\varphi'})$.

Let us check condition (8): $g(s'(i_k, j_k, m)) = g(s(i_k, j_k, m)) = s(g(i_k, j_k), m) = s'(g(i_k, j_k), m)$.

Now let us prove that $\forall g \in \sigma_{\varphi'}$ $(g \in \sigma_{\varphi})$.

Let $m \le p$. Then $g(s(i_k, j_k, m)) = g(s'(i_k, j_k, m)) = s'(g(i_k, j_k), m) = s(g(i_k, j_k), m)$. Let $m > p$, then $g(s(i_k, j_k, m)) = g(s'(i_k', j_k', m - p)) = s'(g(i_k', j_k'), m - p)$.. Since $g(i_k', j_k') \in \gamma^{k}$ and $g(i_k', j_k') \neq g(i_k, j_k)$, then $s'(g(i_k', j_k'), m - p) = g(s(i_k, j_k, m))$.

3. Assume that φ' is obtained from φ by transformation 2b. Condition (1) is obviously true. We will show that (4) is satisfied:

$$((i_t, j_t) \in S'_{(i_k, j_k)}) \Rightarrow ((i_t, j_t) \in S_{(i_k, j_k)}) \vee ((i_t, j_t) \in$$
$$\in S_{(i_k', j_k')}) \Rightarrow (S'_{(i_t, j_t)} = S'_{(i_k, j_k)}) \Rightarrow (S_{(i_t, j_t)} \subseteq S_{(i_k, j_k)}).$$

Condition (2) is obviously satisfied. Let us check condition (3). Consider two cases: $k \le p$ and $k > p$. Let $k \le p$. Then $s'(i, j, k) = s(i, j, k)$, $s'(i', j', k) = s(i', j', k)$. Therefore, $\lambda'(s'(i, j, k)) = \lambda(s'(i', j', k))$. Let $k > p$. In this case, $s'(i, j, k) = s'(i', j', k - p)$, $s'(i', j', k) = s(i, j, k - p)$, $\lambda'(s'(i, j, k)) = \lambda(s(i', j', k - p)) = \lambda(s(i, j, k - p)) = \lambda'(s'(i', j', k))$. We will show that condition (5) is satisfied. Consider the following possible cases: 1) $k \le p$, $r \le p$; 2) $k \le p$, $r > p$; 3) $k > p$, $r \le p$; 4) $k > p$, $r > p$.

In case 1), $s'(i_t, j_t, k) = s(i_t, j_t, k)$, $s'(s'(i_t, j_t, k), r) = s(s(i_t, j_t, k), r)$, $S_{s(i_t, j_t, k)} = S_{(i_t, j_t)}$. If $s'(s'(i_t, j_t, k), r) = s'(i_t, j_t, m)$, then $m \le p$, because for any $m > p$ we have the inclusion $s'(i_t, j_t, m) \in S_{(i_t', j_t')}$. We thus obtain

$$(s(s(i_t, j_t, k), r) = s(i_t, j_t, m)) \equiv (s(s(i_t', j_t', k), r) = s(i_t', j_t', m)) \equiv (s'(s'(i_t', j_t', k), r) = s'(i_t', j_t', m)).$$

In case 2 it is obvious that the elements $s(i_t, j_t, k)$ and $s(i_t', j_t', k)$ form a two-element functional similarity class. If $s'(s'(i_t, j_t, k), r) = s(s(i_t', j_t', k), r - p) = s'(i_t, j_t, m)$,, then $m > p$, because $S_{s(i_t', j_t', k)} \in S_{(i_t', j_t')}$. We thus obtain

$$(s(s(i_t', j_t', k), r - p) = s(i_t', j_t', m - p)) \equiv (s(s(i_t, j_t, k), r - p) = s(i_t, j_t, m - p)) \equiv (s'(s'(i_t', j_t', k), r) = s'(i_t', j_t', m)).$$

Cases 3 and 4 are considered similarly.

Let us show that $\sigma_{\varphi} = \sigma_{\varphi'}$. We will prove that $\forall g \in \sigma_{\varphi}$ $(g \in \sigma_{\varphi'})$.

Let $m \le p$, then $g(s'(i_k, j_k, m)) = g(s(i_k, j_k, m)) = s(g(i_k, j_k), m) = s'(g(i_k, j_k), m)$. For $m > p$, we have $g(s'(i_k, j_k, m)) = g(s(i_k', j_k', m - p)) = s(g(i_k', j_k'), m - p) = s'(g(i_k', j_k'), m - p)$.

Thus, condition (8) is satisfied. Condition (7) is obvious.

The converse inclusion is proved similarly to the inclusion $\sigma_\varphi \subseteq \sigma_{\varphi'}$ for transformation 2a.

We have examined all the transformations. Q.E.D.

LEMMA 5. Let $\sigma_\varphi = \sigma_{\varphi'}$. Then φ' can be obtained from φ by superposition of transformations 1, 2a, and 2b.

Proof. Consider the tuples corresponding to the pair (i_1, j_1) in φ and φ'. By Lemma 3, $S_{(i_1,j_1)} = w_{(i_1,j_1)}$ or $w_{(i_1,j_1)} = S_{(i_1,j_1)} \cup S_{(i_1',j_1')}$, and $S_{(i_1,j_1)} \cap S_{(i_1',j_1')} = \varnothing$ and for any g in σ_φ we have $g(i_1, j_1) \in \{(i_1, j_1), (i_1', j_1')\}$. A similar assertion holds for the tuple $S'_{(i_1,j_1)}$ and the block $w'_{(i_1,j_1)}$. Since blocks are defined in terms of σ_φ, the blocks for φ and φ' are identical. The following cases are possible.

1. $S_{(i_1,j_1)} = S'_{(i_1,j_1)} = w_{(i_1,j_1)}$. Transformation 1 takes $S_{(i_1,j_1)}$ to $S'_{(i_1,j_1)}$. Define ψ^ℓ as follows: $s'(i_1, j_1, k) = s(i_1, j_1, \psi^\ell(k))$, where $\ell = \lambda(i_1, j_1)$. Since $\sigma_\varphi = \sigma_{\varphi'}$, then $s'(i_1', j_1', k) = g(s'(i_1, j_1, k))$, where g is a permutation in σ_φ such that $g(i_1, j_1) = (i_1', j_1')$. Therefore, $s'(i_1', j_1', k) = s(i_1', j_1', \psi^l(k)) = g(s(i_1, j_1, \psi^l(k)))$, i.e., for any (i_1', j_1') such that $\lambda(i_1, j_1) = \lambda(i_1', j_1')$, $S_{(i_1',j_1')}$ goes to $S'_{(i_1',j_1')}$.

2. $S_{(i_1,j_1)} = w_{(i_1,j_1)}$, $S'_{(i_1,j_1)} \cup S'_{(i_1',j_1')} = w_{(i_1,j_1)}$, $\{(i_1, j_1), (i_1, j_1)\}$ is a two-element functional similarity class.

Transformation 1 takes $S_{(i_k,j_k)}$ and $S_{(i_k',j_k')}$ to the form $S'_{(i_k,j_k)} \| S'_{(i_k',j_k')}$ and $S''_{(i_k',j_k')} \| S'_{(i_k,j_k)}$, respectively, where $\|$ denotes concatenation, $w_{(i_1,j_1)} = \gamma^{l_1} \cup \dots \cup \gamma^{l_p}$, $\gamma^{l_k} = \{(i_1, j_1), (i_1, j_1)\}$. Applying transformation 2a to the tuples $S'_{(i_k,j_k)} \| S'_{(i_k',j_k')}$ and $S'_{(i_k',j_k')} \| S'_{(i_k,j_k)}$, we obtain the tuples $S'_{(i_k,j_k)}$ and $S'_{(i_k',j_k')}$.

3. $S'_{(i_1,j_1)} = w_{(i_1,j_1)}$, $S_{(i_1,j_1)} \cup S_{(i_1',j_1')} = w_{(i_1,j_1)}$. This case is similar to case 2.

4. $S'_{(i_1,j_1)} \cup S'_{(i_1',j_1')} = w_{(i_1,j_1)}$, $S_{(i_1,j_1)} \cup S_{(i_1',j_1')} = w_{(i_1,j_1)}$. Applying transformation 2b to one of the signatures, we obtain case 2 or 3. Q.E.D.

The results obtained in Lemmas 4 and 5 can be summarized in the form of a theorem.

THEOREM 1. Admissible functional signatures φ and φ' that satisfy condition (6) and do not contain stationary pairs are σ-equivalent if and only if φ' can be obtained from φ by superposition of transformations 1, 2a, and 2b.

This theorem provides a description of the set of σ-equivalent functional signatures.

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