FETI domain decomposition method to solution of contact problems with large displacements

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1 Introduction

The solution to contact problems between solid bodies poses difficulties to solvers because in general neither the distributions of the contact tractions throughout the areas currently in contact nor the configurations of these areas are known a priori. This implies that the contact problems are inherently strongly nonlinear. Probably the most popular solution method is based on direct iterations with the non-penetration conditions imposed by the penalty method ([Z93] or [W02]). The method enables easily enhance other non-linearity such as in the case of large displacements.

In this paper we are concerned with application of a variant of the FETI domain decomposition method that enforces feasibility of Lagrange multipliers by the penalty [DH04b]. The dual penalty method, which has been shown to be optimal for small displacements is used in inner loop of the algorithm that treats large displacements. We give results of numerical experiments that demonstrate high efficiency of the FETI method

2 Primal Penalty Method

The boundary conditions generated by bodies in contact are formally of the same form as the boundary conditions induced by externally applied surface tractions. However, the difficulties with the contact tractions is that in general we do not know either their distributions throughout the areas currently in contact, or shapes and magnitudes of these areas until we have run the problem. Their evaluations have to be part of the solution.

Consider the frictionless contact henceforth. There exist two basic methods to remove the contact constraints. The first one is the Lagrange multiplier method and the second one the penalty method, or in the sense of this paper the primal penalty method. With the latter method constraints are enforced by the penalization. The penalization of the Kuhn–Tucker conditions in the normal direction is established by introducing the penalty parameter ϵ_n in

$$f_n = \epsilon_n \langle g \rangle \tag{1}$$

where f_n stands for the normal contact force, g denotes the depth of interpenetration of the bodies in contact and $\langle . \rangle = 0.5[(.) + |.|]$ is known as Macauley bracket. It returns the non-negative part of its operand. The normal penalty can be seen as stiffness of a spring placed between corresponding contacting surfaces. The penalty method yields exact solution if the penalty tends to infinity, but otherwise permits certain violation of the constraint that the interpenetration has to be zero. In practice it is necessary to estimate the magnitude of the penalty parameter to limit the penetration, yet it should not be too large to avoid ill-conditioning. The penalty parameter should be increased if the grid is refined.

3 Application of FETI to Contact Problems

Let us briefly outline the fundamental formulae of the FETI method. Consider solid bodies in contact, discretized into a finite element mesh and in addition decomposed into sub-domains. The numerical approximation to the problem in terms of the finite element discretization and auxiliary domain decomposition can be expressed as reads

$$\min \frac{1}{2} u^\top K u - f^\top u \quad \text{subject to} \quad B^I u \le 0 \quad \text{and} \quad B^E u = 0 \tag{2}$$

where A stands for a positive semi-definite stiffness matrix, B^I and B^E denote the full rank matrices which enforce the discretized inequality constraints describing conditions of non-interpenetration of bodies and inter-subdomain equality constraints, respectively, and f stands for the discrete analogue of the linear form $\ell(u)$.

Denoting

$$\lambda = \begin{bmatrix} \lambda^I \\ \lambda^E \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B^I \\ B^E \end{bmatrix},$$

we can write the Lagrangian associated with problem (2) briefly as

$$L(u, \lambda) = \frac{1}{2} u^\top K u - f^\top u + \lambda^\top B u.$$

It is well known that (2) is equivalent to the saddle point problem

$$\text{Find} \quad (\overline{u}, \overline{\lambda}) \quad \text{s.t.} \quad L(\overline{u}, \overline{\lambda}) = \sup_{\lambda_I \geq 0} \inf_{u} L(u, \lambda). \tag{3}$$

After eliminating the primal variables u from (3), we shall get the minimization problem

$$\min \ \Theta(\lambda) \quad \text{s.t.} \quad \lambda_I \geq 0 \quad \text{and} \quad R^\top (f - B^\top \lambda) = 0, \tag{4}$$

where

$$\Theta(\lambda) = \frac{1}{2} \lambda^\top B K^\dagger B^\top \lambda - \lambda^\top B K^\dagger f, \tag{5}$$

$K^\dagger$ denotes a generalized inverse that satisfies $KK^\dagger K = K$, and R denotes the full rank matrix whose columns span the kernel of K.

Even though problem (4) is much more suitable for computations than (2), further improvement may be achieved by adopting some simple observations and the results of Farhat, Mandel, Roux and Tezaur [FMR94, MT96]. Let us denote

$$F = BK^\dagger B^\top, \quad \widetilde{G} = R^\top B^\top, \quad \widetilde{e} = R^\top f, \quad \widetilde{d} = BK^\dagger f,$$

and let $\widetilde{\lambda}$ solve $\widetilde{G}\widetilde{\lambda} = \widetilde{e}$, so that we can transform the problem (4) to minimization on the subset of the vector space by looking for the solution in the form $\lambda = \mu + \widetilde{\lambda}$. Since

$$\frac{1}{2} \lambda^\top F \lambda - \lambda^\top \widetilde{d} = \frac{1}{2} \mu^\top F \mu - \mu^\top (\widetilde{d} - F\widetilde{\lambda}) + \frac{1}{2} \widetilde{\lambda}^\top F \widetilde{\lambda} - \widetilde{\lambda}^\top \widetilde{d},$$

problem (4) is, after returning to the old notation, equivalent to

$$\min \quad \frac{1}{2} \lambda^\top F \lambda - \lambda^\top d \quad \text{s.t.} \quad G\lambda = 0 \quad \text{and} \quad \lambda^I \geq -\widetilde{\lambda}^I \tag{6}$$

where $d = \widetilde{d} - F\widetilde{\lambda}$ and G denotes a matrix arising from the orthonormalization of the rows of $\widetilde{G}$.

Our final step is based on observation that the problem (6) is equivalent to

$$\min \quad \frac{1}{2} \lambda^\top PFP \lambda - \lambda^\top P d \quad \text{s.t.} \quad G\lambda = 0 \quad \text{and} \quad \lambda^I \geq -\widetilde{\lambda}^I \tag{7}$$

where

$$Q = G^\top G \qquad \text{and} \qquad P = I - Q$$

denote the orthogonal projectors on the image space of $G^\top$ and on the kernel of G, respectively. Enhancing the equality constraints in (7) by the penalty into the function

$$\Theta_\rho(\lambda) = \frac{1}{2} \lambda^\top (PFP + \rho Q) \lambda - \lambda^\top P d, \tag{8}$$

we can approximate the solution of (7) by the solution of

$$\min \quad \Theta_\rho(\lambda) \quad \text{s.t.} \quad \lambda^I \geq -\widetilde{\lambda}^I \tag{9}$$

with a sufficiently large penalty parameter ρ. Note that image spaces of the projectors P and Q are invariant subspaces of the Hessian $H^\rho = PFP + \rho Q$ of $\Theta_\rho(\lambda)$.

4 Scalable algorithm based on optimal dual penalty

In this section we shall describe a scalable algorithm for (9). Basic ingredient of the theoretical development is the estimate by Mandel and Tezaur [MT96] who proved that under the assumption on regularity of the discretization and boundedness of H/h, there is a lower bound $\alpha > 0$ on the eigenvalues of PFP restricted to the range of P that is independent of h and H, so that for any vector λ

$$\lambda^\top PFP\lambda \geq \alpha \|P\lambda\|^2. \tag{10}$$

Denoting $\alpha_1 = \min\{\alpha, \rho_0\}$, it follows that for any $\delta, \rho_0 > 0$ and $\rho \geq \rho_0$

$$\delta^\top H^\rho \delta \geq \delta^\top H^{\rho_0} \delta \geq \alpha_1 \|\delta\|^2. \tag{11}$$

Another important ingredient is a recently proposed algorithm for bound constrained quadratic programming called modified proportioning with reduced gradient projections (MPRGP) [DS05]. The MPRGP algorithm with the choice of parameters $\Gamma = 1$ and $\overline{\alpha} \in (0, \|H^\rho\|^{-1}]$ generates the iterations $\{\lambda^k\}$ for the unique solution $\overline{\lambda}$ of (9) so that the rate of convergence in the energy norm defined by $\|\lambda\|^2_{H^\rho} = \lambda^\top H^\rho \lambda$ is given by

$$\|\lambda^k - \overline{\lambda}\|^2_{H^\rho} \leq \frac{2\eta^k}{\alpha_1} \left(\Theta_\rho(\lambda^0) - \Theta_\rho(\overline{\lambda})\right), \qquad \eta = 1 - \frac{\overline{\alpha}\alpha_1}{4}. \tag{12}$$

Theorem 1. *Let C, ρ and ϵ denote given positive numbers, and let $\{\lambda^i_{H,h}\}$ denote the iterations generated by MPRPG algorithm with the initial approximation $\lambda^0 = 0$ for the solution $\overline{\lambda}_{H,h}$ of the problem (9) arising from the sufficiently regular discretization of the continous problem with the decomposition, discretization and penalization parameters H, h and ρ. Then there is an integer k independent of h and H such that $H/h \leq C$ implies*

$$\|\lambda^k_{H,h} - \overline{\lambda}_{H,h}\| \leq \epsilon \|Pd\|. \tag{13}$$

Proof. See [DH04b].

Theorem 1 shows that we can generate efficiently λ that is near to the solution of (9). Its feasibility error is considered in the next theorem.

Theorem 2. *Let C_1 and ρ denote given positive numbers. Then there is a positive constant C such that if $\epsilon > 0$ and $\lambda_{H,h,\rho}$ denotes an approximate solution of the problem (9) arising from the sufficiently regular discretization of the continuous problem with the decomposition, discretization and penalization parameters H, h and ρ, respectively, $H/h \leq C_1$, $\rho \geq \rho_0$ and $\|\nu(\lambda_{H,h,\rho})\| \leq \epsilon \|Pd\|$, then*

$$\|G\lambda_{H,h,\rho}\| \leq C \frac{1+\epsilon}{\sqrt{\rho}} \|Pd\|. \tag{14}$$

Moreover, there is a constant $C_{H,h}$ that depends on H, h such that for any ρ

$$\|G\lambda_{H,h,\rho}\| \leq C_{H,h} \frac{1+\epsilon}{\rho} \|Pd\|. \tag{15}$$

Proof. See [DH04b].

Theorem 2 shows that a prescribed bound on the relative feasibility error (14) may be achieved with the penalty parameter ρ independent of the discretization parameter h. Thus we have shown that *we can get an approximate solution of the problem (7) with prescribed precision in a number of steps that does not depend on the discretization parameter h*. Let us recall that even though the large penalty parameters may destroy conditioning of the Hessian of the Lagrangian, they *need not* slow down the convergence of the conjugate gradient based methods.

5 Contact problems with large displacements

While the FETI method is directly applicable to the solution to contact problems of linearly elastic bodies with small displacements, any other nonlinearity necessitates application of additional method for solution of nonlinear problems. The non-linearity we take into account, apart from the contact, is the one caused by large displacements and finite rotations. To this end we use the total Lagrangian formulation which includes all kinematic non-linear effects. As a strain measure we make use of the Green–Lagrange tensor and as a stress measure the second Piola–Kirchhoff tensor which is work–conjugate with the previously mentioned strain tensor.

The Modified Newton-Raphson method was used as a tool for solution of these nonlinear problems. Hence, the following algorithm was proposed:

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Initial step:
    Assembling of stiffness matrix K and matrix of
    continuity conditions between subdomains B_E
Step 1
    Assembling of external nodal forces vector f_ext
    Prescribing conditions of non-interpenetration of bodies
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    in current configuration B_I, c_I.
Step 2
    Evaluation of internal forces vector f_int stemming from
    stresses
Step 3
    FETI solution of contact problem
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$$\min \ \frac{1}{2}u^T K u \ - \ u^T f \ \ s.t. \ B_I u \leq \ c_I \ \ and \ B_E u = 0$$

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    where vector f is residual between external forces f_ext
    and contact and internal forces f_int.
Step 4
    Test of convergence.
    In negative case go to Step 1, otherwise stop.
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As the suitable stopping criterion the relative change of nodal displacements can be chosen.

6 Hertzian Problem of Contact of Two Cylinders

Consider a classic Hertzian problem, i.e. frictionless and elastic one, of two cylinders with parallel axes in contact according to Figure 1. The radius of the upper cylinder is $R_1 = 1000$ mm and the radius of the lower cylinder is infinite, which means that the lower body is a half-space. The material properties of both bodies were as follows: Young's modulus $E = 2.0 \times 10^{11}$ Pa and Poisson's ratio $\nu = 0.3$. The load $Q = 400$ MN/m is applied along the axis of the upper cylinder. The problem is a two-dimensional one from mathematical point of view, but it was modelled with tri-linear elements as a three-dimensional problem by considering bodies of a finite length. The boundary conditions were imposed in such a way that they generated a plane strain problem. The complete mesh is shown in Figure 1 as well as its detail along the surfaces potentially in contact.

The analytical solution by McEwen can be found in [J85]. The results yielded by both the FETI method in terms of the dual penalty approach and the analytical solution are shown in Figure 2b. It shows distribution of normal contact stress along one half of the contact surface of the lower cylinder from the plane of symmetry upwards. It is obvious that the difference between both solutions is small. Let us notice that the various values of dual penalty was set from 1e+0 to 1e+4 without significant change of the solution. For comparison Figure 2a depicts solution of the same problem but in terms of the primal penalty method. The problem is not semi-coercive but coercive in this case because the primal penalty method cannot treat problems with sub-domains undergoing the rigid body modes. The penalty method was applied with five different magnitudes of the penalty parameters. It can clearly be seen how the quality of solution degrades progressively as the penalty parameter is reduced.

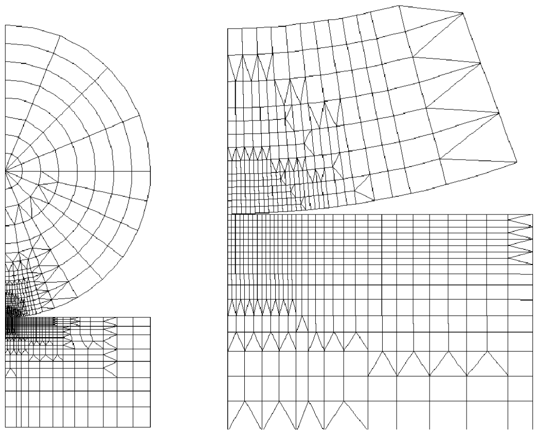

Fig. 1. Hertzian contact problem

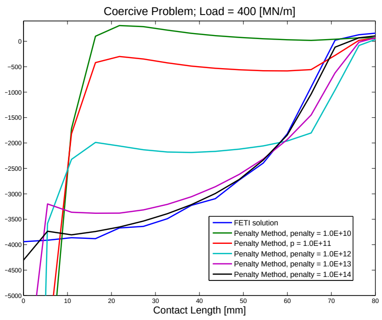

Fig. 2a. Normal contact stresses: Primal penalty method

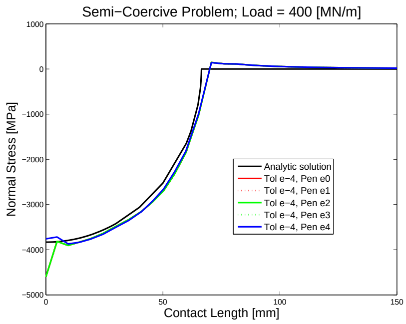

Fig. 2b. Normal contact stresses: Dual penalty (FETI) method

We solved the problem with a load (400 MN/m) and the displacements cannot be regarded as small. Therefore we had to iterate in the outer loop, in the sense of algorithm in section 5, because of the large displacements. The total load was applied in two steps for better convergence.

Figure 3a depicts numbers of conjugate gradients of the inner problem solver needed for convergence at each cycle of the outer loop. Figure 3b demonstrates independency of the number of conjugate gradients for different choices of the value of dual penalty.

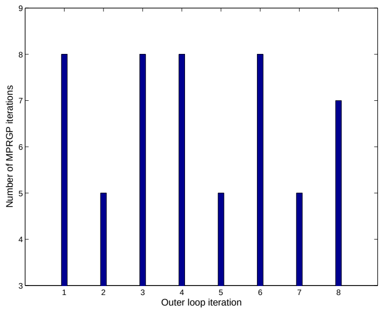

Fig. 3a. Number of iterations of CG.

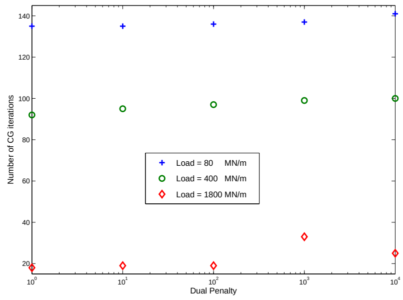

Fig. 3b. Convergence rate.

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