Optimized Restricted Additive Schwarz Methods

Amik St-Cyr[1], Martin J. Gander[2] and Stephen J. Thomas[3]

[1] National Center for Atmospheric Research, Boulder CO, amik@ucar.edu
[2] University of Geneva, Switzerland, martin.gander@math.unige.ch
[3] National Center for Atmospheric Research, Boulder CO, thomas@ucar.edu

Summary. A small modification of the restricted additive Schwarz (RAS) preconditioner at the algebraic level, motivated by continuous optimized Schwarz methods, leads to a greatly improved convergence rate of the iterative solver. The modification is only at the level of the subdomain matrices, and hence easy to do in an existing RAS implementation. Numerical experiments using finite difference and spectral element discretizations of the modified Helmholtz problem $u - \Delta u = f$ illustrate the effectiveness of the new approach.

1 Schwarz Methods at the Algebraic Level

The discretization of an elliptic partial differential equation

$$\mathcal{L}u = f \quad \text{in } \Omega, \quad \mathcal{B}u = g \quad \text{on } \partial\Omega, \tag{1}$$

where $\mathcal{L}$ is an elliptic differential operator, $\mathcal{B}$ is a boundary operator and Ω is a bounded domain, leads to a linear system of equations

$$A\mathbf{u} = \mathbf{f}. \tag{2}$$

A stationary iterative method for (2) is given by

$$\mathbf{u}^{n+1} = \mathbf{u}^n + M^{-1}(\mathbf{f} - A\mathbf{u}^n). \tag{3}$$

An initial guess $\mathbf{u}^0$ is required to start the iteration. Algebraic domain decomposition methods group the unknowns into subsets, $\mathbf{u}_j = R_j\mathbf{u}$, $j = 1, \ldots, J$, where R_j are rectangular matrices. Classical coefficient matrices for subdomain problems are defined by $A_j = R_j A R_j^T$. The additive Schwarz (AS) preconditioner [DW87], and the restricted additive Schwarz (RAS) preconditioner [CS99]) are defined by

$$M_{AS}^{-1} = \sum_{j=1}^{J} R_j^T A_j^{-1} R_j, \quad M_{RAS}^{-1} = \sum_{j=1}^{J} \tilde{R}_j^T A_j^{-1} R_j, \tag{4}$$

where the $\tilde{R}_j$ correspond to a non-overlapping decomposition, i.e. each entry u_l of the vector $\mathbf{u}$ occurs in $\tilde{R}_j\mathbf{u}$ for exactly one j.

The algebraic formulation of Schwarz methods has an important feature: a subdomain matrix A_j is not necessarily the restriction of A to a subdomain j. For example, if A represents a spectral element discretization of a differential operator, then A_j can be obtained from a finite element discretization at the collocation points. Furthermore, subdomain matrices A_j can be chosen to accelerate convergence and this is the focus of the next section.

2 Optimized Restricted Additive Schwarz Methods

Historically, domain decomposition methods were formulated at the continuous level. We consider a decomposition of the original domain Ω in (1) into two overlapping sub-domains Ω_1 and Ω_2, and we denote the interfaces by $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j$, $i \neq j$, and the outer boundaries by $\partial\Omega_j = \partial\Omega \cap \bar{\Omega}_j$. In [Lio88], a parallel Jacobi variant of the classical alternating Schwarz method was introduced for (1),

$$\begin{array}{ll} \mathcal{L}u_1^{n+1} = f \quad \text{in } \Omega_1, & \mathcal{L}u_2^{n+1} = f \quad \text{in } \Omega_2, \\ \mathcal{B}(u_1^{n+1}) = g \quad \text{on } \partial\Omega_1, & \mathcal{B}(u_2^{n+1}) = g \quad \text{on } \partial\Omega_2, \\ u_1^{n+1} = u_2^n \text{ on } \Gamma_{12}, & u_2^{n+1} = u_1^n \text{ on } \Gamma_{21}. \end{array} \tag{5}$$

It was shown in [EG02] that the discrete form of (5), namely

$$A_1\mathbf{u}_1^{n+1} = \mathbf{f}_1 + B_1\mathbf{u}_2^n, \quad A_2\mathbf{u}_2^{n+1} = \mathbf{f}_2 + B_2\mathbf{u}_1^n, \tag{6}$$

is equivalent to RAS in (4). In optimized algorithms, the Dirichlet transmission conditions in (5) are replaced by more effective transmission conditions, which corresponds to replacing the subdomain matrices A_j in (6) by $\tilde{A}_j$ and the transmission matrices B_j by $\tilde{B}_j$, corresponding to optimized transmission conditions, and leads to

$$\tilde{A}_1\mathbf{u}_1^{n+1} = \mathbf{f}_1 + \tilde{B}_1\mathbf{u}_2^n, \quad \tilde{A}_2\mathbf{u}_2^{n+1} = \mathbf{f}_2 + \tilde{B}_2\mathbf{u}_1^n, \tag{7}$$

see Sections 3 and 4 for how to choose $\tilde{A}_j$.

We now shown that, for sufficient overlap, the subdomain matrices A_j in the RAS algorithm (4) can be replaced by the optimized subdomain matrices $\tilde{A}_j$ from (7), to obtain an optimized RAS method (ORAS) equivalent to (7),

$$u^{n+1} = u^n + (\sum_{j=1}^{2} \tilde{R}_j^T \tilde{A}_j^{-1} R_j)(f - Au^n). \tag{8}$$

The additional interface matrices $\tilde{B}_j$ in (7) are not needed in the optimized RAS method (8), which greatly simplifies the transition from RAS to ORAS.

Definition 1 (Consistency). *Let R_j, $j = 1, 2$ be restriction matrices covering the entire discrete domain, and let $\mathbf{f}_j := R_j\mathbf{f}$. We call the matrix splitting R_j, $\tilde{A}_j$, $\tilde{B}_j$, $j = 1, 2$ in (7) consistent, if for all $\mathbf{f}$ and associated solution $\mathbf{u}$ of (2), $\mathbf{u}_1 = R_1\mathbf{u}$ and $\mathbf{u}_2 = R_2\mathbf{u}$ satisfy*

$$\tilde{A}_1\mathbf{u}_1 = \mathbf{f}_1 + \tilde{B}_1\mathbf{u}_2, \quad \tilde{A}_2\mathbf{u}_2 = \mathbf{f}_2 + \tilde{B}_2\mathbf{u}_1. \tag{9}$$

Lemma 1. *Let A in (2) have full rank. For a consistent matrix splitting R_j, $\tilde{A}_j$, $\tilde{B}_j$, $j = 1, 2$, we have the matrix identities*

$$\tilde{A}_1R_1 - \tilde{B}_1R_2 = R_1A, \quad \tilde{A}_2R_2 - \tilde{B}_2R_1 = R_2A. \tag{10}$$

Proof. We only prove the first identity, the second follows analogously. For an arbitrary $\mathbf{f}$, we apply R_1 to equation (2), and obtain, using consistency (9),

$$R_1A\mathbf{u} = R_1\mathbf{f} = \mathbf{f}_1 = \tilde{A}_1\mathbf{u}_1 - \tilde{B}_1\mathbf{u}_2.$$

Now using $\mathbf{u}_1 = R_1\mathbf{u}$ and $\mathbf{u}_2 = R_2\mathbf{u}$ on the right-hand side yields

$$(\tilde{A}_1R_1 - \tilde{B}_1R_2 - R_1A)\mathbf{u} = 0.$$

Because $\mathbf{f}$ was arbitrary, the identity is true for all $\mathbf{u}$ and therefore the first identity in (10) is established.

While the definition of consistency is simple, it has important consequences: if the classical submatrices are used, i.e. $\tilde{A}_j = A_j = R_jAR_j^T$, $j = 1, 2$, then the restriction matrices R_j can be overlapping or non-overlapping, and with the associated B_j, we obtain a consistent splitting R_j, A_j, B_j, $j = 1, 2$. If however other subdomain matrices $\tilde{A}_j$ are employed, then the restriction matrices R_j must be such that the unknowns in u_1 affected by the change in $\tilde{A}_1$ are also available in u_2 to compensate via $\tilde{B}_1$ in equation (9), and similarly for u_2. Hence consistency implies for all non-classical splittings a condition on the overlap in the R_j in RAS. A strictly non-overlapping variant can be obtained when applying standard AS with non-overlapping R_j to the augmented system obtained from (7) at convergence,

$$\begin{bmatrix} \tilde{A}_1 & -\tilde{B}_1 \\ -\tilde{B}_2 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{11}$$

see the non-overlapping spectral element experiments in Section 4 and [SCGT05]. For optimized RAS, a further restriction on the overlap is necessary:

Lemma 2. *Let R_j, $j = 1, 2$, be restriction matrices covering the entire discrete domain, and let $\tilde{R}_j$ be the corresponding RAS versions of these matrices. If $\tilde{B}_1R_2\tilde{R}_1^T = 0$, then $\tilde{B}_1R_2\tilde{R}_2^T = \tilde{B}_1$, and if $\tilde{B}_2R_1\tilde{R}_2^T = 0$, then $\tilde{B}_2R_1\tilde{R}_1^T = \tilde{B}_2$.*

Proof. We first note that by the non-overlapping definition of $\tilde{R}_j$, $j = 1, 2$, the identity matrix I can be written as

$$I = \tilde{R}_1^T \tilde{R}_1 + \tilde{R}_2^T \tilde{R}_2. \tag{12}$$

Now multiplying $\tilde{B}_1 R_2 \tilde{R}_1^T = 0$ on the right by R_1 and substituting the term $\tilde{R}_1^T R_1$ using (12) leads to

$$(\tilde{B}_1 - \tilde{B}_1 R_2 \tilde{R}_2^T) R_2 = 0,$$

which completes the proof, since the fat restriction matrix R_2 has full rank. The second result follows analogously.

Theorem 1. *Let R_j, $\tilde{A}_j$, $\tilde{B}_j$, $j = 1, 2$ be a consistent matrix splitting, and let $\tilde{R}_j$ be the corresponding RAS versions of R_j. If the initial iterates $\mathbf{u}_j^0$, $j = 1, 2$, of the optimized Schwarz method (7) and the initial iterate $\mathbf{u}^0$ of the optimized RAS method (8) satisfy*

$$\mathbf{u}^0 = \tilde{R}_1^T \mathbf{u}_1^0 + \tilde{R}_2^T \mathbf{u}_2^0, \tag{13}$$

and if the overlap condition

$$\tilde{B}_1 R_2 \tilde{R}_1^T = 0, \quad \tilde{B}_2 R_1 \tilde{R}_2^T = 0 \tag{14}$$

is satisfied, then the two methods (7) and (8) generate an equivalent sequence of iterates,

$$\mathbf{u}^n = \tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n. \tag{15}$$

Proof. The proof is by induction. For $n = 0$, we have (15) by assumption (13) on the initial iterates. We now assume that $\mathbf{u}^n = \tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n$, and show that the identity (15) holds for $n + 1$. Applying Lemma 1 to the first term of the sum in (8), we obtain

$$\begin{aligned} \tilde{R}_1^T \tilde{A}_1^{-1} R_1 (\mathbf{f} - A\mathbf{u}^n) &= \tilde{R}_1^T \tilde{A}_1^{-1} (\mathbf{f}_1 - R_1 A \mathbf{u}^n) \\ &= \tilde{R}_1^T \tilde{A}_1^{-1} (\mathbf{f}_1 - (\tilde{A}_1 R_1 - \tilde{B}_1 R_2) \mathbf{u}^n) \\ &= \tilde{R}_1^T (\tilde{A}_1^{-1} \mathbf{f}_1 - R_1 \mathbf{u}^n + \tilde{A}_1^{-1} \tilde{B}_1 R_2 \mathbf{u}^n), \end{aligned} \tag{16}$$

and similarly for the second term of the sum,

$$\tilde{R}_2^T \tilde{A}_2^{-1} R_2 (\mathbf{f} - A\mathbf{u}^n) = \tilde{R}_2^T (\tilde{A}_2^{-1} \mathbf{f}_2 - R_2 \mathbf{u}^n + \tilde{A}_2^{-1} \tilde{B}_2 R_1 \mathbf{u}^n). \tag{17}$$

Substituting these two expressions into (8), and using (12) leads to

$$\mathbf{u}^{n+1} = \tilde{R}_1^T (\tilde{A}_1^{-1} (\mathbf{f}_1 + \tilde{B}_1 R_2 \mathbf{u}^n)) + \tilde{R}_2^T (\tilde{A}_2^{-1} (\mathbf{f}_2 + \tilde{B}_2 R_1 \mathbf{u}^n)).$$

Now replacing by induction hypothesis $\mathbf{u}^n$ by $\tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n$ on the right hand side and applying Lemma 2, we find together with (14)

$$\begin{aligned} \mathbf{u}^{n+1} &= \tilde{R}_1^T (\tilde{A}_1^{-1} (\mathbf{f}_1 + \tilde{B}_1 R_2 (\tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n))) \\ &\quad + \tilde{R}_2^T (\tilde{A}_2^{-1} (\mathbf{f}_2 + \tilde{B}_2 R_1 (\tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n))) \\ &= \tilde{R}_1^T (\tilde{A}_1^{-1} (\mathbf{f}_1 + \tilde{B}_1 \mathbf{u}_2^n)) + \tilde{R}_2^T (\tilde{A}_2^{-1} (\mathbf{f}_2 + \tilde{B}_2 \mathbf{u}_1^n)), \end{aligned}$$

which together with (7) implies $\mathbf{u}^{n+1} = \tilde{R}_1^T \mathbf{u}_1^{n+1} + \tilde{R}_2^T \mathbf{u}_2^{n+1}$.

3 The Schur Complement as Optimal Choice for $\tilde{A}_j$

We show now algebraically what the best choice of $\tilde{A}_j$ is: we partition A from (2) into two blocks with a common interface,

$$A\mathbf{u} = \begin{bmatrix} A_{1i} & C_1 & \\ B_2 & A_\Gamma & B_1 \\ & C_2 & A_{2i} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i} \\ \mathbf{u}_\Gamma \\ \mathbf{u}_{2i} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma \\ \mathbf{f}_{2i} \end{bmatrix},$$

where $\mathbf{u}_{1i}$ and $\mathbf{u}_{2i}$ correspond to the interior unknowns and $\mathbf{u}_\Gamma$ corresponds to the interface unknowns. The classical Schwarz subdomain matrices are in this case

$$A_1 = \begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_\Gamma & B_1 \\ C_2 & A_{2i} \end{bmatrix},$$

and the subdomain solution vectors and the right hand side vectors are

$$\mathbf{u}_1 = \begin{bmatrix} \mathbf{u}_{1i} \\ \mathbf{u}_\Gamma \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} \mathbf{u}_\Gamma \\ \mathbf{u}_{2i} \end{bmatrix}, \quad \mathbf{f}_1 = \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma \end{bmatrix}, \ \mathbf{f}_2 = \begin{bmatrix} \mathbf{f}_\Gamma \\ \mathbf{f}_{2i} \end{bmatrix}.$$

The classical Schwarz iteration (6) would thus be

$$\begin{aligned} \begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i}^{n+1} \\ \mathbf{u}_{1\Gamma}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma - B_1 \mathbf{u}_{2i}^n \end{bmatrix}, \\ \begin{bmatrix} A_\Gamma & B_1 \\ C_2 & A_{2i} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{2\Gamma}^{n+1} \\ \mathbf{u}_{2i}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_\Gamma - B_2 \mathbf{u}_{1i}^n \\ \mathbf{f}_{2i} \end{bmatrix}. \end{aligned} \tag{18}$$

Using a Schur complement to eliminate the unknowns $\mathbf{u}_{2i}$ on the first subdomain at the fixed point, we obtain

$$\begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma - B_1 A_{2i}^{-1} C_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i} \\ \mathbf{u}_{1\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma - B_1 A_{2i}^{-1} \mathbf{f}_{2i} \end{bmatrix},$$

and $\mathbf{f}_{2i}$ can be expressed again using the unknowns of subdomain 2,

$$\mathbf{f}_{2i} = C_2 \mathbf{u}_{2\Gamma} + A_{2i} \mathbf{u}_{2i}.$$

Doing the same on the other subdomain, we obtain the new Schwarz method

$$\begin{aligned} \begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma - B_1 A_{2i}^{-1} C_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i}^{n+1} \\ \mathbf{u}_{1\Gamma}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma - B_1 \mathbf{u}_{2i}^n - B_1 A_{2i}^{-1} C_2 \mathbf{u}_{2\Gamma}^n \end{bmatrix}, \\ \begin{bmatrix} A_\Gamma - B_2 A_{1i}^{-1} C_1 & B_1 \\ C_2 & A_{2i} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{2\Gamma}^{n+1} \\ \mathbf{u}_{2i}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_\Gamma - B_2 \mathbf{u}_{1i}^n - B_2 A_{1i}^{-1} C_1 \mathbf{u}_{1\Gamma}^n \\ \mathbf{f}_{2i} \end{bmatrix}. \end{aligned} \tag{19}$$

This method converges in two steps, since after one solve, the right hand side in both subdomains is the right hand side of the Schur complement system, which is then solved in the next step. The optimal choice for the new subdomain matrices $\tilde{A}_j$, $j = 1, 2$, is therefore to subtract in A_1 from the last diagonal block the Schur complement $B_1 A_{2i}^{-1} C_2$, and from the first diagonal block in A_2 the Schur complement $B_2 A_{1i}^{-1} C_1$. Since these Schur complements are dense, using them significantly increases the cost per iteration. Any approximation of these Schur complements with the same sparsity structure as A_Γ however leads to an optimized Schwarz method with identical cost to the classical Schwarz method (18) per iteration. Approximation of the Schur complement at the algebraic level was extensively studied in [RMSS02]. We show in the next section an approximation based on the PDE which is discretized.

4 Numerical Results

As test problems, we use finite difference and spectral element discretizations of the modified Helmholtz problem in two spatial dimensions with appropriate boundary conditions,

$$\mathcal{L}u = (\eta - \Delta)u = f, \quad \text{in } \Omega. \tag{20}$$

Discretization of (20) using a standard five point finite difference stencil on an equidistant grid on the domain $\Omega = (0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions leads to the matrix problem

$$A^{FD}\mathbf{u} = \mathbf{f}, \quad A^{FD} = \frac{1}{h^2}\begin{bmatrix} T_\eta & -I & \\ -I & T_\eta & \ddots \\ & \ddots & \ddots \end{bmatrix}, \quad T_\eta = \begin{bmatrix} \eta h^2 + 4 & -1 & \\ -1 & \eta h^2 + 4 & \ddots \\ & \ddots & \ddots \end{bmatrix}.$$

The subdomain matrices A_j, $j = 1, 2$ of a classical Schwarz method are of the same form as A^{FD}, just smaller. To obtain the optimized subdomain matrices $\tilde{A}_j$, it suffices according to Section 3 to replace the last diagonal block T_η in A_1 and the first one in A_2 by an approximation of the Schur complements. Based on the discretized PDE, we use here the matrix [Gan03]

$$\tilde{T} = \frac{1}{2}T_\eta + phI + \frac{q}{h}(T_0 - 2I), \quad T_0 := T_\eta|_{\eta=0}, \tag{21}$$

which corresponds to a general optimized transmission condition of order 2 with the two parameters p and q. The optimal choice of the parameters p and q in the new block $\tilde{T}$ depends on the problem parameter η, the overlap in the method, the mesh parameter h and the lowest frequency along the interface, $k_{\min}$. Using the results in [Gan03], one can derive the hierarchy of choices in Table 1 for h small.

Figure 1 illustrates the effect of replacing the interface blocks on the performance of the RAS iteration for the model problem on the unit square with $\eta = 1$ and $h = 1/30$. The asymptotic formulas from [Gan03] were employed for the various choices of the parameters in (21). Clearly, the convergence

	p	q
T0	$\sqrt{\eta}$	0
T2	$\sqrt{\eta}$	$\frac{1}{2\sqrt{\eta}}$
O0, no overlap	$\sqrt{\pi}(k_{\min}^2+\eta)^{1/4}h^{-1/2}$	0
O0, overlap Ch	$2^{-1/3}(k_{\min}^2+\eta)^{1/3}(Ch)^{-1/3}$	0
O2, no overlap	$2^{-1/2}\pi^{1/4}(k_{\min}^2+\eta)^{3/8}h^{-1/4}$	$2^{-1/2}\pi^{-3/4}(k_{\min}^2+\eta)^{-1/8}h^{3/4}$
O2, overlap Ch	$2^{-3/5}(k_{\min}^2+\eta)^{2/5}(Ch)^{-1/5}$	$2^{-1/5}(k_{\min}^2+\eta)^{-1/5}(Ch)^{3/5}$

Table 1. Choices for the parameters p and q in the new interface blocks $\tilde{T}$ in (21). Tj stands for Taylor of order j, and Oj stands for optimized of order j.

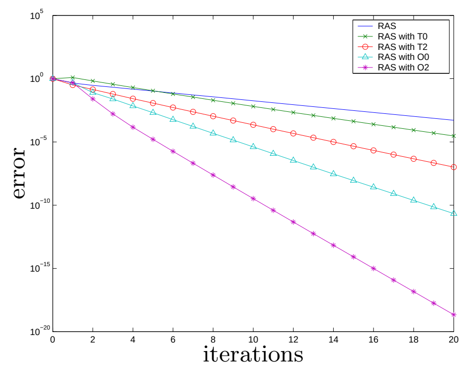

Fig. 1. Convergence curves of classical RAS, compared to the hierarchy of optimized RAS methods: Taylor optimized zero-th order (T0) and second order (T2), and RAS optimized zero-th (O0) and second order (O2).

of RAS is greatly accelerated and the number of operations per iteration is identical.

In a nodal spectral element discretization, the computational domain Ω is partitioned into K elements Ω_k in which u is expanded in terms of the N–th degree Lagrangian interpolants h_i defined in Ronquist [Ron88]. A weak variational problem is obtained by integrating the equation with respect to test functions and directly evaluating inner products using Gaussian quadrature.

The model problem (20) is discretized on the domain $\Omega = (0,2)\times(0,4)$ with periodic boundary conditions and 32 spectral elements. The right hand side is constructed to be C^0 along element boundaries as displayed in Figure 2. Non-overlapping Schwarz methods are well-suited to spectral element discretizations. Here, a zero-th order optimized transmission condition is employed in AS applied to the augmented system. The resulting optimized Schwarz iteration is accelerated by a generalized minimal residual (GMRES) Krylov method [SS86]. Figure 2 also contains a plot of the residual error versus the number of GMRES iterations for diagonal (the inverse mass matrix) and optimized Schwarz preconditioning.

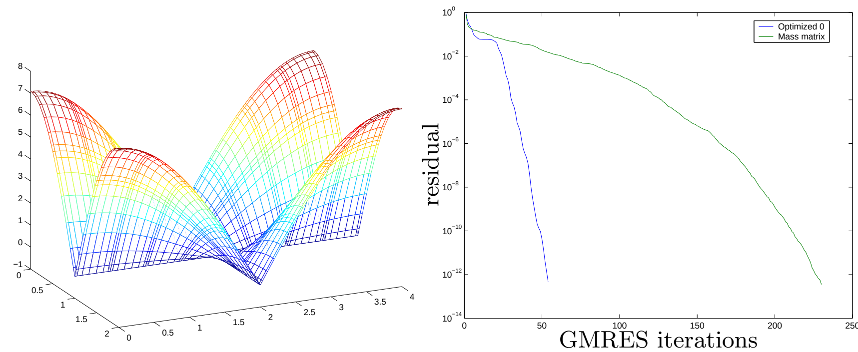

Fig. 2. Left panel: Right hand side of modified Helmholtz problem. Right panel: Residual error versus GMRES iterations.

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