Lower Bounds in Domain Decomposition

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1 Introduction

An important indicator of the efficiency of a domain decomposition preconditioner is the condition number of the preconditioned system. Upper bounds for the condition numbers of the preconditioned systems have been the focus of most analyses in domain decomposition [21, 20, 23]. However, in order to have a fair comparison of two preconditioners, the sharpness of the respective upper bounds must first be established, which means that we need to derive lower bounds for the condition numbers of the preconditioned systems.

In this paper we survey lower bound results for domain decomposition preconditioners [7, 3, 8, 5, 22] that can be obtained within the framework of additive Schwarz preconditioners. We will describe the results in terms of the following model problem.

Find $u_h \in V_h$ such that

$$\int_{\Omega} \nabla u_h \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \qquad \forall\, v \in V_h, \tag{1}$$

where $\Omega = [0,1]^2$, $f \in L_2(\Omega)$, and V_h is the P_1 Lagrange finite element space associated with a uniform triangulation $\mathcal{T}_h$ of Ω. We assume that the length of the horizontal (or vertical) edges of $\mathcal{T}_h$ is a dyadic number $h = 2^{-k}$.

We recall the basic facts concerning additive Schwarz preconditioners in Section 2 and present the lower bound results for one-level and two-level additive Schwarz preconditioners, Bramble-Pasciak-Schatz preconditioner and the FETI-DP preconditioner in Sections 3–6. Section 7 contains some concluding remarks.

2 Additive Schwarz Preconditioners

Let V be a finite dimensional vector space and $A : V \longrightarrow V'$ be an SPD operator, i.e., $\langle Av_1, v_2\rangle = \langle Av_2, v_1\rangle$ $\forall\, v_1, v_2 \in V$ and $\langle Av, v\rangle > 0$ for any

$v \in V \setminus \{0\}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form between a vector space and its dual.

The ingredients for an additive Schwarz preconditioner B for A are (i) auxiliary finite dimensional vector spaces V_j for $1 \leq j \leq J$, (ii) SPD operators $A_j : V_j \longrightarrow V_j'$ and (iii) connection operators $I_j : V_j \longrightarrow V$. The preconditioner $B : V' \longrightarrow V$ is then given by

$$B = \sum_{j=1}^{J} I_j A_j^{-1} I_j^t,$$

where $I_j^t : V' \longrightarrow V_j'$ is the transpose of I_j, i.e. $\langle I_j^t \phi, v \rangle = \langle \phi, I_j v \rangle \; \forall \, \phi \in V'$ and $v \in V_j$.

Under the condition $V = \sum_{j=1}^{J} I_j V_j$, the operator B is SPD and the maximum and minimum eigenvalues of $BA : V \longrightarrow V$ are characterized by the following formulas [26, 1, 25, 14, 21, 8, 23]:

$$\lambda_{\max}(BA) = \max_{v \in V \setminus \{0\}} \frac{\langle Av, v \rangle}{\displaystyle\min_{\substack{v = \sum_{j=1}^{J} I_j v_j \\ v_j \in V_j}} \sum_{j=1}^{J} \langle A_j v_j, v_j \rangle}, \tag{2}$$

$$\lambda_{\min}(BA) = \min_{v \in V \setminus \{0\}} \frac{\langle Av, v \rangle}{\displaystyle\min_{\substack{v = \sum_{j=1}^{J} I_j v_j \\ v_j \in V_j}} \sum_{j=1}^{J} \langle A_j v_j, v_j \rangle}. \tag{3}$$

3 One-Level Additive Schwarz Preconditioner

Let $A_h : V_h \to V_h'$ be defined by

$$\langle A_h v_1, v_2 \rangle = \int_{\Omega} \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall \, v_1, v_2 \in V_h.$$

We can precondition the operator A_h using subdomain solves from an overlapping decomposition, which is created by (i) dividing Ω into $J = H^{-2}$ nonoverlapping squares (H is a dyadic number $\gg h$) and (ii) enlarging the nonoverlapping subdomains by an amount of δ ($\leq H$) so that each of the overlapping subdomains $\Omega_1, \ldots, \Omega_J$ is the union of triangles from $\mathcal{T}_h$ (cf. Figure 1). We take the auxiliary space $V_j \subset H_0^1(\Omega)$ to be the finite element space associated with the triangulation of Ω_j by triangles from $\mathcal{T}_h$, and define the SPD operator $A_j : V_j \longrightarrow V_j'$ by

$$\langle A_j v_1, v_2 \rangle = \int_{\Omega_j} \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall \, v_1, v_2 \in V_j.$$

The space V_j is connected to V_h by the trivial extension map I_j and the one-level additive Schwarz preconditioner [19] B_{OL} for A_h is defined by

$$B_{OL} = \sum_{j=1}^{J} I_j A_j^{-1} I_j^t. \tag{4}$$

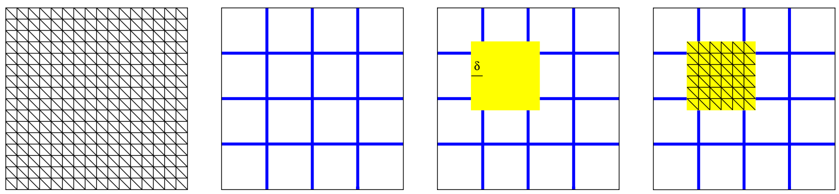

Fig. 1. An overlapping domain decomposition

It is well-known that the preconditioner B_{OL} does not scale. Here we give a lower bound for the condition number $\kappa(B_{OL}A_h)$ that explains this phenomenon. We use the notation $A \lesssim B$ ($B \gtrsim A$) to represent the inequality $A \leq (\text{constant})B$, where the positive constant is independent of h, J, δ and H. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

Theorem 1. *Under the condition $\delta \approx H$, it holds that*

$$\kappa(B_{OL}A_h) = \lambda_{\max}(B_{OL}A_h)/\lambda_{\min}(B_{OL}A_h) \gtrsim J. \tag{5}$$

Proof. Since the connection maps I_j preserve the energy norm (in other words, $\langle A_h I_j v, I_j v\rangle = \langle A_j v, v\rangle \ \forall\, v \in V_j$), it follows immediately from (2) that

$$\lambda_{\max}(B_{OL}A_h) \geq 1. \tag{6}$$

Let $v_* \in H_0^1(\Omega)$ be the piecewise linear function with respect to the triangulation of Ω of mesh size $1/4$ such that v_* equals 1 on the four central squares (cf. the first figure in Figure 2). Since v_* is independent of h, we have

$$\langle A_h v, v\rangle = |v_*|^2_{H^1(\Omega)} \approx 1 \tag{7}$$

as $h \downarrow 0$. We will show that, for this function $v_* \in V_h$, the estimate

$$\sum_{j=1}^{J} \langle A_j v_j, v_j\rangle \gtrsim J\langle A_h v_*, v_*\rangle \tag{8}$$

holds whenever

$$v_* = \sum_{j=1}^{J} I_j v_j \qquad \text{and} \qquad v_j \in V_j \quad \text{for} \quad 1 \leq j \leq J. \tag{9}$$

It follows immediately from (3), (8) and (9) that

$$\lambda_{\min}(B_{OL}A_h) \lesssim 1/J, \tag{10}$$

which together with (6) implies (5).

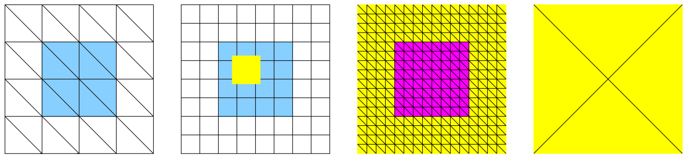

Fig. 2. Subdomains for Theorem 1

In order to derive (8), we first focus on a single subdomain Ω_j that overlaps with the square where v_* is identically 1 (cf. the second figure in Figure 2), and without loss of generality, assume that $\delta = H/4$. Condition (9) then implies $v_j = 1$ in the central area of Ω_j (cf. the third figure of Figure 2).

We can construct a weak interpolation operator Π from $H^1(\Omega_j)$ into the space of functions that are piecewise linear with respect to the triangulation of Ω_j by its two diagonals (cf. the fourth figure of Figure 2). For $v \in H^1(\Omega)$, we define the value of Πv at the four corners of Ω_j to be the mean of v on $\partial\Omega$ and the value of Πv at the center of Ω_j to be the mean of v on the central area of Ω_j. It follows that Πv_j equals 1 at the center of Ω_J and vanishes identically on $\partial\Omega_j$. A simple calculation shows that $|\Pi v_j|^2_{H^1(\Omega_j)} \approx 1$. On the other hand, the weak interpolation operator satisfies the estimate $|\Pi v_j|_{H^1(\Omega)} \lesssim |v_j|_{H^1(\Omega)}$. We conclude that

$$\langle A_j v_j, v_j\rangle = |v_j|^2_{H^1(\Omega_j)} \gtrsim 1. \tag{11}$$

Since there are $J/4$ such subdomains, the estimate (8) follows from (7) and (11).

Remark 1. The estimate (5) implies that, for a given tolerance, the number of iterations for the preconditioned conjugate gradient method grows at the rate of $O(\sqrt{J}) = O(1/H)$, a phenomenon that has been observed numerically [21].

4 Two-Level Additive Schwarz Preconditioner

To obtain scalability for the additive Schwarz overlapping domain decomposition preconditioner, Dryja and Widlund [10] developed a two-level additive Schwarz preconditioner by introducing a coarse space.

Let $\mathcal{T}_H$ be a coarse triangulation of Ω obtained by adding diagonals to the underlying nonoverlapping squares whose sides are of length H (cf. the second figure in Figure 1) and $V_H \subset H_0^1(\Omega)$ be the corresponding P_1 finite element space. The coarse space V_H is connected to V_h by the natural injection I_H, and $A_H : V_H \longrightarrow V_H'$ is defined by

$$\langle A_H v_1, v_2\rangle = \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \qquad \forall\, v_1, v_2 \in V_H.$$

The two-level preconditioner $B_{TL} : V_h' \longrightarrow V_h$ is then given by

$$B_{TL} = I_H A_H^{-1} I_H^t + B_{OL} = I_H A_H^{-1} I_H^t + \sum_{j=1}^J I_j A_j^{-1} I_j^t. \tag{12}$$

It follows from the well-known estimate [11]

$$\kappa(B_{TL} A_h) \lesssim 1 + \frac{H}{\delta} \tag{13}$$

that B_{TL} is an optimal preconditioner when $\delta \approx H$ (the case of generous overlap). However, in the case of small overlap where $\delta \ll H$, the number $1 + (H/\delta)$ becomes significant and it is natural to ask whether the estimate (13) can be improved. That the estimate (13) is sharp is established by the following lower bound result [3].

Theorem 2. *In the case of minimal overlap where $\delta = h$, it holds that*

$$\kappa(B_{TL} A_h) \gtrsim \frac{H}{h}. \tag{14}$$

We will sketch the derivation of (14) in the remaining part of this section and refer to [3] for the details.

First observe that, by comparing (4) and (12), the estimate

$$\lambda_{\max}(B_{TL} A_h) \geq \lambda_{\max}(B_{OL} A_h) \geq 1 \tag{15}$$

follows immediately from (2) and (6).

In the other direction, it suffices to construct a finite element function $v_* \in V_h$ such that, for any decomposition $v_* = I_H v_H + \sum_{j=j}^J I_j v_j$ where $v_H \in V_H$ and $v_j \in V_j$,

$$\frac{H}{h} \langle A_h v_*, v_*\rangle \lesssim \langle A_H v_H, v_H\rangle + \sum_{j=1}^J \langle A_j v_j, v_j\rangle. \tag{16}$$

The estimate $\lambda_{\min}(B_{TL} A_h) \lesssim h/H$ then follows from (3) and (16), and together with (15) it implies (14).

Since the subdomains are almost nonoverlapping when $\delta = h$, we can construct v_* using techniques from nonoverlapping domain decomposition. Let $\hat{\Omega}_j$ $(1 \leq j \leq J)$ be the underlying nonoverlapping decomposition of Ω

(cf. the second figure in Figure 1) from which we construct the overlapping decomposition, and $\Gamma = \bigcup_{j=1}^{J} \partial\hat{\Omega}_j \setminus \partial\Omega$ be the interface of $\hat{\Omega}_1, \ldots, \hat{\Omega}_J$. The space $V_h(\Gamma)$ of discrete harmonic functions is defined by

$$V_h(\Gamma) = \{v \in V_h : \int_\Omega \nabla v \cdot \nabla w \, dx = 0 \quad \forall\, w \in V_h, w\big|_\Gamma = 0\}.$$

We will choose v_* from $V_h(\Gamma)$. Note that a discrete harmonic function is uniquely determined by its restriction on Γ.

Let E be an edge of length H shared by two nonoverlapping subdomains $\hat{\Omega}_1$ and $\hat{\Omega}_2$. Let g be a function defined on E such that (i) g is piecewise linear with respect to the uniform subdivision of E of mesh size $H/8$, (ii) g is identically zero within a distance of $H/4$ from either one of the endpoints of E, (iii) g is $L_2(E)$-orthogonal to all polynomials on E of degree ≤ 1. (It is easy to see that such a function g exists by a dimension argument.) We then define $v_* \in V_h(\Gamma)$ to be g on E and 0 on $\Gamma \setminus E$.

It follows from property (ii) of g and standard properties of discrete harmonic functions [2, 6, 23] that

$$\begin{aligned}\langle A_h v_*, v_*\rangle = |v_*|^2_{H^1(\Omega)} &\approx \sum_{j=1}^{2} |v_*|^2_{H^{1/2}(\partial\hat{\Omega}_j)} \\ &\approx |g|^2_{H^{1/2}(E)} \approx \frac{1}{H}\|g\|^2_{L_2(E)} = \frac{1}{H}\|v_*\|^2_{L_2(E)}. \end{aligned} \tag{17}$$

Suppose $v_* = I_H v_H + \sum_{j=1}^{J} I_j v_j$ where $v_H \in V_H$ and $v_j \in V_j$ for $1 \leq j \leq J$. Let E_c be the set of points in E whose distance from the endpoints of E exceed $H/4$. Since $v_H\big|_E$ is a polynomial of degree ≤ 1, property (iii) of g implies that

$$\|v_*\|^2_{L_2(E_c)} \leq \|v_* - v_0\|^2_{L_2(E_c)} = \|\sum_{j=1}^{J} v_j\|^2_{L_2(E_c)} = \|v_1 + v_2\|^2_{L_2(E_c)}, \tag{18}$$

where we have also used the fact that $v_j = 0$ on E_c for $j \neq 1, 2$ because $\delta = h$.

Finally, since v_1 (resp. v_2) vanishes on $\partial\Omega_1$ (resp. $\partial\Omega_2$) which is within one layer of elements from E, a simple calculation shows that

$$\|v_j\|^2_{L_2(E_c)} \lesssim h|v_j|^2_{H^1(\Omega_j)} = h\langle A_j v_j, v_j\rangle \quad \text{for} \quad j = 1, 2. \tag{19}$$

The estimate (16) follows from (17)–(19).

Remark 2. Theorem 2 also holds for nonconforming finite elements [7] and mortar elements [22]. It can also be extended to fourth order problems [8, 7] in which case the right-hand side of (14) becomes $(H/h)^3$.

5 Bramble-Pasciak-Schatz Preconditioner

Let Γ be the interface of a nonoverlapping decomposition of Ω and $V_h(\Gamma)$ be the space of discrete harmonic functions as described in Section 4. By a parallel subdomain solve, we can reduce (1) to the following problem.

Find $\bar{u}_h \in V_h(\Gamma)$ such that

$$\langle S_h \bar{u}_h, v\rangle = \int_\Omega f v\, dx \qquad \forall\, v \in V_h(\Gamma),$$

and the Schur complement operator $S_h : V_h(\Gamma) \longrightarrow V_h(\Gamma)'$, defined by

$$\langle S_h v_1, v_2\rangle = \int_\Omega \nabla v_1 \cdot \nabla v_2\, dx \qquad \forall\, v_1, v_2 \in V_h(\Gamma),$$

is the operator that needs a preconditioner.

The auxiliary spaces for the Bramble-Pasciak-Schatz preconditioner [2] are the coarse space V_H introduced in Section 4, and the edge spaces $V_\ell = \{v \in V_h(\Gamma) :\ v = 0 \text{ on } \Gamma \setminus E_\ell\}$ associated with the edges E_ℓ of the interface Γ. The space V_H is equipped with the SPD operator A_H introduced in Section 4, and is connected to $V_h(\Gamma)$ by the map I_H that maps $v \in V_H$ to the discrete harmonic function that agrees with v on Γ. The edge space V_ℓ is connected to $V_h(\Gamma)$ by the natural injection I_j, and is equipped with the Schur complement operator $S_\ell : V_\ell \longrightarrow V_\ell'$ defined by

$$\langle S_\ell v_1, v_2\rangle = \int_\Omega \nabla v_1 \cdot \nabla v_2\, dx \qquad \forall\, v_1, v_2 \in V_\ell.$$

The preconditioner $B_{BPS} : V_h(\Gamma)' \longrightarrow V_h(\Gamma)$ is then given by

$$B_{BPS} = I_H A_H^{-1} I_H + \sum_{\ell=1}^{L} I_\ell S_\ell^{-1} I_\ell^t.$$

The sharpness of the well-known estimate [2]

$$\kappa(B_{BPS} S_h) \lesssim \left(1 + \ln \frac{H}{h}\right)^2 \tag{20}$$

follows from the following lower bound result [8].

Theorem 3. *It holds that*

$$\kappa(B_{BPS} S_h) \gtrsim \left(1 + \ln \frac{H}{h}\right)^2. \tag{21}$$

Since the natural injection I_ℓ preserves the energy norm, it follows immediately from (2) that

$$\lambda_{\max}(B_{BPS} S_h) \geq 1. \tag{22}$$

To complete the proof of (21), it suffices to construct $v_* \in V_h(\Gamma)$ such that, for the unique decomposition $v_* = I_H v_H + \sum_{\ell=1}^{L} v_\ell$ where $v_H \in V_H$ and $v_\ell \in V_\ell$,

$$\langle A_H v_H, v_H \rangle + \sum_{\ell=1}^{L} \langle S_\ell v_\ell, v_\ell \rangle \gtrsim \Big(1 + \ln \frac{H}{h}\Big)^2 \langle S_h v_*, v_* \rangle, \tag{23}$$

which together with (3) implies that $\lambda_{\min}(B_{BPS} S_h) \lesssim \Big(1 + \ln \frac{H}{h}\Big)^{-2}$ and thus, in view of (22), completes the proof of (21). Below we will sketch the construction of v_* and refer to [8] for the details.

Since the derivation of (20) depends crucially on the discrete Sobolev inequality [2, 6, 23], v_* is intimately related to piecewise linear functions on an interval with special property with respect to the Sobolev norm of order $\frac{1}{2}$. Let $I = (0, 1)$. A key observation in this direction is that

$$|v|^2_{H^{1/2}_{00}(I)} \approx \sum_{n=1}^{\infty} n |v_n|^2 \qquad \forall\, v \in H^{1/2}_{00}(I), \tag{24}$$

where $\sum_{n=1}^{\infty} v_n \sin(n\pi x)$ is the Fourier sine-series expansion of v.

Let $\mathcal{T}_\rho$ ($\rho = 2^{-k}$) be a uniform dyadic subdivision of I and $\mathcal{L}_\rho \subset H^1_0(I)$ be the space of piecewise linear functions on I (with respect to $\mathcal{T}_\rho$) that vanish at 0 and 1. The special piecewise linear functions that we need come from the functions S_N ($N = 2^k = \rho^{-1}$) defined by

$$S_N(x) = \sum_{n=1}^{N} \Big(\frac{1}{4n-3}\Big) \sin\big((4n-3)\pi x\big). \tag{25}$$

From (24) and (25) we find

$$|S_N|^2_{H^{1/2}_{00}(I)} \approx \ln N \approx |\ln \rho|, \tag{26}$$

and a direct calculation shows that

$$|S_N|^2_{H^1(I)} \approx 1. \tag{27}$$

Now we define $\sigma_\rho \in \mathcal{L}_\rho$ to be the nodal interpolant of S_N. It follows from (26), (27) and an interpolation error estimate that

$$|\sigma_\rho|^2_{H^{1/2}_{00}(I)} \approx |\ln \rho|. \tag{28}$$

Remark 3. Since $\|\sigma_\rho\|_{L_\infty(I)} = \sigma_\rho(1/2) = S_N(1/2) \approx \ln N = |\ln \rho|$, the estimate (28) implies the sharpness of the discrete Sobolev inequality.

Let σ^{I}_ρ be the piecewise linear interpolant of S_N with respect to the coarse subdivision $\{0, 1/2, 1\}$ of I. Then a calculation using (24) yields

$$|\sigma_\rho - \sigma_\rho^I|^2_{H^{1/2}_{00}(0,1/2)} = |\sigma_\rho - \sigma_\rho^I|^2_{H^{1/2}_{00}(1/2,1)} \approx |\ln\rho|^3. \tag{29}$$

Finally we take $\rho = h/2H$ and $g(x) = \sigma_\rho\big((x+H)/2H\big)$. Then g is a continuous piecewise linear function on $[-H, H]$ with respect to the uniform partition of mesh size h. Note that S_N is symmetric with respect to the midpoint $1/2$ and hence g is symmetric with respect to 0. We can now define $v_* \in V_h(\Gamma)$ as follows: (i) v_* vanishes on Γ except on the two line segments P_1P_2 and P_3P_4 (each of length $2H$) that form the interface of the four nonoverlapping subdomains $\Omega_1, \ldots, \Omega_4$ (cf. the first figure in Figure 3), and (ii) $v_* = g$ on P_1P_2 and P_3P_4.

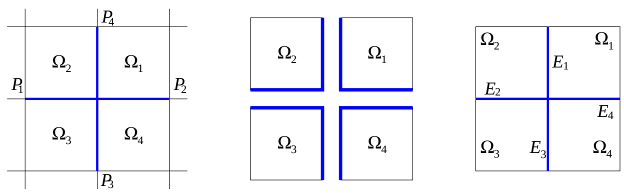

Fig. 3. The four subdomains associated with v_*

It is clear that $v_* = 0$ outside the four subdomains and, by the symmetry of g, $v_* = g$ on one half of $\partial\Omega_j$ (represented by the thick lines in the second figure in Figure 3) and vanishes at the other half, for $1 \le j \le 4$. Therefore, we have, from (28) and standard properties of discrete harmonic functions,

$$\begin{aligned}\langle S_h v_*, v_*\rangle &= \sum_{j=1}^{4} |v_*|^2_{H^1(\Omega_j)} \approx \sum_{j=1}^{4} |v_*|^2_{H^{1/2}(\partial\Omega_j)} \\ &\approx |g|^2_{H^{1/2}_{00}(-H,H)} = |\sigma_\rho|^2_{H^{1/2}_{00}(0,1)} \approx |\ln\rho| \approx \ln\frac{H}{h}.\end{aligned} \tag{30}$$

The function v_* admits a unique decomposition $v_* = I_H v_H + \sum_{\ell=1}^4 v_\ell$, where $v_H \in V_H$, $v_\ell \in V(E_\ell)$ and E_ℓ $(1 \le j \le 4)$ are the interfaces of $\Omega_1, \ldots, \Omega_4$ (cf. the third figure in Figure 3). On each E_ℓ, $v_\ell = v - I_H v_H$ agrees with $g - g^I$, where g^I is the linear polynomial that agrees with g at the two endpoints of E_ℓ. Therefore it follows from (29) that

$$\langle S_\ell v_\ell, v_\ell\rangle \approx |\ln\rho|^3 \approx \Big(\ln\frac{H}{h}\Big)^3 \qquad \text{for} \quad 1 \le \ell \le 4, \tag{31}$$

and the estimate (23) follows from (30) and (31).

6 FETI-DP Preconditioner

Let $\Omega_1, \ldots, \Omega_J$ be a nonoverlapping decomposition of Ω aligned with $\mathcal{T}_h$ (cf. the first two figures in Figure 4) and $\tilde V_h = \{v \in L_2(\Omega) : v$ is a standard P_1

finite element function on each subdomain, v is not required to be continuous on the interface Γ except at the cross points and $v = 0$ on $\partial\Omega\}$. In the Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) approach [13], the problem (1) is rewritten as

$$\begin{aligned} \sum_{j=1}^{J} \int_{\Omega_j} \nabla u_h \cdot \nabla v \, dx + \langle \phi, v \rangle &= \int_{\Omega} f v \, dx \qquad \forall \, v \in \tilde{V}_h, \\ \langle \mu, u_h \rangle \qquad\qquad\qquad &= 0 \qquad\qquad\quad \forall \, \mu \in M_h, \end{aligned} \tag{32}$$

where $M_h \subset \tilde{V}_h'$ is the space of Lagrange multipliers that enforce the continuity of v along the interface Γ. More precisely, for each node p on Γ that is not a cross point, we have a multiplier $\mu_p \in \tilde{V}_h'$ defined by $\langle \mu_p, v \rangle = (v|_{\Omega_J})(p) - (v|_{\Omega_k})(p)$, where Ω_j and Ω_k are the two subdomains whose interface contains p, and the space M_h is spanned by all such μ_p's.

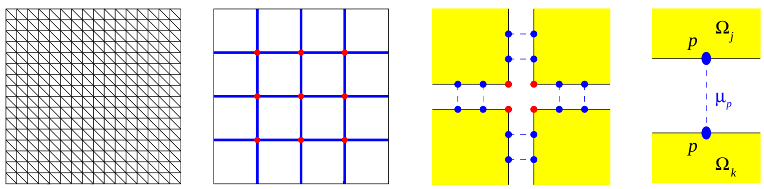

Fig. 4. FETI

By solving local SPD problems (associated with the subdomains) and a global SPD problem (associated with the cross points), the unknown u_h can be eliminated from (32), and the resulting system for ϕ involved the operator $\hat{\mathbb{S}}_h : M_h \longrightarrow M_h'$ defined by $\hat{\mathbb{S}}_h = R^t \tilde{S}_h^{-1} R$, where $R : M_h \longrightarrow [\tilde{V}_h(\Gamma)]'$ is the restriction map, $\tilde{V}_h(\Gamma)$ is the subspace of $\tilde{V}_h$ consisting of discrete harmonic functions, and $\tilde{S}_h : \tilde{V}_h(\Gamma) \longrightarrow \tilde{V}_h(\Gamma)'$ is the corresponding Schur complement operator.

Let V_j ($1 \le j \le J$) be the space of discrete harmonic functions on Ω_j that vanish at the corners of Ω_j and $S_j : V_j \longrightarrow V_j'$ be the Schur complement operator (which is SPD). The dual spaces V_j' are the auxiliary spaces of the additive Schwarz preconditioner for $\hat{\mathbb{S}}_h$ developed in [18]. Each V_j' is connected to M_h by the operator I_j defined by $\langle I_j \psi, \tilde{v} \rangle = \frac{1}{2} \langle \psi, v \rangle \; \forall \, v \in V_j$, where $\tilde{v} \in \tilde{V}_h$ is the trivial extension of v. The preconditioner of Mandel and Tezaur is given by

$$B_{DP} = \sum_{j=1}^{J} I_j S_j I_j^t,$$

and the condition number estimate

$$\kappa(B_{DP} \hat{\mathbb{S}}_h) \lesssim \Big(1 + \ln \frac{H}{h} \Big)^2 \tag{33}$$

was established in [18]. The sharpness of (33) is a consequence of the following lower bound result [4].

Theorem 4. *It holds that*

$$\kappa(B_{DP}\hat{\mathbb{S}}_h) \gtrsim \Big(1+\ln\frac{H}{h}\Big)^2.$$

Since the operator $B_{DP}\hat{\mathbb{S}}_h$ is essentially dual to the operator $B_{BPS}S_h$, Theorem 4 is derived using the special piecewise linear functions from Section 5 and duality arguments. Details can be found in [4].

7 Concluding Remarks

We present two dimensional results in this paper for simplicity. But the generalization of the results of Sections 3 and 4 to three dimensions is straightforward, and the results in Section 5 have been generalized [5] to three dimensions (wire-basket algorithm [9]) and Neumann-Neumann algorithms [12]. Since the balancing domain decomposition by constraint (BDDC) method has the same condition number as the FETI-DP method [17, 15], the sharpness of the condition number estimate for BDDC [16] also follows from Theorem 4.

We would also like to mention that the special discrete harmonic function v_* constructed in Section 5 has been used in the derivation of an upper bound for the three-level BDDC method [24].

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