# On the maximum of two log-correlated fields: the logarithms of the characteristic polynomial of the Circular Beta Ensemble and the Riemann zeta function

Joseph Najnudel
Joint work with Reda Chhaibi and Thomas Madaule (for the Circular Beta Ensemble)

Extreme values in Number Theory and Probability

June 2019

# The Circular Beta Ensemble

- We first consider the Circular Beta Ensemble (C$\beta$E), corresponding to $n$ points on the unit circle $\mathbb{U}$, whose probability density with respect to the uniform measure on $\mathbb{U}^n$ is given by

$$C_{n,\beta} \prod_{1 \leq j,k \leq n} |\lambda_j - \lambda_k|^{\beta},$$

for some $\beta > 0$.
- For $\beta = 2$, one gets the distribution of the eigenvalues of a Haar-distributed matrix on the unitary group $U(n)$. Other matrix models has been found by Killip and Nenciu in 2004 for general $\beta$.

# The Circular Beta Ensemble

- We first consider the Circular Beta Ensemble (CβE), corresponding to $n$ points on the unit circle $\mathbb{U}$, whose probability density with respect to the uniform measure on $\mathbb{U}^n$ is given by

$$C_{n,\beta} \prod_{1 \leq j,k \leq n} |\lambda_j - \lambda_k|^{\beta},$$

for some $\beta > 0$.

- For $\beta = 2$, one gets the distribution of the eigenvalues of a Haar-distributed matrix on the unitary group $U(n)$. Other matrix models has been found by Killip and Nenciu in 2004 for general $\beta$.

- If $(\lambda_j^{-1})_{1 \leq j \leq n}$ are the eigenvalues of a random matrix, one can consider the characteristic polynomial:

$$X_n(z) = \prod_{j=1}^{n} (1 - \lambda_j z),$$

and its logarithm

$$\log X_n(z) = \sum_{j=1}^{n} \log(1 - \lambda_j z),$$

which can be well-defined in a continuous way, except on the half-lines $\lambda_j^{-1}[1, \infty)$.

- We will be interested in the extremal values of $\log X_n(z)$ on the unit circle.

- If $(\lambda_j^{-1})_{1 \leq j \leq n}$ are the eigenvalues of a random matrix, one can consider the characteristic polynomial:

$$X_n(z) = \prod_{j=1}^{n} (1 - \lambda_j z),$$

  and its logarithm

$$\log X_n(z) = \sum_{j=1}^{n} \log(1 - \lambda_j z),$$

  which can be well-defined in a continuous way, except on the half-lines $\lambda_j^{-1}[1, \infty)$.
- We will be interested in the extremal values of $\log X_n(z)$ on the unit circle.

- It can be proven that $\left(\sqrt{\beta/2}\log X_n(z)\right)_{z\in\mathbb{D}}$ ($\mathbb{D}$ being the open unit disc) tends in distribution to a complex Gaussian holomorphic function: for $\beta = 2$, it is a direct consequence of a result by Diaconis and Shahshahani (1994) on the moments of the traces of the CUE.
- This Gaussian function $\mathbb{G}$ has the following covariance structure:

$$\mathbb{E}[\overline{\mathbb{G}(z)}\mathbb{G}(z')] = \log\left(\frac{1}{1-\overline{z}z'}\right).$$

- The variance of $\mathbb{G}$ goes to infinity when $|z| \to 1$, and for $z \in \mathbb{U}$, $\log X_n(z)$ does not converge in distribution.

- It can be proven that $\left(\sqrt{\beta/2}\log X_n(z)\right)_{z\in\mathbb{D}}$ ($\mathbb{D}$ being the open unit disc) tends in distribution to a complex Gaussian holomorphic function: for $\beta = 2$, it is a direct consequence of a result by Diaconis and Shahshahani (1994) on the moments of the traces of the CUE.
- This Gaussian function $\mathbb{G}$ has the following covariance structure:

$$\mathbb{E}[\overline{\mathbb{G}(z)}\mathbb{G}(z')] = \log\left(\frac{1}{1-\overline{z}z'}\right).$$

- The variance of $\mathbb{G}$ goes to infinity when $|z| \to 1$, and for $z \in \mathbb{U}$, $\log X_n(z)$ does not converge in distribution.

- It can be proven that $\left(\sqrt{\beta/2}\log X_n(z)\right)_{z\in\mathbb{D}}$ ($\mathbb{D}$ being the open unit disc) tends in distribution to a complex Gaussian holomorphic function: for $\beta = 2$, it is a direct consequence of a result by Diaconis and Shahshahani (1994) on the moments of the traces of the CUE.
- This Gaussian function $\mathbb{G}$ has the following covariance structure:

$$\mathbb{E}[\overline{\mathbb{G}(z)}\mathbb{G}(z')] = \log\left(\frac{1}{1-\bar{z}z'}\right).$$

- The variance of $\mathbb{G}$ goes to infinity when $|z| \to 1$, and for $z \in \mathbb{U}$, $\log X_n(z)$ does not converge in distribution.

- When $n$ goes to infinity,

$$\sqrt{\frac{\beta}{2\log n}} \log X_n(z) \underset{n\to\infty}{\longrightarrow} \mathcal{N}^{\mathbb{C}},$$

where $\mathcal{N}^{\mathbb{C}}$ denotes a complex Gaussian variable $Z$ such that

$$\mathbb{E}[Z] = \mathbb{E}[Z^2] = 0, \ \mathbb{E}[|Z|^2] = 1.$$

For $\beta = 2$, this result has been proven by Keating and Snaith (2000).

- Without normalization, $(\sqrt{\beta/2} \log X_n(z))_{z \in \mathbb{U}}$ tends in distribution to a complex Gaussian field on the unit circle, whose correlation between points $z, z' \in \mathbb{U}$ is given by $\log |z - z'|$. Note that this field is not defined on single points, since the correlation has a logarithmic singularity when $z'$ goes to $z$.

- When $n$ goes to infinity,

$$\sqrt{\frac{\beta}{2\log n}} \log X_n(z) \underset{n\to\infty}{\longrightarrow} \mathcal{N}^{\mathbb{C}},$$

  where $\mathcal{N}^{\mathbb{C}}$ denotes a complex Gaussian variable $Z$ such that

$$\mathbb{E}[Z] = \mathbb{E}[Z^2] = 0, \ \mathbb{E}[|Z|^2] = 1.$$

  For $\beta = 2$, this result has been proven by Keating and Snaith (2000).
- Without normalization, $(\sqrt{\beta/2}\log X_n(z))_{z\in\mathbb{U}}$ tends in distribution to a complex Gaussian field on the unit circle, whose correlation between points $z, z' \in \mathbb{U}$ is given by $\log|z - z'|$. Note that this field is not defined on single points, since the correlation has a logarithmic singularity when $z'$ goes to $z$.

- The logarithm of the characteristic polynomial, multiplied by $\sqrt{\beta/2}$, is a rather complex (yet integrable) regularization of the log-correlated Gaussian field given above.
- In this regularization, the correlation of the field saturates when $|z - z'|$ is of order $1/n$, which is consistent with the result by Keating and Snaith.
- For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log \log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.
- Such result has been proven for Gaussian regularizations (by Madaule in 2015), for branching random walks and branching Brownian motion.

- The logarithm of the characteristic polynomial, multiplied by $\sqrt{\beta/2}$, is a rather complex (yet integrable) regularization of the log-correlated Gaussian field given above.
- In this regularization, the correlation of the field saturates when $|z - z'|$ is of order $1/n$, which is consistent with the result by Keating and Snaith.
- For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log \log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.
- Such result has been proven for Gaussian regularizations (by Madaule in 2015), for branching random walks and branching Brownian motion.

- The logarithm of the characteristic polynomial, multiplied by $\sqrt{\beta/2}$, is a rather complex (yet integrable) regularization of the log-correlated Gaussian field given above.
- In this regularization, the correlation of the field saturates when $|z - z'|$ is of order $1/n$, which is consistent with the result by Keating and Snaith.
- For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log \log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.
- Such result has been proven for Gaussian regularizations (by Madaule in 2015), for branching random walks and branching Brownian motion.

- The logarithm of the characteristic polynomial, multiplied by $\sqrt{\beta/2}$, is a rather complex (yet integrable) regularization of the log-correlated Gaussian field given above.
- In this regularization, the correlation of the field saturates when $|z - z'|$ is of order $1/n$, which is consistent with the result by Keating and Snaith.
- For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log \log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.
- Such result has been proven for Gaussian regularizations (by Madaule in 2015), for branching random walks and branching Brownian motion.

# Statement of the main result

- For $\beta = 2$, Fyodorov, Hiary and Keating (2012), have given a conjecture on the maximum of the characteristic polynomial, which is the following:

$$\sup_{z \in \mathbb{U}} \log |X_n(z)| - \left( \log n - \frac{3}{4} \log \log n \right) \underset{n \to \infty}{\longrightarrow} \frac{1}{2}(K_1 + K_2),$$

in distribution, where $K_1$ and $K_2$ are two independent Gumbel random variables.

- In November 2015, Arguin, Belius and Bourgade have proven that

$$\frac{\sup_{z \in \mathbb{U}} \log |X_n(z)|}{\log n} \underset{n \to \infty}{\longrightarrow} 1$$

in probability.

# Statement of the main result

- For $\beta = 2$, Fyodorov, Hiary and Keating (2012), have given a conjecture on the maximum of the characteristic polynomial, which is the following:

$$\sup_{z \in \mathbb{U}} \log |X_n(z)| - \left( \log n - \frac{3}{4} \log \log n \right) \underset{n \to \infty}{\longrightarrow} \frac{1}{2}(K_1 + K_2),$$

in distribution, where $K_1$ and $K_2$ are two independent Gumbel random variables.

- In November 2015, Arguin, Belius and Bourgade have proven that

$$\frac{\sup_{z \in \mathbb{U}} \log |X_n(z)|}{\log n} \underset{n \to \infty}{\longrightarrow} 1$$

in probability.

- In Feburary 2016, Paquette and Zeitouni have proven:

$$\frac{\sup_{z\in\mathbb{U}} \log|X_n(z)| - \log n}{\log\log n} \underset{n\to\infty}{\longrightarrow} -\frac{3}{4}.$$

- We expect that the conjecture of Fyodorov, Hiary and Keating can be generalized to β ensembles:

$$\sqrt{\beta/2}\sup_{z\in\mathbb{U}} \log|X_n(z)| - \left(\log n - \frac{3}{4}\log\log n\right) \underset{n\to\infty}{\longrightarrow} K,$$

where $K$ is a limiting random variable. It may be possible that $2K$ is the sum two independent Gumbel variables, but we have no argument supporting such a statement.

- In Feburary 2016, Paquette and Zeitouni have proven:

$$\frac{\sup_{z \in \mathbb{U}} \log |X_n(z)| - \log n}{\log \log n} \underset{n \to \infty}{\longrightarrow} -\frac{3}{4}.$$

- We expect that the conjecture of Fyodorov, Hiary and Keating can be generalized to $\beta$ ensembles:

$$\sqrt{\beta/2} \sup_{z \in \mathbb{U}} \log |X_n(z)| - \left( \log n - \frac{3}{4} \log \log n \right) \underset{n \to \infty}{\longrightarrow} K,$$

where $K$ is a limiting random variable. It may be possible that $2K$ is the sum two independent Gumbel variables, but we have no argument supporting such a statement.

Such a result seems very challenging. However, we have proven the following result

## Theorem

*The families of random variables:*

$$\left(\sqrt{\beta/2}\sup_{z\in\mathbb{U}}\Re\log X_n(z)-\left(\log n-\frac{3}{4}\log\log n\right)\right)_{n\geq 2},$$

$$\left(\sqrt{\beta/2}\sup_{z\in\mathbb{U}}\Im\log X_n(z)-\left(\log n-\frac{3}{4}\log\log n\right)\right)_{n\geq 2}$$

*are tight.*

The statement on the imaginary part gives information on the number of eigenvalues lying on arcs of the unit circle.

We deduce the following:

## Corollary

*For $z_1, z_2 \in \mathbb{U}$, let $N(z_1, z_2)$ be the number of points $\lambda_j$ lying on the arc coming counterclockwise from $z_1$ to $z_2$, and $N_0(z_1, z_2)$ its expectation (i.e. the length of the arc multiplied by $n/2\pi$). Then,*

$$\left(\pi\sqrt{\beta/8} \sup_{z_1,z_2\in\mathbb{U}} |N(z_1,z_2) - N_0(z_1,z_2)| - \left(\log n - \frac{3}{4}\log\log n\right)\right)_{n\geq 2}$$

*is tight.*

# Orthogonal polynomials on the unit circle

If $\nu$ is a probability measure on the unit circle, the Gram-Schmidt procedure applied on $L^2(\nu)$ to the sequence $(z^k)_{k \geq 0}$ gives a sequence $(\Phi_k)_{0 \leq k < m}$ of monic orthogonal polynomials, $m$ being the (finite or infinite) cardinality of the support of $\nu$. If $m < \infty$, the procedure stops after $\Phi_{m-1}$ since all $L^2(\mu)$ is spanned: we then define

$$\Phi_m(z) := \prod_{\lambda \in \mathrm{Supp}(\nu)} (z - \lambda),$$

which vanishes in $L^2(\mu)$. Moreover, we define $\Phi_k^*(z) := z^k \overline{\Phi_k^*(1/\bar{z})}$.

- There exists a sequence $(\alpha_j)_{0 \leq j < m}$ of complex numbers, $|\alpha_j| = 1$ if $j = m - 1 < \infty$, $|\alpha_j| < 1$ otherwise, called *Verblunsky coefficients*, such that the polynomials above satisfy the so-called *Szegö recursion*: for $j < m$,

$$\Phi_{j+1}(z) = z\Phi_j(z) - \overline{\alpha_j}\Phi_j^*(z),$$

$$\Phi_{j+1}^*(z) = -\alpha_j z\Phi_j(z) + \Phi_j^*(z).$$

- Moreover, Killip and Nenciu have found an explicit probability distribution for the Verblunsky coefficients, for which one can recover the characteristic polynomial of the Circular Beta Ensemble.

- There exists a sequence $(\alpha_j)_{0 \leq j < m}$ of complex numbers, $|\alpha_j| = 1$ if $j = m - 1 < \infty$, $|\alpha_j| < 1$ otherwise, called *Verblunsky coefficients*, such that the polynomials above satisfy the so-called *Szegö recursion*: for $j < m$,

$$\Phi_{j+1}(z) = z\Phi_j(z) - \overline{\alpha_j}\Phi_j^*(z),$$

$$\Phi_{j+1}^*(z) = -\alpha_j z\Phi_j(z) + \Phi_j^*(z).$$

- Moreover, Killip and Nenciu have found an explicit probability distribution for the Verblunsky coefficients, for which one can recover the characteristic polynomial of the Circular Beta Ensemble.

- Let $(\alpha_j)_{j\geq 0}$, $\eta$ be independent complex random variables, rotationally invariant, such that $|\alpha_j|^2$ is Beta$(1, (\beta/2)(j+1))$-distributed and $|\eta| = 1$ a.s.
- Let $(\Phi_j, \Phi_j^*)_{j\geq 0}$ be the sequence of polynomials obtained from the Verblunsky coefficients $(\alpha_j)_{j\geq 0}$ and the Szegö recursion.
- Then, we have the equality in distribution:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

- If we couple the polynomials in such a way that we have acutally an equality, then

$$\left(\sup_{z\in\mathbb{U}} |\log X_n(z) - \log \Phi_{n-1}^*(z)|\right)_{n\geq 1}$$

is tight: it is then sufficient to study the extreme values of $\log \Phi_n^*$ instead of $\log X_n$.

- Let $(\alpha_j)_{j \geq 0}$, $\eta$ be independent complex random variables, rotationally invariant, such that $|\alpha_j|^2$ is Beta$(1, (\beta/2)(j+1))$-distributed and $|\eta| = 1$ a.s.
- Let $(\Phi_j, \Phi_j^*)_{j \geq 0}$ be the sequence of polynomials obtained from the Verblunsky coefficients $(\alpha_j)_{j \geq 0}$ and the Szegö recursion.
- Then, we have the equality in distribution:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

- If we couple the polynomials in such a way that we have acutally an equality, then

$$\left( \sup_{z \in \mathbb{U}} |\log X_n(z) - \log \Phi_{n-1}^*(z)| \right)_{n \geq 1}$$

is tight: it is then sufficient to study the extreme values of $\log \Phi_n^*$ instead of $\log X_n$.

- Let $(\alpha_j)_{j \geq 0}$, $\eta$ be independent complex random variables, rotationally invariant, such that $|\alpha_j|^2$ is Beta$(1, (\beta/2)(j+1))$-distributed and $|\eta| = 1$ a.s.
- Let $(\Phi_j, \Phi_j^*)_{j \geq 0}$ be the sequence of polynomials obtained from the Verblunsky coefficients $(\alpha_j)_{j \geq 0}$ and the Szegö recursion.
- Then, we have the equality in distribution:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

- If we couple the polynomials in such a way that we have acutally an equality, then

$$\left( \sup_{z \in \mathbb{U}} |\log X_n(z) - \log \Phi_{n-1}^*(z)| \right)_{n \geq 1}$$

is tight: it is then sufficient to study the extreme values of $\log \Phi_n^*$ instead of $\log X_n$.

- Let $(\alpha_j)_{j\geq 0}$, $\eta$ be independent complex random variables, rotationally invariant, such that $|\alpha_j|^2$ is Beta$(1,(\beta/2)(j+1))$-distributed and $|\eta| = 1$ a.s.
- Let $(\Phi_j, \Phi_j^*)_{j\geq 0}$ be the sequence of polynomials obtained from the Verblunsky coefficients $(\alpha_j)_{j\geq 0}$ and the Szegö recursion.
- Then, we have the equality in distribution:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

- If we couple the polynomials in such a way that we have acutally an equality, then

$$\left(\sup_{z\in\mathbb{U}} |\log X_n(z) - \log \Phi_{n-1}^*(z)|\right)_{n\geq 1}$$

is tight: it is then sufficient to study the extreme values of $\log \Phi_n^*$ instead of $\log X_n$.

- The recursion can be rewritten by using the *deformed Verblunsky coefficients* $(\gamma_j)_{j \geq 0}$, which have the same modulii as $(\alpha_j)_{j \geq 0}$ and the same joint distribution.
- We have, for $\theta \in [0, 2\pi)$,

$$\log \Phi_k^*(e^{i\theta}) = \sum_{j=0}^{k-1} \log \left(1 - \gamma_j e^{i\psi_j(\theta)}\right).$$

- The so-called *relative Prüfer phases* $(\psi_k)_{k \geq 0}$ satisfy:

$$\psi_k(\theta) = (k+1)\theta - 2 \sum_{j=0}^{k-1} \log \left( \frac{1 - \gamma_j e^{i\psi_j(\theta)}}{1 - \gamma_j} \right).$$

- The recursion can be rewritten by using the *deformed Verblunsky coefficients* $(\gamma_j)_{j\geq 0}$, which have the same modulii as $(\alpha_j)_{j\geq 0}$ and the same joint distribution.
- We have, for $\theta \in [0, 2\pi)$,

$$\log \Phi_k^*(e^{i\theta}) = \sum_{j=0}^{k-1} \log\left(1 - \gamma_j e^{i\psi_j(\theta)}\right).$$

- The so-called *relative Prüfer phases* $(\psi_k)_{k\geq 0}$ satisfy:

$$\psi_k(\theta) = (k+1)\theta - 2\sum_{j=0}^{k-1} \log\left(\frac{1-\gamma_j e^{i\psi_j(\theta)}}{1-\gamma_j}\right).$$

- The recursion can be rewritten by using the *deformed Verblunsky coefficients* $(\gamma_j)_{j\geq 0}$, which have the same modulii as $(\alpha_j)_{j\geq 0}$ and the same joint distribution.
- We have, for $\theta \in [0, 2\pi)$,

$$\log \Phi_k^*(e^{i\theta}) = \sum_{j=0}^{k-1} \log\left(1 - \gamma_j e^{i\psi_j(\theta)}\right).$$

- The so-called *relative Prüfer phases* $(\psi_k)_{k\geq 0}$ satisfy:

$$\psi_k(\theta) = (k+1)\theta - 2\sum_{j=0}^{k-1} \log\left(\frac{1 - \gamma_j e^{i\psi_j(\theta)}}{1 - \gamma_j}\right).$$

# Sketch of proof of a non-sharp upper bound

- In order to bound $\Re \log \Phi_n^*$ and $\Im \log \Phi_n^*$ on the unit circle, it is sufficient to bound these quantities on $2n$ points.
- Indeed, if $\mathbb{U}_m$ denotes the set of $m$-th roots of unity, we have for all polynomials $Q$ of degree at most $n$:

$$\sup_{z \in \mathbb{U}} |Q(z)| \leq 14 \sup_{z \in \mathbb{U}_{2n}} |Q(z)|.$$

- If $Q(0) = 1$ and $Q$ has all roots outside the unit disc, then

$$\sup_{z \in \mathbb{U}} \operatorname{Arg}(Q(z)) \leq \sup_{z \in \mathbb{U}_n} \operatorname{Arg}(Q(z)) + 2\pi.$$

# Sketch of proof of a non-sharp upper bound

- In order to bound $\Re \log \Phi_n^*$ and $\Im \log \Phi_n^*$ on the unit circle, it is sufficient to bound these quantities on $2n$ points.
- Indeed, if $\mathbb{U}_m$ denotes the set of $m$-th roots of unity, we have for all polynomials $Q$ of degree at most $n$:

$$\sup_{z \in \mathbb{U}} |Q(z)| \leq 14 \sup_{z \in \mathbb{U}_{2n}} |Q(z)|.$$

- If $Q(0) = 1$ and $Q$ has all roots outside the unit disc, then

$$\sup_{z \in \mathbb{U}} \operatorname{Arg}(Q(z)) \leq \sup_{z \in \mathbb{U}_n} \operatorname{Arg}(Q(z)) + 2\pi.$$

# Sketch of proof of a non-sharp upper bound

- In order to bound $\Re \log \Phi_n^*$ and $\Im \log \Phi_n^*$ on the unit circle, it is sufficient to bound these quantities on $2n$ points.
- Indeed, if $\mathbb{U}_m$ denotes the set of $m$-th roots of unity, we have for all polynomials $Q$ of degree at most $n$:

$$\sup_{z \in \mathbb{U}} |Q(z)| \leq 14 \sup_{z \in \mathbb{U}_{2n}} |Q(z)|.$$

- If $Q(0) = 1$ and $Q$ has all roots outside the unit disc, then

$$\sup_{z \in \mathbb{U}} \operatorname{Arg}(Q(z)) \leq \sup_{z \in \mathbb{U}_n} \operatorname{Arg}(Q(z)) + 2\pi.$$

- For any $z \in \mathbb{U}$, we have the equality in distribution:

$$\log \Phi_k^*(z) = \sum_{j=0}^{k-1} \log(1 - \gamma_j),$$

- By computing and then estimating the exponential moments of this sum of independent random variables, we get for $s > 0, t \in \mathbb{R}$,

$$\mathbb{E}[e^{s\Re \log \Phi_k^*(z) + t\Im \log \Phi_k^*(z)}] \leq (ke)^{(s^2+t^2)/(2\beta)}.$$

- Using a Chernoff bound with $s = \sqrt{2\beta}$, $t = 0$, we deduce that for $n \to \infty$,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}} \Re \log \Phi_n^*(z) \geq \log n + h(n)\right) = o(1/n)$$

and the same for the imaginary part.

- For any $z \in \mathbb{U}$, we have the equality in distribution:

$$\log \Phi_k^*(z) = \sum_{j=0}^{k-1} \log(1 - \gamma_j),$$

- By computing and then estimating the exponential moments of this sum of independent random variables, we get for $s > 0, t \in \mathbb{R}$,

$$\mathbb{E}[e^{s \Re \log \Phi_k^*(z) + t \Im \log \Phi_k^*(z)}] \leq (ke)^{(s^2+t^2)/(2\beta)}.$$

- Using a Chernoff bound with $s = \sqrt{2\beta}$, $t = 0$, we deduce that for $n \to \infty$,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}} \Re \log \Phi_n^*(z) \geq \log n + h(n)\right) = o(1/n)$$

and the same for the imaginary part.

- For any $z \in \mathbb{U}$, we have the equality in distribution:

$$\log \Phi_k^*(z) = \sum_{j=0}^{k-1} \log(1 - \gamma_j),$$

- By computing and then estimating the exponential moments of this sum of independent random variables, we get for $s > 0, t \in \mathbb{R}$,

$$\mathbb{E}[e^{s\Re \log \Phi_k^*(z) + t\Im \log \Phi_k^*(z)}] \leq (ke)^{(s^2+t^2)/(2\beta)}.$$

- Using a Chernoff bound with $s = \sqrt{2\beta}$, $t = 0$, we deduce that for $n \to \infty$,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}}\Re \log \Phi_n^*(z) \geq \log n + h(n)\right) = o(1/n)$$

and the same for the imaginary part.

- Using a union bound on the $2n$-th roots of unity,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}} \sup_{z \in \mathbb{U}} \Re \log \Phi_n^*(z) \leq \log n + h(n)\right) \underset{n \to \infty}{\longrightarrow} 1,$$

which gives a weak version of the upper bound stated above.

- Moreover, if we define

$$\mathcal{B}_n := \{\lfloor e^j \rfloor, 0 \leq j \leq \lfloor \log n \rfloor\} \cup \{n\},$$

then

$$\mathbb{P}\left(\forall k \in \mathcal{B}_n, \sup_{z \in \mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log\log n + h(n)\right) \underset{n \to \infty}{\longrightarrow} 1.$$

- This estimate is useful in order to prove a sharper upper bound.

- Using a union bound on the $2n$-th roots of unity,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}} \sup_{z \in \mathbb{U}} \Re \log \Phi_n^*(z) \leq \log n + h(n)\right) \underset{n\to\infty}{\longrightarrow} 1,$$

  which gives a weak version of the upper bound stated above.

- Moreover, if we define

$$\mathcal{B}_n := \{\lfloor e^j \rfloor, 0 \leq j \leq \lfloor \log n \rfloor\} \cup \{n\},$$

  then

$$\mathbb{P}\left(\forall k \in \mathcal{B}_n, \sup_{z \in \mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log \log n + h(n)\right) \underset{n\to\infty}{\longrightarrow} 1.$$

- This estimate is useful in order to prove a sharper upper bound.

- Using a union bound on the $2n$-th roots of unity,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}}\sup_{z\in\mathbb{U}}\Re\log\Phi_n^*(z)\leq\log n+h(n)\right)\underset{n\to\infty}{\longrightarrow}1,$$

which gives a weak version of the upper bound stated above.

- Moreover, if we define

$$\mathcal{B}_n:=\{\lfloor e^j\rfloor, 0\leq j\leq\lfloor\log n\rfloor\}\cup\{n\},$$

then

$$\mathbb{P}\left(\forall k\in\mathcal{B}_n,\sup_{z\in\mathbb{U}}\Re\log\Phi_k^*(z)\leq\log k+\log\log n+h(n)\right)\underset{n\to\infty}{\longrightarrow}1.$$

- This estimate is useful in order to prove a sharper upper bound.

# Sketch of proof of a sharper upper bound

- We will prove that

$$\mathbb{P}\left(\forall k \in \mathcal{B}_n, \sup_{z \in \mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log\log n + h(n),\right.$$

$$\left.\sup_{z \in \mathbb{U}} \Re \log \Phi_n^*(z) \geq \log n - \frac{3}{4}\log\log n + \frac{3}{2}\log\log\log n + h(n)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

- By doing a union bound on $\mathbb{U}_{2n}$, it is sufficient to prove that the probability of the same event for a single $z \in \mathbb{U}$ is $o(1/n)$ when $n$ goes to infinity.

# Sketch of proof of a sharper upper bound

- We will prove that

$$\mathbb{P}\Bigg(\forall k \in \mathcal{B}_n, \sup_{z \in \mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log\log n + h(n),$$

$$\sup_{z \in \mathbb{U}} \Re \log \Phi_n^*(z) \geq \log n - \frac{3}{4}\log\log n + \frac{3}{2}\log\log\log n + h(n)\Bigg) \underset{n\to\infty}{\longrightarrow} 0.$$

- By doing a union bound on $\mathbb{U}_{2n}$, it is sufficient to prove that the probability of the same event for a single $z \in \mathbb{U}$ is $o(1/n)$ when $n$ goes to infinity.

- For fixed $z \in \mathbb{U}$, $(\log \Phi_k^*(z))_{k \geq 0}$ is a random walk with independent increments, given by $\log(1 - \gamma_k)$.
- We have an equality in law:

$$\log(1 - \gamma_k) = \log \left( 1 - e^{i\Theta_k} \sqrt{\frac{E_k}{E_k + \Gamma_k}} \right)$$

where $(E_k)_{k \geq 0}$, $(\Gamma_k)_{k \geq 0}$, $(\Theta_k)_{k \geq 0}$ are independent variables, respectively exponentially distributed, Gamma of parameter $(\beta/2)(k+1)$ and uniform on $[0, 2\pi)$.
- If we replace $E_k + \Gamma_k$ by $2/(\beta(k+1))$ and $\log(1-y)$ by $-y$, we get a Gaussian variable of variance $1/(\beta(k+1))$.

- For fixed $z \in \mathbb{U}$, $(\log \Phi_k^*(z))_{k \geq 0}$ is a random walk with independent increments, given by $\log(1-\gamma_k)$.
- We have an equality in law:

$$\log(1-\gamma_k) = \log\left(1 - e^{i\Theta_k}\sqrt{\frac{E_k}{E_k+\Gamma_k}}\right)$$

where $(E_k)_{k\geq 0}$, $(\Gamma_k)_{k\geq 0}$, $(\Theta_k)_{k\geq 0}$ are independent variables, respectively exponentially distributed, Gamma of parameter $(\beta/2)(k+1)$ and uniform on $[0,2\pi)$.

- If we replace $E_k + \Gamma_k$ by $2/(\beta(k+1))$ and $\log(1-y)$ by $-y$, we get a Gaussian variable of variance $1/(\beta(k+1))$.

- For fixed $z \in \mathbb{U}$, $(\log \Phi_k^*(z))_{k \geq 0}$ is a random walk with independent increments, given by $\log(1-\gamma_k)$.
- We have an equality in law:

$$\log(1-\gamma_k) = \log\left(1 - e^{i\Theta_k}\sqrt{\frac{E_k}{E_k+\Gamma_k}}\right)$$

  where $(E_k)_{k\geq 0}$, $(\Gamma_k)_{k\geq 0}$, $(\Theta_k)_{k\geq 0}$ are independent variables, respectively exponentially distributed, Gamma of parameter $(\beta/2)(k+1)$ and uniform on $[0, 2\pi)$.
- If we replace $E_k + \Gamma_k$ by $2/(\beta(k+1))$ and $\log(1-y)$ by $-y$, we get a Gaussian variable of variance $1/(\beta(k+1))$.

- One can prove that $(\sqrt{\beta/2}\,\Phi_k^*(z))_{k\geq 0, z\in\mathbb{U}}$ can be coupled, with an a.s. bounded difference, with a field $(Z_k(z))_{k\geq 0, z\in\mathbb{U}}$, with complex Gaussian marginals, with independent increments for fixed $\theta$:

$$Z_k(e^{i\theta}) := \sum_{j=0}^{k-1} \frac{\mathcal{N}_j^{\mathbb{C}} e^{i\psi_j(\theta)}}{\sqrt{j+1}}.$$

- In this way, we can deduce that it is essentially sufficient to show ($N$ corresponding to $\log n$), for a Brownian motion $W$ that:

$$\mathbb{P}\Big(\forall j \in \{1,2,\dots,N-1\}, W_j \leq \sqrt{2}\,(j+\log N + h(N)),$$

$$W_N \geq \sqrt{2}\left(N - \frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\Big) = o(e^{-N}).$$

- One can prove that $(\sqrt{\beta/2}\,\Phi_k^*(z))_{k\geq 0, z\in\mathbb{U}}$ can be coupled, with an a.s. bounded difference, with a field $(Z_k(z))_{k\geq 0, z\in\mathbb{U}}$, with complex Gaussian marginals, with independent increments for fixed $\theta$:

$$Z_k(e^{i\theta}) := \sum_{j=0}^{k-1} \frac{\mathcal{N}_j^{\mathbb{C}} e^{i\psi_j(\theta)}}{\sqrt{j+1}}.$$

- In this way, we can deduce that it is essentially sufficient to show ($N$ corresponding to $\log n$), for a Brownian motion $W$ that:

$$\mathbb{P}\Big(\forall j \in \{1,2,\dots,N-1\}, W_j \leq \sqrt{2}\,(j+\log N + h(N))\,,$$

$$W_N \geq \sqrt{2}\left(N - \frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\Big) = o(e^{-N}).$$

- Using Girsanov's theorem, it is enough to show

$$\mathbb{P}\left(\forall j \in \{1,2,\dots,N-1\}, W_j \leq \sqrt{2}\,(\log N + h(N)),\right.$$

$$\left. W_N \geq \sqrt{2}\left(-\frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\right) = O\left(N^{-3/2}(\log N)^3\right).$$

- This result can be deduced from a suitable version of the ballot theorem, or from the joint law of a Brownian motion and its past supremum.

- Using Girsanov's theorem, it is enough to show

$$\mathbb{P}\left(\forall j \in \{1,2,\ldots,N-1\}, W_j \leq \sqrt{2}\,(\log N + h(N)),\right.$$

$$\left. W_N \geq \sqrt{2}\left(-\frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\right) = O\left(N^{-3/2}(\log N)^3\right).$$

- This result can be deduced from a suitable version of the ballot theorem, or from the joint law of a Brownian motion and its past supremum.

# Strategy for a lower bound

- In order to get a sharp lower bound, we would have to show that with high probability, there exists $\theta \in [0, 2\pi)$ such that

$$\Re Z_n(e^{i\theta}) \geq \log n - \frac{3}{4} \log \log n - h(n).$$

- Let $E_n(\theta)$ be any event implying the previous inequality. It is sufficient to show:

$$\mathbb{P}(N_n > 0) \underset{n \to \infty}{\longrightarrow} 1,$$

where $N_n$ is the number of $j \in \{0, \ldots, n-1\}$ such that $E_n(e^{2i\pi j/n})$ occurs.

# Strategy for a lower bound

- In order to get a sharp lower bound, we would have to show that with high probability, there exists $\theta \in [0, 2\pi)$ such that

$$\Re Z_n(e^{i\theta}) \geq \log n - \frac{3}{4} \log \log n - h(n).$$

- Let $E_n(\theta)$ be any event implying the previous inequality. It is sufficient to show:

$$\mathbb{P}(N_n > 0) \underset{n \to \infty}{\longrightarrow} 1,$$

where $N_n$ is the number of $j \in \{0, \ldots, n-1\}$ such that $E_n(e^{2i\pi j/n})$ occurs.

- Paley-Zygmund inequality implies that

$$\mathbb{P}(N_n > 0) \geq \frac{(\mathbb{E}[N_n])^2}{\mathbb{E}[N_n^2]}.$$

- Hence it is enough to show:

$$\mathbb{E}[N_n^2] \leq (\mathbb{E}[N_n])^2 (1 + o(1)),$$

and then to have a suitable lower bound of $\mathbb{E}[N_n]$ and a suitable upper bound of $\mathbb{E}[N_n^2]$.

- For that, we need to choose events $E_n(\theta)$, in such a way that their probability is not too small and that $E_n(\theta)$ and $E_n(\theta')$ are not too much correlated if $\theta$ is not too close to $\theta'$.

- Paley-Zygmund inequality implies that

$$\mathbb{P}(N_n > 0) \geq \frac{(\mathbb{E}[N_n])^2}{\mathbb{E}[N_n^2]}.$$

- Hence it is enough to show:

$$\mathbb{E}[N_n^2] \leq (\mathbb{E}[N_n])^2 (1 + o(1)),$$

and then to have a suitable lower bound of $\mathbb{E}[N_n]$ and a suitable upper bound of $\mathbb{E}[N_n^2]$.

- For that, we need to choose events $E_n(\theta)$, in such a way that their probability is not too small and that $E_n(\theta)$ and $E_n(\theta')$ are not too much correlated if $\theta$ is not too close to $\theta'$.

- Paley-Zygmund inequality implies that

$$\mathbb{P}(N_n > 0) \geq \frac{(\mathbb{E}[N_n])^2}{\mathbb{E}[N_n^2]}.$$

- Hence it is enough to show:

$$\mathbb{E}[N_n^2] \leq (\mathbb{E}[N_n])^2 (1 + o(1)),$$

and then to have a suitable lower bound of $\mathbb{E}[N_n]$ and a suitable upper bound of $\mathbb{E}[N_n^2]$.

- For that, we need to choose events $E_n(\theta)$, in such a way that their probability is not too small and that $E_n(\theta)$ and $E_n(\theta')$ are not too much correlated if $\theta$ is not too close to $\theta'$.

- The event $E_n(\theta)$ corresponds to the fact that the random walk $(\Re Z_k(\theta))_{k \in \mathcal{B}_n}$ stays in a suitably chosen envelope.
- Since the Prüfer phases incrase by approximately $\theta$ at each step, for $\theta \in [0, \pi]$, the increments of the random walks $(Z_k(0))_{k \in \mathcal{B}_n}$ and $(Z_k(\theta))_{k \in \mathcal{B}_n}$ are "roughly similar" for $k \leq 1/\theta$ and "roughly independent" afterwards.
- We can then do similar computations as for branching Gaussian random walks.

- The event $E_n(\theta)$ corresponds to the fact that the random walk $(\Re Z_k(\theta))_{k \in \mathcal{B}_n}$ stays in a suitably chosen envelope.
- Since the Prüfer phases incrase by approximately $\theta$ at each step, for $\theta \in [0, \pi]$, the increments of the random walks $(Z_k(0))_{k \in \mathcal{B}_n}$ and $(Z_k(\theta))_{k \in \mathcal{B}_n}$ are "roughly similar" for $k \leq 1/\theta$ and "roughly independent" afterwards.
- We can then do similar computations as for branching Gaussian random walks.

- The event $E_n(\theta)$ corresponds to the fact that the random walk $(\Re Z_k(\theta))_{k \in \mathcal{B}_n}$ stays in a suitably chosen envelope.
- Since the Prüfer phases incrase by approximately $\theta$ at each step, for $\theta \in [0, \pi]$, the increments of the random walks $(Z_k(0))_{k \in \mathcal{B}_n}$ and $(Z_k(\theta))_{k \in \mathcal{B}_n}$ are "roughly similar" for $k \leq 1/\theta$ and "roughly independent" afterwards.
- We can then do similar computations as for branching Gaussian random walks.

# The Riemann zeta function

- The Riemann zeta function is a complex function which naturally appears in the distribution of prime numbers.
- For $\Re(s) > 1$, it is defined by

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \, \text{prime}} (1 - p^{-s})^{-1}.$$

- It can be uniquely extended to a holomorphic (i.e. everywhere differentiable) function from $\mathbb{C} \backslash \{1\}$ to $\mathbb{C}$.
- This function is $1/(s-1) + O(1)$ in the neighborhood of $s = 1$, and it has infinitely many zeros.
- The zeros are the even negative integers (called trivial zeros), and infinitely many zeros whose real part is in $(0, 1)$ (called non-trivial zeros).

# The Riemann zeta function

- The Riemann zeta function is a complex function which naturally appears in the distribution of prime numbers.
- For $\Re(s) > 1$, it is defined by

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \,\text{prime}} (1 - p^{-s})^{-1}.$$

- It can be uniquely extended to a holomorphic (i.e. everywhere differentiable) function from $\mathbb{C} \backslash \{1\}$ to $\mathbb{C}$.
- This function is $1/(s-1) + O(1)$ in the neighborhood of $s = 1$, and it has infinitely many zeros.
- The zeros are the even negative integers (called trivial zeros), and infinitely many zeros whose real part is in $(0, 1)$ (called non-trivial zeros).

# The Riemann zeta function

- The Riemann zeta function is a complex function which naturally appears in the distribution of prime numbers.
- For $\Re(s) > 1$, it is defined by

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \,\text{prime}} (1 - p^{-s})^{-1}.$$

- It can be uniquely extended to a holomorphic (i.e. everywhere differentiable) function from $\mathbb{C} \backslash \{1\}$ to $\mathbb{C}$.
- This function is $1/(s-1) + O(1)$ in the neighborhood of $s = 1$, and it has infinitely many zeros.
- The zeros are the even negative integers (called trivial zeros), and infinitely many zeros whose real part is in $(0, 1)$ (called non-trivial zeros).

# The Riemann zeta function

- The Riemann zeta function is a complex function which naturally appears in the distribution of prime numbers.
- For $\Re(s) > 1$, it is defined by

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \, \text{prime}} (1 - p^{-s})^{-1}.$$

- It can be uniquely extended to a holomorphic (i.e. everywhere differentiable) function from $\mathbb{C} \backslash \{1\}$ to $\mathbb{C}$.
- This function is $1/(s-1) + O(1)$ in the neighborhood of $s = 1$, and it has infinitely many zeros.
- The zeros are the even negative integers (called trivial zeros), and infinitely many zeros whose real part is in $(0, 1)$ (called non-trivial zeros).

# The Riemann zeta function

- The Riemann zeta function is a complex function which naturally appears in the distribution of prime numbers.
- For $\Re(s) > 1$, it is defined by

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \, \text{prime}} (1 - p^{-s})^{-1}.$$

- It can be uniquely extended to a holomorphic (i.e. everywhere differentiable) function from $\mathbb{C} \backslash \{1\}$ to $\mathbb{C}$.
- This function is $1/(s-1) + O(1)$ in the neighborhood of $s = 1$, and it has infinitely many zeros.
- The zeros are the even negative integers (called trivial zeros), and infinitely many zeros whose real part is in $(0, 1)$ (called non-trivial zeros).

- The non-trivial zeros are symmetrically distributed with respect to the axis $\Re(s) = 1/2$. The Riemann hypothesis states that they are all on the critical line.
- The behavior of $\zeta$ on the critical line $\Re(s) = 1/2$ has been intensively studied, and in particular the order of magnitude of its growth when $t \to \infty$ . The Riemann hypothesis implies the so-called *Lindelöf hypothesis*, stating that for any $\varepsilon > 0$, $|\zeta(1/2 + it)| = O((1 + |t|)^{\varepsilon})$.
- The Lindelöf hypothesis is still open today, the best result in this direction is due to Bourgain [2014], who has shown that the bound is true for all $\varepsilon > 13/84$.

- The non-trivial zeros are symmetrically distributed with respect to the axis $\Re(s) = 1/2$. The Riemann hypothesis states that they are all on the critical line.
- The behavior of $\zeta$ on the critical line $\Re(s) = 1/2$ has been intensively studied, and in particular the order of magnitude of its growth when $t \to \infty$ . The Riemann hypothesis implies the so-called *Lindelöf hypothesis*, stating that for any $\varepsilon > 0$, $|\zeta(1/2 + it)| = O((1 + |t|)^{\varepsilon})$.
- The Lindelöf hypothesis is still open today, the best result in this direction is due to Bourgain [2014], who has shown that the bound is true for all $\varepsilon > 13/84$.

- The non-trivial zeros are symmetrically distributed with respect to the axis $\Re(s) = 1/2$. The Riemann hypothesis states that they are all on the critical line.
- The behavior of $\zeta$ on the critical line $\Re(s) = 1/2$ has been intensively studied, and in particular the order of magnitude of its growth when $t \to \infty$ . The Riemann hypothesis implies the so-called *Lindelöf hypothesis*, stating that for any $\varepsilon > 0$, $|\zeta(1/2 + it)| = O((1 + |t|)^{\varepsilon})$.
- The Lindelöf hypothesis is still open today, the best result in this direction is due to Bourgain [2014], who has shown that the bound is true for all $\varepsilon > 13/84$.

- Under the Riemann hypothesis, it is known (in particular from results by Littlewood [1924], Montgomery [1977], Balasubramanianm, Ramachandra [1977], Titshmarch [1986], Soundararajan [2008], Bodarenko, Seip [2017], de la Bretèche, Tenenbaum [2018]) that for $t$ large enough,

$$|\zeta(1/2+it)| = O(e^{\log t/\log\log t}),$$

$$|\zeta(1/2+it)| \neq O\left(e^{\sqrt{(1-\varepsilon)\log t \log\log\log t/\log\log t}}\right)$$

for all $\varepsilon > 0$.

- Farmer, Gonek and Hughes [2007] have conjectured that

$$\max_{t\in[0,T]} \log|\zeta(1/2+it)| \sim_{T\to\infty} \sqrt{(1/2)\log T \log\log T}.$$

- Under the Riemann hypothesis, it is known (in particular from results by Littlewood [1924], Montgomery [1977], Balasubramanianm, Ramachandra [1977], Titshmarch [1986], Soundararajan [2008], Bodarenko, Seip [2017], de la Bretèche, Tenenbaum [2018]) that for $t$ large enough,

$$|\zeta(1/2+it)| = O(e^{\log t/\log\log t}),$$

$$|\zeta(1/2+it)| \neq O\left(e^{\sqrt{(1-\varepsilon)\log t \log\log\log t/\log\log t}}\right)$$

for all $\varepsilon > 0$.

- Farmer, Gonek and Hughes [2007] have conjectured that

$$\max_{t\in[0,T]} \log|\zeta(1/2+it)| \sim_{T\to\infty} \sqrt{(1/2)\log T \log\log T}.$$

- It is difficult to control the extreme values of $\zeta$ on large intervals. From now, we will consider random intervals of fixed length, and use this randomness to apply probabilistic results.
- Fyodorov, Hiary and Keating [2012] have made a very precise conjecture about the order of magnitude of the maximum of $\log|\zeta|$ on such intervals
- The conjecture can be stated as follows: for $h > 0$ fixed, $T > 0$, $U$ uniformly distributed on $[0, 1]$,

$$\max_{t \in [UT-h, UT+h]} \log|\zeta(1/2 + it)| - \left(\log\log T - \frac{3}{4}\log\log\log T\right) \underset{T\to\infty}{\longrightarrow} K,$$

when $K$ is a random variable.

- It is difficult to control the extreme values of $\zeta$ on large intervals. From now, we will consider random intervals of fixed length, and use this randomness to apply probabilistic results.
- Fyodorov, Hiary and Keating [2012] have made a very precise conjecture about the order of magnitude of the maximum of $\log|\zeta|$ on such intervals
- The conjecture can be stated as follows: for $h > 0$ fixed, $T > 0$, $U$ uniformly distributed on $[0, 1]$,

$$\max_{t \in [UT-h, UT+h]} \log|\zeta(1/2 + it)| - \left(\log\log T - \frac{3}{4}\log\log\log T\right) \underset{T\to\infty}{\longrightarrow} K,$$

when $K$ is a random variable.

- It is difficult to control the extreme values of $\zeta$ on large intervals. From now, we will consider random intervals of fixed length, and use this randomness to apply probabilistic results.
- Fyodorov, Hiary and Keating [2012] have made a very precise conjecture about the order of magnitude of the maximum of $\log |\zeta|$ on such intervals
- The conjecture can be stated as follows: for $h > 0$ fixed, $T > 0$, $U$ uniformly distributed on $[0, 1]$,

$$\max_{t \in [UT-h, UT+h]} \log |\zeta(1/2 + it)| - \left(\log \log T - \frac{3}{4} \log \log \log T\right) \underset{T \to \infty}{\longrightarrow} K,$$

when $K$ is a random variable.

- In November 2016, in the setting of the Riemann function, we have proven the following: for all $\varepsilon > 0$, unconditionally,

$$\max_{t \in [UT-h, UT+h]} \Re \log \zeta(1/2 + it) \leq (1+\varepsilon) \log\log T,$$

and under the Riemann hypothesis,

$$\max_{t \in [UT-h, UT+h]} \Re \log \zeta(1/2 + it) \geq (1-\varepsilon) \log\log T.$$

with probability tending to 1 when $T$ goes to infinity.

- We have proven, under the Riemann hypothesis, the same upper bound and the same lower bound for the imaginary part of $\log \zeta$. This gives information on the fluctuations of the distribution of the zeros of $\zeta$ on random intervals of the critical line.
- In December 2016, Arguin, Belius, Bourgade, Raziwill, Soundararajan, managed to get rid of the Riemann hypothesis for the lower bound on $\Re \log \zeta$. In June 2019, the upper bound had been improved by Harper: $\log \log T - (3/4) \log \log \log T + (3/2 + o(1)) \log \log \log \log T$.
- However, it is not known for the moment if the results on $\Im \log \zeta$ occur unconditionally and/or can be improved.

- We have proven, under the Riemann hypothesis, the same upper bound and the same lower bound for the imaginary part of $\log \zeta$. This gives information on the fluctuations of the distribution of the zeros of $\zeta$ on random intervals of the critical line.
- In December 2016, Arguin, Belius, Bourgade, Raziwill, Soundararajan, managed to get rid of the Riemann hypothesis for the lower bound on $\Re \log \zeta$. In June 2019, the upper bound had been improved by Harper: $\log \log T - (3/4) \log \log \log T + (3/2 + o(1)) \log \log \log \log T$.
- However, it is not known for the moment if the results on $\Im \log \zeta$ occur unconditionally and/or can be improved.

- We have proven, under the Riemann hypothesis, the same upper bound and the same lower bound for the imaginary part of $\log \zeta$. This gives information on the fluctuations of the distribution of the zeros of $\zeta$ on random intervals of the critical line.
- In December 2016, Arguin, Belius, Bourgade, Raziwill, Soundararajan, managed to get rid of the Riemann hypothesis for the lower bound on $\Re \log \zeta$. In June 2019, the upper bound had been improved by Harper: $\log \log T - (3/4) \log \log \log T + (3/2 + o(1)) \log \log \log \log T$.
- However, it is not known for the moment if the results on $\Im \log \zeta$ occur unconditionally and/or can be improved.

# Averaging of $\log |\zeta|$ on the critical line

- For $\Re(s) > 1$, we have

$$\log \zeta(s) = \sum_{n \geq 1} \ell(n) n^{-s}$$

  where $\ell(n) = 1/k$ if $n$ is the $k$-th power of a prime and $\ell(n) = 0$ otherwise.

- If $\varphi$ is a nonnegative function with integral 1, and if $H > 1$, we get

$$\int_{-\infty}^{\infty} \varphi(t) \log \zeta(s + itH^{-1}) dt = \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi}(H^{-1} \log n).$$

# Averaging of $\log|\zeta|$ on the critical line

- For $\Re(s) > 1$, we have

$$\log \zeta(s) = \sum_{n \geq 1} \ell(n) n^{-s}$$

  where $\ell(n) = 1/k$ if $n$ is the $k$-th power of a prime and $\ell(n) = 0$ otherwise.

- If $\varphi$ is a nonnegative function with integral 1, and if $H > 1$, we get

$$\int_{-\infty}^{\infty} \varphi(t) \log \zeta(s + itH^{-1}) dt = \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi}(H^{-1} \log n).$$

- If we take $\widehat{\varphi}$ compactly supported, the last sum is supported in $n \leq e^{O(H)}$. By analytic continuation arguments, one shows that *under the Riemann hypothesis*, for $H$ sufficiently small with respect to the argument of $s$, the equality remains true up to a bounded error term, when $\Re(s) \in [1/2, 1)$.
- With high probability, it is possible to take, for some fixed $\delta \in (0, 1/2)$, $H = \lfloor (\log T)^{1-\delta} \rfloor$, if $s = 1/2 + it$, $t \in [UT - h, UT + h]$.
- Averaging $\Im \log \zeta$ tends to smooth its behavior, and then to decrease its maximum.
- It is possible to show that one can replace the smooth cutoff of the sum with $\widehat{\varphi}$ by a sharp cutoff, and remove the powers of primes with exponents at least 2, by doing an error $o(\log \log T)$ on the maximum with high probability.

- If we take $\widehat{\varphi}$ compactly supported, the last sum is supported in $n \leq e^{O(H)}$. By analytic continuation arguments, one shows that *under the Riemann hypothesis*, for $H$ sufficiently small with respect to the argument of $s$, the equality remains true up to a bounded error term, when $\Re(s) \in [1/2, 1)$.
- With high probability, it is possible to take, for some fixed $\delta \in (0, 1/2)$, $H = \lfloor (\log T)^{1-\delta} \rfloor$, if $s = 1/2 + it$, $t \in [UT - h, UT + h]$.
- Averaging $\Im \log \zeta$ tends to smooth its behavior, and then to decrease its maximum.
- It is possible to show that one can replace the smooth cutoff of the sum with $\widehat{\varphi}$ by a sharp cutoff, and remove the powers of primes with exponents at least 2, by doing an error $o(\log \log T)$ on the maximum with high probability.

- If we take $\widehat{\varphi}$ compactly supported, the last sum is supported in $n \leq e^{O(H)}$. By analytic continuation arguments, one shows that *under the Riemann hypothesis*, for $H$ sufficiently small with respect to the argument of $s$, the equality remains true up to a bounded error term, when $\Re(s) \in [1/2, 1)$.
- With high probability, it is possible to take, for some fixed $\delta \in (0, 1/2)$, $H = \lfloor (\log T)^{1-\delta} \rfloor$, if $s = 1/2 + it$, $t \in [UT - h, UT + h]$.
- Averaging $\Im \log \zeta$ tends to smooth its behavior, and then to decrease its maximum.
- It is possible to show that one can replace the smooth cutoff of the sum with $\widehat{\varphi}$ by a sharp cutoff, and remove the powers of primes with exponents at least 2, by doing an error $o(\log \log T)$ on the maximum with high probability.

- If we take $\widehat{\varphi}$ compactly supported, the last sum is supported in $n \leq e^{O(H)}$. By analytic continuation arguments, one shows that *under the Riemann hypothesis*, for $H$ sufficiently small with respect to the argument of $s$, the equality remains true up to a bounded error term, when $\Re(s) \in [1/2, 1)$.
- With high probability, it is possible to take, for some fixed $\delta \in (0, 1/2)$, $H = \lfloor (\log T)^{1-\delta} \rfloor$, if $s = 1/2 + it$, $t \in [UT - h, UT + h]$.
- Averaging $\Im \log \zeta$ tends to smooth its behavior, and then to decrease its maximum.
- It is possible to show that one can replace the smooth cutoff of the sum with $\widehat{\varphi}$ by a sharp cutoff, and remove the powers of primes with exponents at least 2, by doing an error $o(\log \log T)$ on the maximum with high probability.

- Because of these considerations, it is enough to prove the following result, in order to get the lower bound in our main theorem: with high probability, the supremum of

$$\Im \sum_{p \in \mathcal{P}, p \leq e^H} p^{-1/2 - i(UT+t)},$$

for $t \in [-h, h]$ is larger than $(1-\varepsilon) \log \log T$, if $\delta$ is taken sufficiently small depending on $\varepsilon$.

# Correlation structure

- For distinct primes $p$, the phases $p^{-iUT}$ tend in law to i.i.d., uniform variables on the unit circle $X_p$.
- It is then natural to compare the previous random variables by

$$\Im \sum_{p \in \mathcal{P}, p \leq e^H} X_p p^{-1/2-it}.$$

- For $t, t' \in [-h, h]$, the covariance of these random variables is given for $\Theta_p$ i.i.d. uniform on $[0, 2\pi]$

$$\sum_{p \in \mathcal{P}, p \leq e^H} p^{-1} \mathbb{E}[\sin(\Theta_p - t \log p) \sin(\Theta_p - t' \log p)]$$

$$= \frac{1}{2} \sum_{p \in \mathcal{P}, p \leq e^H} p^{-1} \cos((t - t') \log p).$$

# Correlation structure

- For distinct primes $p$, the phases $p^{-iUT}$ tend in law to i.i.d., uniform variables on the unit circle $X_p$.
- It is then natural to compare the previous random variables by

$$\Im \sum_{p \in \mathcal{P}, p \leq e^H} X_p p^{-1/2-it}.$$

- For $t, t' \in [-h, h]$, the covariance of these random variables is given for $\Theta_p$ i.i.d. uniform on $[0, 2\pi]$

$$\sum_{p \in \mathcal{P}, p \leq e^H} p^{-1} \mathbb{E}[\sin(\Theta_p - t \log p) \sin(\Theta_p - t' \log p)]$$

$$= \frac{1}{2} \sum_{p \in \mathcal{P}, p \leq e^H} p^{-1} \cos((t - t') \log p).$$

# Correlation structure

- For distinct primes $p$, the phases $p^{-iUT}$ tend in law to i.i.d., uniform variables on the unit circle $X_p$.
- It is then natural to compare the previous random variables by

$$\Im \sum_{p \in \mathcal{P}, p \leq e^H} X_p p^{-1/2-it}.$$

- For $t, t' \in [-h, h]$, the covariance of these random variables is given for $\Theta_p$ i.i.d. uniform on $[0, 2\pi]$

$$\sum_{p \in \mathcal{P}, p \leq e^H} p^{-1} \mathbb{E}[\sin(\Theta_p - t \log p) \sin(\Theta_p - t' \log p)]$$

$$= \frac{1}{2} \sum_{p \in \mathcal{P}, p \leq e^H} p^{-1} \cos((t - t') \log p).$$

- If $(t-t')\log p$ is small the cosine is always close to 1, whereas, when it is large, it oscillates so it is natural to expect that it is close to 0 in average.
- Hence, the covariance is expected to be close to

$$\frac{1}{2} \sum_{p \in \mathcal{P}, p \leq e^{\min(H, (|t-t'|^{-1})}} p^{-1} \sim \frac{1}{2} \log(\min(|t-t'|^{-1}, (\log T)^{1-\delta}).$$

- The covariance is then logarithmic in the distance between the points, with a saturation when $|t-t'|$ is of order $(\log T)^{-(1-\delta)}$ with $\delta$ arbitrarily small. We then have roughly the same structure as for the $C\beta E$, with $n$ replaced by $\log T$. It is then natural to expect a similar result for the maximum of an interval of fixed size.

- If $(t - t') \log p$ is small the cosine is always close to 1, whereas, when it is large, it oscillates so it is natural to expect that it is close to 0 in average.
- Hence, the covariance is expected to be close to

$$\frac{1}{2} \sum_{p \in \mathcal{P}, p \leq e^{\min(H, (|t-t'|^{-1})}} p^{-1} \sim \frac{1}{2} \log(\min(|t-t'|^{-1}, (\log T)^{1-\delta}).$$

- The covariance is then logarithmic in the distance between the points, with a saturation when $|t - t'|$ is of order $(\log T)^{-(1-\delta)}$ with $\delta$ arbitrarily small. We then have roughly the same structure as for the $C\beta E$, with $n$ replaced by $\log T$. It is then natural to expect a similar result for the maximum of an interval of fixed size.

- If $(t - t') \log p$ is small the cosine is always close to 1, whereas, when it is large, it oscillates so it is natural to expect that it is close to 0 in average.
- Hence, the covariance is expected to be close to

$$\frac{1}{2} \sum_{p \in \mathcal{P}, p \leq e^{\min(H, (|t-t'|^{-1})}} p^{-1} \sim \frac{1}{2} \log(\min(|t-t'|^{-1}, (\log T)^{1-\delta}).$$

- The covariance is then logarithmic in the distance between the points, with a saturation when $|t - t'|$ is of order $(\log T)^{-(1-\delta)}$ with $\delta$ arbitrarily small. We then have roughly the same structure as for the $C\beta E$, with $n$ replaced by $\log T$. It is then natural to expect a similar result for the maximum of an interval of fixed size.

Thank you for your attention!