Applications of multiplicative chaos: extreme values of logarithmically correlated fields

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June 17 – Extreme values in Number Theory and Probability IHP

Based on joint work with T. Claeys, B. Fahs, G. Lambert; J. Junnila and E. Saksman

Goal of the talk

- Basic setting. X_N centered stoch. proc. on $\Omega \subset \mathbb{R}^d$:

$$\mathbb{E}X_N(x)X_N(y) = \min\big(\log|x-y|^{-1}, \sigma_N^2\big) + \mathcal{O}(1)$$

and $\sigma_N \to \infty$ as $N \to \infty$.

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- Tools. **Assume** that corresponding multiplicative chaos measure exists:

$$\int_A \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}} dx \xrightarrow{d} \mu_\gamma(A)$$

  for all $0 < \gamma < \sqrt{2d}$ and $A \subset \Omega$ Borel.

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  for all $0 < \gamma < \sqrt{2d}$ and $A \subset \Omega$ Borel.
- Other approaches/tools exist too (see Louis-Pierre's minicourse, Adam's talk, and Joseph's talk).

Examples (either known or conjectured)

- Riemann zeta (partly conjecture): For $\omega \sim \mathrm{Unif}[0,1]$ and $x \in \mathbb{R}$

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- Eigenvalue counting function of the GUE (CFLW): For $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ eigenvalues of a $N \times N$ GUE matrix (suitably normalized) and $x \in (-1,1)$

$$X_N(x) = \sqrt{2}\pi\left(\sum_{j=1}^{N} \mathbf{1}\{\lambda_j \leq x\} - N\int_{-1}^{x} \tfrac{2}{\pi}\sqrt{1-u^2}\,du\right).$$

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- The Ginibre ensemble (Bourgade, Dubach, and Hartung): For G_N $N \times N$ complex Ginibre (suitably normalized) and $z \in \mathbb{C}$, $|z| < 1$

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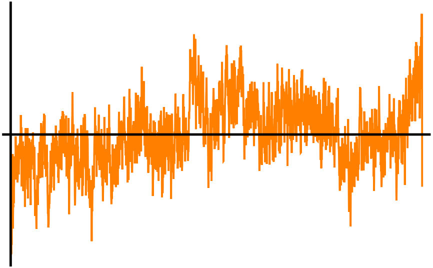
- See also Reda's talk.

What kind of beasts are these (fields in $d = 1, 2$)?



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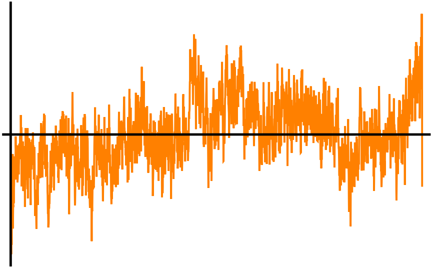

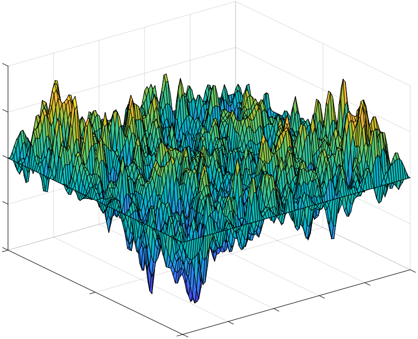

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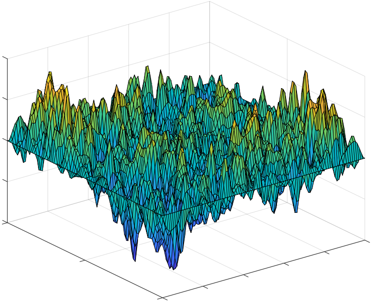

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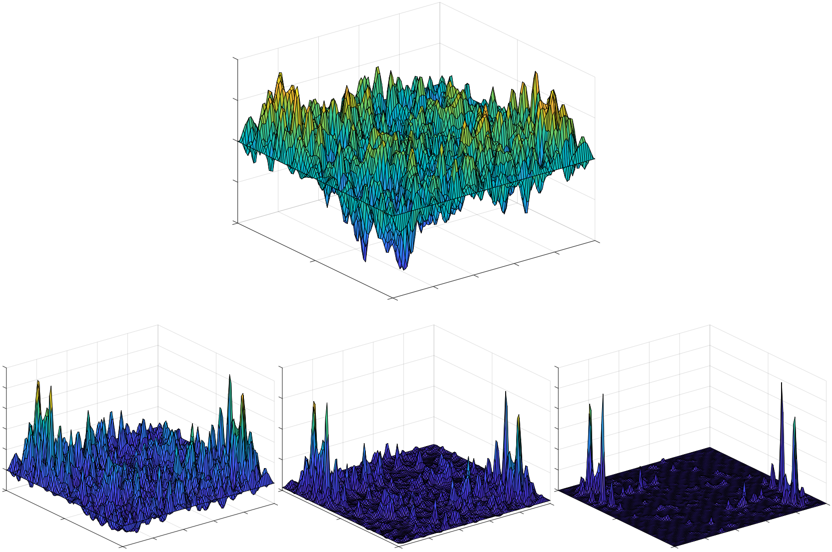

Thick points – heuristics based on Gaussian case

Much known about $\frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}}$ and extrema in Gaussian setting (goes back to Kahane 80s, Duplantier-Sheffield $\sim$ 2010, Berestycki $\sim$ 2015, ...).

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- Expected: $\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}$ lives on **"γ-thick points"** (random set):

$$\left\{x \in \Omega : X_N(x) \approx \gamma \mathbb{E}X_N(x)^2\right\}.$$

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- Expect: $\mu_\gamma = 0$ for $\gamma \geq \sqrt{2d}$ and no thick points to live on so

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Standing assumptions for X_N

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Assumption (Close to being a centered Gaussian of variance σ_N^2)

For each $\alpha > 0$ and $K \subset \Omega$ compact, $\exists c = c(\alpha, K), C = C(\alpha, K) > 0$:

$$c e^{\frac{\alpha^2}{2}\sigma_N^2} \leq \mathbb{E} e^{\alpha X_N(x)} \leq C e^{\frac{\alpha^2}{2}\sigma_N^2} \qquad \text{for all} \qquad x \in K$$

for some $\sigma_N \to \infty$ (independent of x, α, K).

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for some $\sigma_N \to \infty$ (independent of x, α, K).

Assumption (Non-triviality of chaos)

For $0 < \gamma < \sqrt{2d}$, $K \subset \Omega$ compact with non-empty interior, and some random variable $\mu_\gamma(K)$ which **is almost surely finite and positive**

$$\int_K \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}} dx \xrightarrow{d} \mu_\gamma(K).$$

Thick points – rigorous definitions and results

Define for $\gamma > 0$

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Theorem

For any $\epsilon > 0$, $0 < \gamma < \sqrt{2d}$ and $K \subset \Omega$ compact

$$\int_{(K\cap T_N(\gamma-\epsilon))\setminus T_N(\gamma+\epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx \xrightarrow{d} \mu_\gamma(K).$$

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Interpretation: only points x with $X_N(x) \approx \gamma \sigma_N^2 \approx \gamma \mathbb{E} X_N(x)^2$ contribute to $\frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}}$.

Thick points – proof

Proof.

$$
\begin{aligned}
\mathbb{E}\int_{K\setminus T_N(\gamma-\epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx &= \mathbb{E}\int_K \mathbf{1}\{X_N(x) < (\gamma-\epsilon)\sigma_N^2\}\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx \\
&\leq e^{\epsilon(\gamma-\epsilon)\sigma_N^2}\int_K \frac{\mathbb{E}e^{(\gamma-\epsilon)X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx \\
&\leq \frac{C(\gamma-\epsilon,K)}{c(\gamma,K)}|K|e^{-\frac{\epsilon^2}{2}\sigma_N^2} \to 0.
\end{aligned}
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Similarly (using again approx Gaussian assumption)

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Thus for some $\mathcal{E}_N$ with $\mathbb{E}|\mathcal{E}_N|\to 0$ (can thus use Slutsky's theorem)

$$\int_{(K\cap T_N(\gamma-\epsilon))\setminus T_N(\gamma+\epsilon)}\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx = \int_K \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx + \mathcal{E}_N. \qquad \square$$

Lower bound for the maximum

Corollary

For any $\epsilon > 0$ and $K \subset \Omega$ compact with non-empty interior.

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{x\in K} X_N(x) \geq (\sqrt{2d} - \epsilon)\sigma_N^2\right) = 1.$$

Lower bound for the maximum: proof

Proof.

Let $\alpha < \gamma < \sqrt{2d}$ and note that for every $\epsilon > 0$ and $K \subset \Omega$ compact

$$\begin{aligned}\mathbb{P}\left(\max_{x\in K} X_N(x) \geq \alpha\sigma_N^2\right) &\geq \mathbb{P}\left(T_N(\alpha)\cap K \neq \emptyset\right)\\ &\geq \mathbb{P}\left(\int_{T_N(\alpha)\cap K} \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}dx > \epsilon\right)\\ &\to \mathbb{P}(\mu_\gamma(K) > \epsilon).\end{aligned}$$

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If K has non-empty interior, then (non-triviality of chaos assumption)

$$\liminf_{N\to\infty} \mathbb{P}\left(\max_{x \in K} X_N(x) \geq \alpha\sigma_N^2\right) \geq \mathbb{P}(\mu_\gamma(K) > \epsilon) \to 1$$

as $\epsilon \to 0$. □

Upper bound for the maximum: assumptions

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Assumption (Local scale of regularity)

There exist deterministic $C, c > 0$, such that for each $x \in \Omega$, there exists a (possibly random) compact $K_x \subset \Omega$ with $|K_x| \geq ce^{-d\sigma_N^2}$ and

$$X_N(t) \geq X_N(x) - C \qquad \textit{for all} \qquad t \in K_x.$$

Upper bound for the maximum

Theorem

For any $\epsilon > 0$ *and* $K \subset \Omega$ *compact with non-empty interior.*

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{x\in K} X_N(x) \leq (\sqrt{2d} + \epsilon)\sigma_N^2\right) = 1.$$

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- Assume that there is some $x_* \in K$ such that $X_N(x_*) \geq (\sqrt{2d} + \epsilon)\sigma_N^2$.

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- By regularity assumption and approx Gaussian assumption

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\begin{aligned}
\int_K \frac{e^{(\sqrt{2d}-\epsilon)X_N(x)}}{\mathbb{E}e^{(\sqrt{2d}-\epsilon)X_N(x)}}dx &\geq e^{(\sqrt{2d}-\epsilon)[(\sqrt{2d}+\epsilon)\sigma_N^2 - C]} \int_{K_{x_*}} \frac{1}{\mathbb{E}e^{(\sqrt{2d}-\epsilon)X_N(x)}}dx \\
&\geq \widetilde{C} e^{(2d-\epsilon^2)\sigma_N^2} e^{-\frac{(\sqrt{2d}-\epsilon)^2}{2}\sigma_N^2} e^{-d\sigma_N^2} \\
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- By assumption of non-triviality (finiteness) of chaos, the probability of this tends to zero.

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$\square$

Again, other approaches exist.

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Also various kinds of complex multiplicative chaos exists: formally $e^{\gamma X(x)+i\beta Y(x)}$ where X, Y log-correlated.

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Theorem (Saksman, W. 2016)

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Theorem (Junnila, Saksman, W. 2018)

For $\sigma, \widetilde{\sigma}$ independent realizations of a spin configuration of the critical Ising model with $+$ b.c. on $\Omega \cap \delta\mathbb{Z}^2$, as $\delta \to 0$

$$\delta^{-1/4}\sigma(x)\widetilde{\sigma}(x) \xrightarrow{d} f_\Omega(x)\mathrm{Re}\text{``}e^{i\frac{1}{\sqrt{2}}X_\Omega(x)}\text{''}$$

for a suitable deterministic f_Ω and X_Ω being the GFF on Ω.

Challenges/open questions

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- What is the analogue of thick points for complex multiplicative chaos?
- In other words, what x do $\zeta(\frac{1}{2} + i\omega T + ix)$ and $\sigma(x)\widetilde{\sigma}(x)$ live on?