Applications of multiplicative chaos: extreme values of logarithmically correlated fields

Christian Webb

Aalto University, Finland

June 17 – Extreme values in Number Theory and Probability IHP

Based on joint work with T. Claeys, B. Fahs, G. Lambert; J. Junnila and E. Saksman
Goal of the talk

- **Basic setting.** $X_N$ centered stochastic process on $\Omega \subset \mathbb{R}^d$:

\[
\mathbb{E}X_N(x)X_N(y) = \min \left( \log |x - y|^{-1}, \sigma_N^2 \right) + \mathcal{O}(1)
\]

and $\sigma_N \to \infty$ as $N \to \infty$. 

- Logarithmically correlated field – though not necessarily Gaussian!

- **Main questions.**
  - Understand extrema of $X_N$: e.g. $\max_{x \in X_N} x$ as $N \to \infty$?

- **Tools.**
  - Assume that corresponding multiplicative chaos measure exists:
    \[
    \int_A e^{\gamma X_N(x)} \mathbb{E}e^{\gamma X_N(x)} \, dx \to \mu_{\gamma}(A)
    \] 
    for all $0 < \gamma < \sqrt{2d}$ and $A \subset \Omega$ Borel.

- Other approaches/tools exist too (see Louis-Pierre's minicourse, Adam's talk, and Joseph's talk).
Goal of the talk

- Basic setting. $X_N$ centered stoch. proc. on $\Omega \subset \mathbb{R}^d$:

\[
\mathbb{E}X_N(x)X_N(y) = \min \left( \log |x - y|^{-1}, \sigma_N^2 \right) + O(1)
\]

and $\sigma_N \to \infty$ as $N \to \infty$.

- Logarithmically correlated field – though not necessarily Gaussian!

Other approaches/tools exist too (see Louis-Pierre's minicourse, Adam's talk, and Joseph's talk).
Goal of the talk

- Basic setting. $X_N$ centered stochastic process on $\Omega \subset \mathbb{R}^d$:

$$\mathbb{E}X_N(x)X_N(y) = \min\left( \log |x - y|^{-1}, \sigma_N^2 \right) + O(1)$$

and $\sigma_N \to \infty$ as $N \to \infty$.

- Logarithmically correlated field – though not necessarily Gaussian!

- Main questions. Understand extrema of $X_N$: e.g. $\max_x X_N(x)$ as $N \to \infty$?
Goal of the talk

• Basic setting. \( X_N \) centered stoch. proc. on \( \Omega \subset \mathbb{R}^d \):

\[
\mathbb{E}X_N(x)X_N(y) = \min\left( \log |x - y|^{-1}, \sigma_N^2 \right) + O(1)
\]

and \( \sigma_N \to \infty \) as \( N \to \infty \).

• Logarithmically correlated field – though not necessarily Gaussian!

• Main questions. Understand extrema of \( X_N \): e.g. \( \max_x X_N(x) \) as \( N \to \infty \)?

• Tools. **Assume** that corresponding multiplicative chaos measure exists:

\[
\int_A \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}} \, dx \xrightarrow{d} \mu_\gamma(A)
\]

for all \( 0 < \gamma < \sqrt{2d} \) and \( A \subset \Omega \) Borel.
Goal of the talk

- **Basic setting.** $X_N$ centered stoch. proc. on $\Omega \subset \mathbb{R}^d$:

\[
\mathbb{E}X_N(x)X_N(y) = \min \left( \log |x - y|^{-1}, \sigma_N^2 \right) + O(1)
\]

and $\sigma_N \to \infty$ as $N \to \infty$.

- **Logarithmically correlated field** – though not necessarily Gaussian!

- **Main questions.** Understand extrema of $X_N$: e.g. $\max_x X_N(x)$ as $N \to \infty$?

- **Tools.** *Assume* that corresponding multiplicative chaos measure exists:

\[
\int_A \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}} \, dx \overset{d}{\to} \mu_{\gamma}(A)
\]

for all $0 < \gamma < \sqrt{2d}$ and $A \subset \Omega$ Borel.

- **Other approaches/tools** exist too (see Louis-Pierre’s minicourse, Adam’s talk, and Joseph’s talk).
Examples (either known or conjectured)

- **Riemann zeta (partly conjecture):** For $\omega \sim \text{Unif}[0, 1]$ and $x \in \mathbb{R}$
  \[
  X_N(x) = \sqrt{2} \log |\zeta(\frac{1}{2} + i\omega N + ix)|
  \]
Examples (either known or conjectured)

- **Riemann zeta (partly conjecture):** For $\omega \sim \text{Unif}[0, 1]$ and $x \in \mathbb{R}$

  \[
  X_N(x) = \sqrt{2} \log \left| \zeta \left( \frac{1}{2} + \delta_N + i\omega N + ix \right) \right|
  \]
Examples (either known or conjectured)

• Riemann zeta (partly conjecture): For $\omega \sim \text{Unif}[0, 1]$ and $x \in \mathbb{R}$

$$X_N(x) = \sqrt{2} \log |\zeta(\frac{1}{2} + \delta_N + i\omega N + ix)|$$

• Eigenvalue counting function of the GUE (CFLW): For $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ eigenvalues of a $N \times N$ GUE matrix (suitably normalized) and $x \in (-1, 1)$

$$X_N(x) = \sqrt{2\pi} \left( \sum_{j=1}^{N} 1\{\lambda_j \leq x\} - N \int_{-1}^{x} \frac{2}{\pi} \sqrt{1 - u^2} du \right).$$

• See also Reda’s talk.
Examples (either known or conjectured)

• **Riemann zeta (partly conjecture):** For $\omega \sim \text{Unif}[0, 1]$ and $x \in \mathbb{R}$

$$X_N(x) = \sqrt{2} \log |\zeta(\frac{1}{2} + \delta_N + i\omega N + ix)|$$

• **Eigenvalue counting function of the GUE (CFLW):** For $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ eigenvalues of a $N \times N$ GUE matrix (suitably normalized) and $x \in (-1, 1)$

$$X_N(x) = \sqrt{2\pi} \left( \sum_{j=1}^{N} 1\{\lambda_j \leq x\} - N \int_{-1}^{x} \frac{2}{\pi} \sqrt{1 - u^2} du \right).$$

• **The Ginibre ensemble (Bourgade, Dubach, and Hartung):** For $G_N$ $N \times N$ complex Ginibre (suitably normalized) and $z \in \mathbb{C}, |z| < 1$

$$X_N(z) = \sqrt{2} \log |\det(z - G_N)| - \frac{1}{\sqrt{2}} N(|z|^2 - 1)$$

• See also Reda's talk.
Examples (either known or conjectured)

- **Riemann zeta (partly conjecture):** For $\omega \sim \text{Unif}[0, 1]$ and $x \in \mathbb{R}$

  $X_N(x) = \sqrt{2} \log |\zeta(\frac{1}{2} + \delta_N + i\omega N + ix)|$

- **Eigenvalue counting function of the GUE (CFLW):** For $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ eigenvalues of a $N \times N$ GUE matrix (suitably normalized) and $x \in (-1, 1)$

  $X_N(x) = \sqrt{2\pi} \left( \sum_{j=1}^{N} \mathbf{1}\{\lambda_j \leq x\} - N \int_{-1}^{x} \frac{2}{\pi} \sqrt{1 - u^2} du \right)$

- **The Ginibre ensemble (Bourgade, Dubach, and Hartung):** For $G_N$ $N \times N$ complex Ginibre (suitably normalized) and $z \in \mathbb{C}$, $|z| < 1$

  $X_N(z) = \sqrt{2} \log |\det(z - G_N)| - \frac{1}{\sqrt{2}} N(|z|^2 - 1)$

- See also Reda's talk.
What kind of beasts are these (fields in $d = 1, 2$)?
What kind of beasts are these (fields in \( d = 1, 2 \))?
What kind of beasts are these (realizations of the field and chaos for $\gamma = 0.5, 1, 2$)?
What kind of beasts are these (realizations of the field and chaos for $\gamma = 0.5, 1, 2$)?
Thick points – heuristics based on Gaussian case

Much known about $\frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}}$ and extrema in Gaussian setting (goes back to Kahane 80s, Duplantier-Sheffield ~ 2010, Berestycki ~ 2015, ...).
Thick points – heuristics based on Gaussian case

Much known about \( \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \) and extrema in Gaussian setting (goes back to Kahane 80s, Duplantier-Sheffield ∼ 2010, Berestycki ∼ 2015, ...).

- Expected: \( \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \) lives on “\( \gamma \)-thick points” (random set):

\[
\{ x \in \Omega : X_N(x) \approx \gamma \mathbb{E} X_N(x)^2 \}.
\]
Thick points – heuristics based on Gaussian case

Much known about $\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}$ and extrema in Gaussian setting (goes back to Kahane 80s, Duplantier-Sheffield \sim 2010, Berestycki \sim 2015, ...).

• Expected: $\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}$ lives on “$\gamma$-thick points” (random set):

$$\{x \in \Omega : X_N(x) \approx \gamma \mathbb{E}X_N(x)^2\}.$$  

• Interpretation: $\mu_\gamma$ encodes “extreme level sets”.

6/16
Thick points – heuristics based on Gaussian case

Much known about $\frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}}$ and extrema in Gaussian setting (goes back to Kahane 80s, Duplantier-Sheffield $\sim$ 2010, Berestycki $\sim$ 2015, ...).

- Expected: $\frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}}$ lives on “$\gamma$-thick points” (random set):

\[ \{ x \in \Omega : X_N(x) \approx \gamma \mathbb{E} X_N(x)^2 \} . \]

- Interpretation: $\mu_\gamma$ encodes “extreme level sets”.
- Expect: $\mu_\gamma$ non-trivial for $\gamma < \sqrt{2d}$, so $\gamma$-thick points exist and

\[ \max_x X_N(x) \geq \sqrt{2d \mathbb{E} X_N(x)^2} . \]
Thick points – heuristics based on Gaussian case

Much known about $\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}$ and extrema in Gaussian setting (goes back to Kahane 80s, Duplantier-Sheffield $\sim$ 2010, Berestycki $\sim$ 2015, ...).

- Expected: $\frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}}$ lives on “γ-thick points” (random set):

$$\{x \in \Omega : X_N(x) \approx \gamma \mathbb{E} X_N(x)^2\}.$$

- Interpretation: $\mu_\gamma$ encodes “extreme level sets”.
- Expect: $\mu_\gamma$ non-trivial for $\gamma < \sqrt{2d}$, so γ-thick points exist and

$$\max_x X_N(x) \geq \sqrt{2d \mathbb{E} X_N(x)^2}.$$ 

- Expect: $\mu_\gamma = 0$ for $\gamma \geq \sqrt{2d}$ and no thick points to live on so

$$\max_x X_N(x) \leq \sqrt{2d \mathbb{E} X_N(x)^2}.$$
Standing assumptions for $X_N$

Before going to rigorous claims, need assumptions on $X_N$. 

Assumption (Close to being a centered Gaussian of variance $\sigma^2$)

For each $\alpha > 0$ and $K \subset \Omega$ compact, $\exists c = c(\alpha, K)$, $C = C(\alpha, K) > 0$:

$$ce^{\alpha^2/2\sigma^2} \leq Ee^{\alpha X_N(x)} \leq Ce^{\alpha^2/2\sigma^2}$$

for all $x \in K$ for some $\sigma_N \to \infty$ (independent of $x, \alpha, K$).

Assumption (Non-triviality of chaos)

For $0 < \gamma < \sqrt{2d}$, $K \subset \Omega$ compact with non-empty interior, and some random variable $\mu_\gamma(K)$ which is almost surely finite and positive

$$\int_K e^{\gamma X_N(x)} dx \to \mu_\gamma(K).$$
Standing assumptions for $X_N$

Before going to rigorous claims, need assumptions on $X_N$.

Assumption (Close to being a centered Gaussian of variance $\sigma_N^2$)

For each $\alpha > 0$ and $K \subset \Omega$ compact, \( \exists c = c(\alpha, K), C = C(\alpha, K) > 0 : \)

\[
    ce^{\frac{\alpha^2}{2} \sigma_N^2} \leq \mathbb{E} e^{\alpha X_N(x)} \leq Ce^{\frac{\alpha^2}{2} \sigma_N^2} 
\]

for all $x \in K$ for some $\sigma_N \to \infty$ (independent of $x, \alpha, K$).
Standing assumptions for $X_N$

Before going to rigorous claims, need assumptions on $X_N$.

**Assumption (Close to being a centered Gaussian of variance $\sigma_N^2$)**

For each $\alpha > 0$ and $K \subset \Omega$ compact, $\exists c = c(\alpha, K), C = C(\alpha, K) > 0$:

$$ce^{\frac{\alpha^2}{2}\sigma_N^2} \leq \mathbb{E}e^{\alpha X_N(x)} \leq Ce^{\frac{\alpha^2}{2}\sigma_N^2}$$

for all $x \in K$

for some $\sigma_N \to \infty$ (independent of $x, \alpha, K$).

**Assumption (Non-triviality of chaos)**

For $0 < \gamma < \sqrt{2d}$, $K \subset \Omega$ compact with non-empty interior, and some random variable $\mu_\gamma(K)$ which is almost surely finite and positive

$$\int_K \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}} \, dx \overset{d}{\to} \mu_\gamma(K).$$
Thick points – rigorous definitions and results

Define for $\gamma > 0$

$$T_N(\gamma) = \{ x \in \Omega : X_N(x) \geq \gamma \sigma_N^2 \}.$$
Thick points – rigorous definitions and results

Define for $\gamma > 0$

$$T_N(\gamma) = \{x \in \Omega : X_N(x) \geq \gamma \sigma_N^2\}.$$ 

**Theorem**

For any $\epsilon > 0$, $0 < \gamma < \sqrt{2d}$ and $K \subset \Omega$ compact

$$\int_{(K \cap T_N(\gamma - \epsilon)) \setminus T_N(\gamma + \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E}e^{\gamma X_N(x)}} \, dx \xrightarrow{d} \mu_\gamma(K).$$
Thick points – rigorous definitions and results

Define for $\gamma > 0$

$$T_N(\gamma) = \{ x \in \Omega : X_N(x) \geq \gamma \sigma^2_N \}.$$ 

Theorem

For any $\epsilon > 0$, $0 < \gamma < \sqrt{2d}$ and $K \subset \Omega$ compact

$$\int_{(K \cap T_N(\gamma - \epsilon)) \setminus T_N(\gamma + \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \xrightarrow{d} \mu_\gamma(K).$$

Interpretation: only points $x$ with $X_N(x) \approx \gamma \sigma^2_N \approx \gamma \mathbb{E} X_N(x)^2$ contribute to $\frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}}$. 

Thick points – proof

Proof.

\[ \mathbb{E} \int_{K \setminus T_N(\gamma - \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx = \mathbb{E} \int_K 1\{X_N(x) < (\gamma - \epsilon)\sigma^2_N\} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \]

\[ \leq e^{\epsilon(\gamma - \epsilon)\sigma^2_N} \int_K \frac{\mathbb{E} e^{(\gamma - \epsilon)X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \]

\[ \leq \frac{C(\gamma - \epsilon, K)}{c(\gamma, K)} |K| e^{-\frac{\epsilon^2}{2}\sigma^2_N} \to 0. \]
Thick points – proof

Proof.

\[
\mathbb{E} \int_{K \setminus T_N(\gamma - \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx = \mathbb{E} \int_K 1\{X_N(x) < (\gamma - \epsilon)\sigma_N^2\} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \\
\leq e^{\epsilon(\gamma - \epsilon)\sigma_N^2} \int_K \frac{\mathbb{E} e^{(\gamma - \epsilon)X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \\
\leq \frac{C(\gamma - \epsilon, K)}{c(\gamma, K)} |K| e^{-\frac{\epsilon^2}{2} \sigma_N^2} \to 0.
\]

Similarly (using again approx Gaussian assumption)

\[
\mathbb{E} \int_{K \cap T_N(\gamma + \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \to 0.
\]
Thick points – proof

Proof.

\[
\mathbb{E} \int_{K \setminus T_N(\gamma - \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx = \mathbb{E} \int_K \mathbf{1}\{X_N(x) < (\gamma - \epsilon)\sigma_N^2\} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \\
\leq e^{\epsilon(\gamma - \epsilon)\sigma_N^2} \int_K \frac{\mathbb{E} e^{(\gamma - \epsilon)X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \\
\leq \frac{C(\gamma - \epsilon, K)}{c(\gamma, K)} |K| e^{\frac{-\epsilon^2}{2} \sigma_N^2} \to 0.
\]

Similarly (using again approx Gaussian assumption)

\[
\mathbb{E} \int_{K \cap T_N(\gamma + \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx \to 0.
\]

Thus for some \( \mathcal{E}_N \) with \( \mathbb{E} |\mathcal{E}_N| \to 0 \) (can thus use Slutsky’s theorem)

\[
\int_{(K \cap T_N(\gamma - \epsilon)) \setminus T_N(\gamma + \epsilon)} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx = \int_K \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx + \mathcal{E}_N. \quad \square
\]
Lower bound for the maximum

Corollary

For any $\epsilon > 0$ and $K \subset \Omega$ compact with non-empty interior.

$$\lim_{N \to \infty} \mathbb{P} \left( \max_{x \in K} X_N(x) \geq (\sqrt{2d} - \epsilon)\sigma_N^2 \right) = 1.$$
Proof.

Let $\alpha < \gamma < \sqrt{2d}$ and note that for every $\epsilon > 0$ and $K \subset \Omega$ compact

$$
\mathbb{P} \left( \max_{x \in K} X_N(x) \geq \alpha \sigma_N^2 \right) \geq \mathbb{P} \left( T_N(\alpha) \cap K \neq \emptyset \right)
$$

$$
\geq \mathbb{P} \left( \int_{T_N(\alpha) \cap K} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx > \epsilon \right)
$$

$$
\rightarrow \mathbb{P} \left( \mu_\gamma(K) > \epsilon \right).
$$
Lower bound for the maximum: proof

Proof.

Let $\alpha < \gamma < \sqrt{2d}$ and note that for every $\epsilon > 0$ and $K \subset \Omega$ compact

$$\mathbb{P} \left( \max_{x \in K} X_N(x) \geq \alpha \sigma_N^2 \right) \geq \mathbb{P} \left( T_N(\alpha) \cap K \neq \emptyset \right)$$

$$\geq \mathbb{P} \left( \int_{T_N(\alpha) \cap K} \frac{e^{\gamma X_N(x)}}{\mathbb{E} e^{\gamma X_N(x)}} \, dx > \epsilon \right)$$

$$\rightarrow \mathbb{P}(\mu_\gamma(K) > \epsilon).$$

If $K$ has non-empty interior, then (non-triviality of chaos assumption)

$$\liminf_{N \to \infty} \mathbb{P} \left( \max_{x \in K} X_N(x) \geq \alpha \sigma_N^2 \right) \geq \mathbb{P}(\mu_\gamma(K) > \epsilon) \rightarrow 1$$

as $\epsilon \to 0$. \qed
Upper bound for the maximum: assumptions

Upper bound requires $X_N$ to be regular enough on scale $e^{-\sigma_N^2}$ – need further assumptions.
Upper bound for the maximum: assumptions

Upper bound requires $X_N$ to be regular enough on scale $e^{-\sigma_N^2}$ – need further assumptions.

**Assumption (Local scale of regularity)**

There exist deterministic $C, c > 0$, such that for each $x \in \Omega$, there exists a (possibly random) compact $K_x \subset \Omega$ with $|K_x| \geq ce^{-d\sigma^2_N}$ and

$$X_N(t) \geq X_N(x) - C \quad \text{for all} \quad t \in K_x.$$
Upper bound for the maximum

**Theorem**

For any $\epsilon > 0$ and $K \subset \Omega$ compact with non-empty interior.

$$\lim_{N \to \infty} \mathbb{P} \left( \max_{x \in K} X_N(x) \leq (\sqrt{2d} + \epsilon) \sigma_N^2 \right) = 1.$$
Upper bound for the maximum: proof

Proof.

- Assume that there is some $x_* \in K$ such that $X_N(x_*) \geq (\sqrt{2d} + \epsilon) \sigma_N^2$. 

- By regularity assumption and approx Gaussian assumption

$$
\int_K e^{\left(\sqrt{2d} - \epsilon\right) X_N(x_*)} \leq e^{\left(\sqrt{2d} - \epsilon\right) \left(\sqrt{2d} + \epsilon\right) \sigma_N^2 - C} \int_K x_*^1 e^{\left(\sqrt{2d} - \epsilon\right) X_N(x_*)} dx \geq \tilde{C} e^{2d - \epsilon^2} \sigma_N^2 e^{-\left(\sqrt{2d} - \epsilon\right)^2} \sigma_N^2 \rightarrow \infty .
$$

- By assumption of non-triviality (finiteness) of chaos, the probability of this tends to zero.

Again, other approaches exist.
Proof.

- Assume that there is some $x_\ast \in K$ such that $X_N(x_\ast) \geq (\sqrt{2d} + \epsilon)\sigma_N^2$.
- By regularity assumption and approx Gaussian assumption

$$
\int_K \frac{e^{(\sqrt{2d}-\epsilon)X_N(x)}}{\mathbb{E}e^{(\sqrt{2d}-\epsilon)X_N(x)}} \, dx \geq e^{(\sqrt{2d}-\epsilon)[(\sqrt{2d}+\epsilon)\sigma_N^2 - C]} \int_{K_{x_\ast}} \frac{1}{\mathbb{E}e^{(\sqrt{2d}-\epsilon)X_N(x)}} \, dx
$$

$$
\geq \tilde{C} e^{(2d-\epsilon^2)\sigma_N^2} e^{-(\frac{\sqrt{2d}-\epsilon)^2}{2}\sigma_N^2} e^{-d\sigma_N^2}
$$

$$
\geq \tilde{C} e^{(\sqrt{2d}\epsilon - \frac{\epsilon^2}{2})\sigma_N^2} \to \infty.
$$
Upper bound for the maximum: proof

Proof.

- Assume that there is some \( x_\ast \in K \) such that \( X_N(x_\ast) \geq (\sqrt{2d} + \epsilon)\sigma_N^2 \).
- By regularity assumption and approx Gaussian assumption

\[
\int_K \frac{e^{(\sqrt{2d} - \epsilon)X_N(x)}}{\mathbb{E}e^{(\sqrt{2d} - \epsilon)X_N(x)}} \, dx \geq e^{(\sqrt{2d} - \epsilon)[(\sqrt{2d} + \epsilon)\sigma_N^2 - C]} \int_{K_{x_\ast}} \frac{1}{\mathbb{E}e^{(\sqrt{2d} - \epsilon)X_N(x)}} \, dx
\]

\[
\geq \bar{C} e^{(2d - \epsilon^2)\sigma_N^2} e^{-\frac{(\sqrt{2d} - \epsilon)^2}{2} \sigma_N^2} e^{-d\sigma_N^2}
\]

\[
\geq \bar{C} e^{(\sqrt{2d} - \frac{\epsilon^2}{2})\sigma_N^2} \to \infty.
\]

- By assumption of non-triviality (finiteness) of chaos, the probability of this tends to zero.
Upper bound for the maximum: proof

Proof.

- Assume that there is some $x^* \in K$ such that $X_N(x^*) \geq (\sqrt{2d} + \epsilon)\sigma_N^2$.
- By regularity assumption and approx Gaussian assumption

\[
\int_K \frac{e^{\left(\sqrt{2d}-\epsilon\right)X_N(x)}}{E e^{\left(\sqrt{2d}-\epsilon\right)X_N(x)}} \, dx \geq e^{\left(\sqrt{2d}-\epsilon\right)[\left(\sqrt{2d}+\epsilon\right)\sigma_N^2 - C]} \int_{K_{x^*}} \frac{1}{E e^{\left(\sqrt{2d}-\epsilon\right)X_N(x)}} \, dx
\]

\[
\geq \tilde{C} e^{(2d-\epsilon^2)\sigma_N^2 - \frac{(\sqrt{2d}-\epsilon)^2}{2}\sigma_N^2} e^{-d\sigma_N^2}
\]

\[
\geq \tilde{C} e^{\left(\sqrt{2d}\epsilon - \frac{\epsilon^2}{2}\right)\sigma_N^2} \to \infty.
\]

- By assumption of non-triviality (finiteness) of chaos, the probability of this tends to zero.

Again, other approaches exist.
Also various kinds of complex multiplicative chaos exists: formally
\[ e^{\gamma X(x) + i\beta Y(x)} \] where \( X, Y \) log-correlated.

Theorem (Saksman, W. 2016)

For \( \omega \sim \text{Unif}[0, 1] \), as \( T \to \infty \),
\[ \zeta(\frac{1}{2} + i\omega T + ix) \to e^{\gamma X(x) + i\beta Y(x)} \]
for suitable correlated non-Gaussian log-correlated \( X, Y \).

Theorem (Junnila, Saksman, W. 2018)

For \( \sigma, \tilde{\sigma} \) independent realizations of a spin configuration of the critical Ising model with \(+\) b.c. on \( \Omega \cap \delta \mathbb{Z}^2 \), as \( \delta \to 0 \)
\[ \delta^{-1/4} \sigma(x) \tilde{\sigma}(x) \to f_{\Omega}(x) \Re "e^{i 1/\sqrt{2} X_{\Omega}(x)}" \]
for a suitable deterministic \( f_{\Omega} \) and \( X_{\Omega} \) being the GFF on \( \Omega \).
Also various kinds of complex multiplicative chaos exists: formally \( e^{\gamma X(x) + i\beta Y(x)} \) where \( X, Y \) log-correlated.

**Theorem (Saksman, W. 2016)**

*For \( \omega \sim \text{Unif}[0, 1] \), as \( T \to \infty \),

\[
\zeta\left(\frac{1}{2} + i\omega T + ix\right) \xrightarrow{d} e^{X(x) + iY(x)}
\]

for suitable correlated non-Gaussian log-cor \( X, Y \).
Also various kinds of complex multiplicative chaos exists: formally $e^{\gamma X(x) + i\beta Y(x)}$ where $X$, $Y$ log-correlated.

Theorem (Saksman, W. 2016)

For $\omega \sim \text{Unif}[0, 1]$, as $T \to \infty$,

$$\zeta\left(\frac{1}{2} + i\omega T + ix\right) \sim \mathcal{D} \xrightarrow{d} e^{X(x) + iY(x)}$$

for suitable correlated non-Gaussian log-cor $X$, $Y$.

Theorem (Junnila, Saksman, W. 2018)

For $\sigma, \tilde{\sigma}$ independent realizations of a spin configuration of the critical Ising model with $+ b.c.$ on $\Omega \cap \delta\mathbb{Z}^2$, as $\delta \to 0$

$$\delta^{-1/4} \sigma(x)\tilde{\sigma}(x) \xrightarrow{d} f_\Omega(x) \text{Re}\left\{e^{i\frac{1}{\sqrt{2}}X_\Omega(x)}\right\}$$

for a suitable deterministic $f_\Omega$ and $X_\Omega$ being the GFF on $\Omega$. 
Challenges/open questions

- What is the analogue of thick points for complex multiplicative chaos?
Challenges/open questions

- What is the analogue of thick points for complex multiplicative chaos?
- In other words, what $x$ do $\zeta(\frac{1}{2} + i\omega T + ix)$ and $\sigma(x)\tilde{\sigma}(x)$ live on?