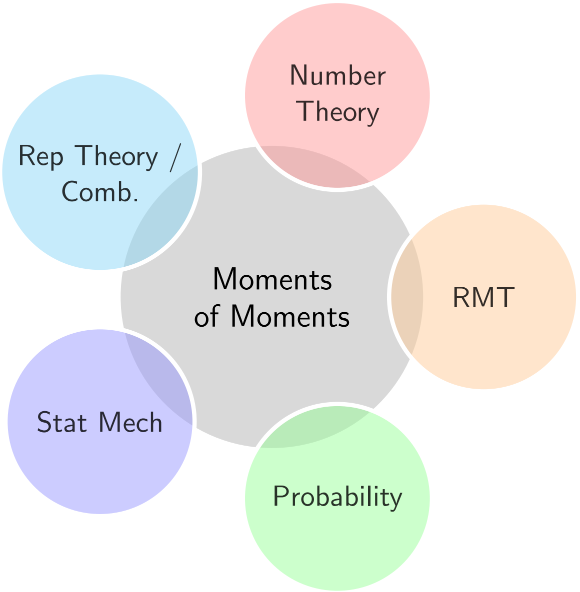
Moments of Moments

Emma Bailey
University of Bristol

Joint work with Jon Keating

arXiv:1807.06605 (to appear in CMP)


Number Theory
Rep Theory / Comb.
Moments of Moments
RMT
Stat Mech
Probability

Number Theoretic Motivation

Number Theoretic Motivation

Consider moments of the zeta function,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt.$$

Number Theoretic Motivation

Consider moments of the zeta function,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt.$$

Conjecture

For $\beta \in \mathbb{R}^+$,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt \sim f(\beta)c_\zeta(\beta)\left(\log \tfrac{T}{2\pi}\right)^{\beta^2},$$

as $T \to \infty$.

Number Theoretic Motivation

Consider moments of the zeta function,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt.$$

Conjecture

For $\beta \in \mathbb{R}^+$,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt \sim f(\beta) c_\zeta(\beta) \left(\log \tfrac{T}{2\pi}\right)^{\beta^2},$$

as $T \to \infty$.

- $f(\beta)$ is a known arithmetic function

Number Theoretic Motivation

Consider moments of the zeta function,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt.$$

Conjecture

For $\beta \in \mathbb{R}^+$,

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt \sim f(\beta) c_\zeta(\beta) \left(\log \tfrac{T}{2\pi}\right)^{\beta^2},$$

as $T \to \infty$.

- $f(\beta)$ is a known arithmetic function
- $c_\zeta(\beta)$ is another function depending on β.

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Then recall

$$\frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2\beta} dt$$

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A,\theta) = \det(I - Ae^{-i\theta}).$$

Instead

$$\int_{\mathrm{U}(N)} |P_N(A,\theta)|^{2\beta} dA$$

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A,\theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

$$\int_{\mathrm{U}(N)} |P_N(A,\theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{(\Gamma(j+\beta))^2}$$

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

$$\int_{\mathrm{U}(N)} |P_N(A, \theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2\beta)}{(\Gamma(j + \beta))^2} \sim c_U(\beta) N^{\beta^2}$$

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A,\theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

$$\int_{\mathrm{U}(N)} |P_N(A,\theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{(\Gamma(j+\beta))^2} \sim c_U(\beta) N^{\beta^2}$$

where

$$c_U(\beta) = \frac{G^2(\beta+1)}{G(2\beta+1)},$$

with $G(s)$ the Barnes G-function and if $\beta \in \mathbb{N}$,

$$c_U(\beta) = \prod_{j=0}^{\beta-1} \frac{j!}{(j+\beta)!}.$$

For $A \in \mathrm{CUE}_N$ ($A \in \mathrm{U}(N)$ with Haar measure) set

$$P_N(A,\theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

$$\int_{\mathrm{U}(N)} |P_N(A,\theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{(\Gamma(j+\beta))^2} \sim c_U(\beta) N^{\beta^2}$$

where

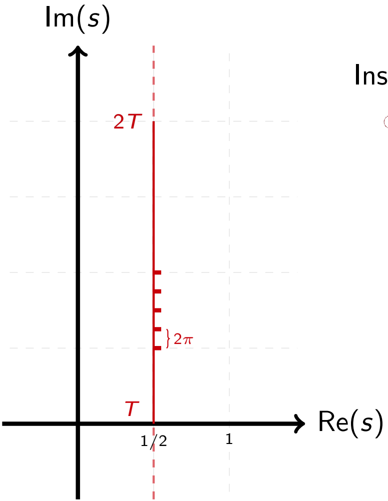
$$c_U(\beta) = \frac{G^2(\beta+1)}{G(2\beta+1)},$$

with $G(s)$ the Barnes G-function and if $\beta \in \mathbb{N}$,

$$c_U(\beta) = \prod_{j=0}^{\beta-1} \frac{j!}{(j+\beta)!}.$$

Conjecture: $c_U(\beta) = c_\zeta(\beta)$.

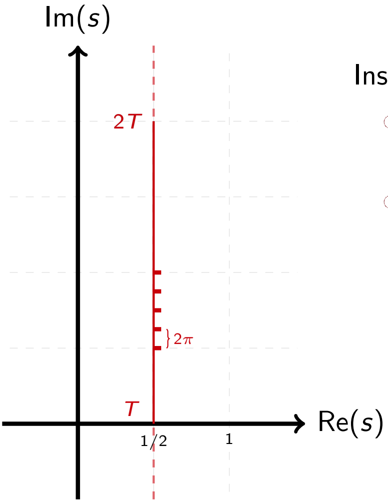
Short vs long intervals



Instead

- Consider fluctuations of moments of $\zeta(1/2+it)$ over short ranges

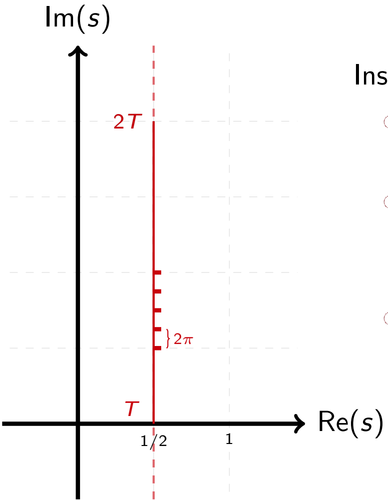
Short vs long intervals



Instead

- Consider fluctuations of moments of $\zeta(1/2 + it)$ over short ranges
- For a fixed short range, model by a single matrix $A \in \mathrm{U}(N)$ where $N \sim \log t/2\pi$

Short vs long intervals



Instead

- Consider fluctuations of moments of $\zeta(1/2+it)$ over short ranges
- For a fixed short range, model by a single matrix $A \in U(N)$ where $N \sim \log t/2\pi$
- Average fluctuations over many short intervals

Moments of Moments

Moments of Moments

$\mathrm{MoM}_N(k, \beta)$

Set

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

Moments of Moments

$\mathrm{MoM}_N(k,\beta)$

Set

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

Conjecture (Fyodorov & Keating)

As $N\to\infty$,

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2)N^{k\beta^2} & k\beta^2<1 \\ c(k,\beta)N^{k^2\beta^2-k+1} & k\beta^2>1, \end{cases}$$

where $G(s)$ is the Barnes G-function and $c(k,\beta)$ is some complicated function of k and β.

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k,\beta) N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases}$$

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k,\beta) N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases}$$

If $k \in \mathbb{N}$,

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k.$$

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k,\beta) N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases}$$

If $k \in \mathbb{N}$,

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k.$$

- Integrand can be expressed as a Toeplitz determinant

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k,\beta) N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases}$$

If $k \in \mathbb{N}$,

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k.$$

- Integrand can be expressed as a Toeplitz determinant
- As $N \to \infty$ and when $\theta_1, \ldots, \theta_k$ are distinct and fixed, can use Fisher-Hartwig

$$\text{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k,\beta) N^{k^2\beta^2 - k + 1} & k\beta^2 > 1. \end{cases}$$

If $k \in \mathbb{N}$,

$$\text{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k.$$

- Integrand can be expressed as a Toeplitz determinant
- As $N \to \infty$ and when $\theta_1, \ldots, \theta_k$ are distinct and fixed, can use Fisher-Hartwig
- When $k\beta^2 < 1$, can then use Selberg to recover conjecture in this range

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k,\beta) N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases}$$

If $k \in \mathbb{N}$,

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k.$$

- Integrand can be expressed as a Toeplitz determinant
- As $N \to \infty$ and when $\theta_1, \ldots, \theta_k$ are distinct and fixed, can use Fisher-Hartwig
- When $k\beta^2 < 1$, can then use Selberg to recover conjecture in this range
- However, if $k\beta^2 \geq 1$, then the expression diverges - coalescence of singularities becomes important

Previous results

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

Previous results

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

- $k=1$, $\beta > -1/2$: follows from Keating and Snaith, 2000 *CMP*

Previous results

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

- $k=1$, $\beta > -1/2$: follows from Keating and Snaith, 2000 *CMP*
- $k=1$, $\beta \in \mathbb{N}$: alternative proof from Bump and Gamburd, 2006 *CMP*

Previous results

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

- $k=1$, $\beta > -1/2$: follows from Keating and Snaith, 2000 *CMP*
- $k=1$, $\beta \in \mathbb{N}$: alternative proof from Bump and Gamburd, 2006 *CMP*
- $k=2$ and $\beta \in \mathbb{N}$: can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 *Mathematische Zeitschrift*

Previous results

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

- $k=1$, $\beta > -1/2$: follows from Keating and Snaith, 2000 *CMP*
- $k=1$, $\beta \in \mathbb{N}$: alternative proof from Bump and Gamburd, 2006 *CMP*
- $k=2$ and $\beta \in \mathbb{N}$: can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 *Mathematische Zeitschrift*
- $k=2$ all β: Claeys and Krasovsky establish correct powers of N, and relate $c(2,\beta)$ to Painlevé, 2015 *Duke*

Previous results

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right).$$

- $k=1$, $\beta > -1/2$: follows from Keating and Snaith, 2000 *CMP*
- $k=1$, $\beta \in \mathbb{N}$: alternative proof from Bump and Gamburd, 2006 *CMP*
- $k=2$ and $\beta \in \mathbb{N}$: can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 *Mathematische Zeitschrift*
- $k=2$ all β: Claeys and Krasovsky establish correct powers of N, and relate $c(2,\beta)$ to Painlevé, 2015 *Duke*
- $k\beta^2$ small: Webb, and Nikula, Saksman and Webb get consistent results

Results

Consider the case when $k, \beta \in \mathbb{N}$.

Results

Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\begin{aligned}\mathrm{MoM}_N(k,\beta) &= \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right)\\ &= \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\mathbb{E}\Big(\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta}\Big)d\theta_1\cdots d\theta_k.\end{aligned}$$

Results

Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\mathbb{E}\Big(\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta}\Big)d\theta_1\cdots d\theta_k.$$

Results

Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E}\Big(\prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} \Big) d\theta_1 \cdots d\theta_k.$$

Also $k\beta^2 > 1$ so we expect $\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2 - k + 1}$.

Results

Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E}\Big(\prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} \Big) d\theta_1 \cdots d\theta_k.$$

Also $k\beta^2 > 1$ so we expect $\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$.

Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_N(k,\beta)$ is a polynomial in N.

Results

Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E}\Big(\prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} \Big) d\theta_1 \cdots d\theta_k.$$

Also $k\beta^2 > 1$ so we expect $\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$.

Theorem [B.-Keating (2018)]

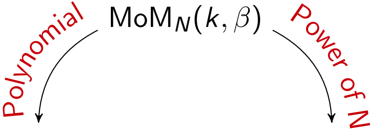
Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_N(k,\beta)$ is a polynomial in N.

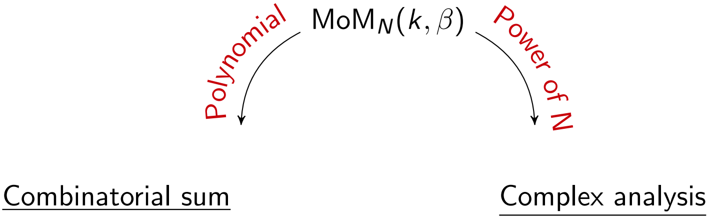
Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then for $c(k,\beta)$, an explicit function of k, β,

$$\mathrm{MoM}_N(k,\beta) = c(k,\beta) N^{k^2\beta^2-k+1} + O(N^{k^2\beta^2-k}).$$

$$\mathrm{MoM}_N(k,\beta)$$


$\mathrm{MoM}_N(k, \beta)$
Polynomial
Power of N


$\mathrm{MoM}_N(k, \beta)$
Polynomial
Power of N
Combinatorial sum
Complex analysis

$\mathrm{MoM}_N(k,\beta)$

Polynomial → Combinatorial sum

Power of N → Complex analysis

Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith

Complex analysis

$\text{MoM}_N(k, \beta)$

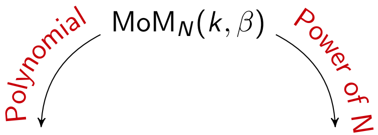
Polynomial

Power of N

Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith
- L'Hôpital

Complex analysis



Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith
- L'Hôpital
- Bump and Gamburd SSYT

Complex analysis

$\mathrm{MoM}_N(k, \beta)$

Polynomial ↙ ↘ Power of N

Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith
- L'Hôpital
- Bump and Gamburd SSYT

Complex analysis

- Exact representation of $\mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta}$

$\mathrm{MoM}_N(k, \beta)$

Polynomial → **Combinatorial sum**

Power of N → **Complex analysis**

Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith
- L'Hôpital
- Bump and Gamburd SSYT

Complex analysis

- Exact representation of $\mathbb{E}\prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta}$
- Multiple contour integrals

$\mathrm{MoM}_N(k, \beta)$

Polynomial → | Power of N →

Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith
- L'Hôpital
- Bump and Gamburd SSYT

Complex analysis

- Exact representation of $\mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta}$
- Multiple contour integrals
- Leading order analysis

Aside

Representation-theoretic approach

Aside

Representation-theoretic approach

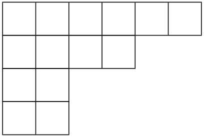
Partition

A *partition* λ is a sequence $(\lambda_1, ..., \lambda_k)$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Partition

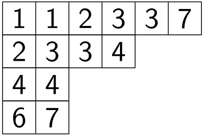
A *partition* λ is a sequence $(\lambda_1, ..., \lambda_k)$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Take the partition $\lambda = (6, 4, 2, 2)$. Then λ corresponds to the Young diagram



SSYT

For λ a partition, a *semistandard Young tableau (SSYT)* of shape λ is an array $T = (T_{ij})_{1\leq i\leq \ell(\lambda), 1\leq j\leq \lambda_i}$ of positive integers such that $T_{i,j} \leq T_{i,j+1}$ and $T_{ij} < T_{i+1,j}$. It is common to write SSYTs in a Young diagram; e.g.



is a SSYT of shape $(6, 4, 2, 2)$.

SSYT

For λ a partition, a *semistandard Young tableau (SSYT)* of shape λ is an array $T = (T_{ij})_{1\le i\le \ell(\lambda), 1\le j\le \lambda_i}$ of positive integers such that $T_{i,j} \le T_{i,j+1}$ and $T_{ij} < T_{i+1,j}$. It is common to write SSYTs in a Young diagram; e.g.

1	1	2	3	3	7
2	3	3	4		
4	4				
6	7				

is a SSYT of shape $(6, 4, 2, 2)$. T has *type* $t = (t_1, t_2, ...)$ if T has t_i parts equal to i. The SSYT above has type $(2, 2, 4, 3, 0, 1, 2)$.

SSYT

For λ a partition, a *semistandard Young tableau (SSYT)* of shape λ is an array $T = (T_{ij})_{1 \le i \le \ell(\lambda), 1 \le j \le \lambda_i}$ of positive integers such that $T_{i,j} \le T_{i,j+1}$ and $T_{ij} < T_{i+1,j}$. It is common to write SSYTs in a Young diagram; e.g.

1	1	2	3	3	7
2	3	3	4		
4	4				
6	7				

is a SSYT of shape $(6, 4, 2, 2)$. T has *type* $t = (t_1, t_2, ...)$ if T has t_i parts equal to i. The SSYT above has type $(2, 2, 4, 3, 0, 1, 2)$.
It is common to use the multivariate notation

$$x^T = x_1^{t_1(T)} x_2^{t_2(T)} \cdots ,$$

so for the example SSYT above,

$$x^T = x_1^2 x_2^2 x_3^4 x_4^3 x_6 x_7^2 .$$

Schur functions

The combinatorial definition of *Schur functions* is as follows:
For a partition λ, the Schur function in the variables $x_1, \ldots, x_r$ indexed by λ is a multivariable polynomial defined by

$$s_\lambda(x_1, \ldots, x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},$$

where the sum is over all SSYTs T whose entries belong to the set $\{1, \ldots, r\}$ (i.e. $t_i(T) = 0$ for $i > r$).

Schur functions

The combinatorial definition of *Schur functions* is as follows:
For a partition λ, the Schur function in the variables $x_1, ..., x_r$ indexed by λ is a multivariable polynomial defined by

$$s_\lambda(x_1, ..., x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},$$

where the sum is over all SSYTs T whose entries belong to the set $\{1, ..., r\}$ (i.e. $t_i(T) = 0$ for $i > r$).
Take $\lambda = (2, 1) \vdash 3$.

Schur functions

The combinatorial definition of *Schur functions* is as follows:
For a partition λ, the Schur function in the variables $x_1, ..., x_r$ indexed by λ is a multivariable polynomial defined by

$$s_\lambda(x_1, ..., x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},$$

where the sum is over all SSYTs T whose entries belong to the set $\{1, ..., r\}$ (i.e. $t_i(T) = 0$ for $i > r$).
Take $\lambda = (2,1) \vdash 3$. Then to calculate $s_\lambda(x_1, x_2, x_3)$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

Schur functions

The combinatorial definition of *Schur functions* is as follows:
For a partition λ, the Schur function in the variables $x_1, ..., x_r$ indexed by λ is a multivariable polynomial defined by

$$s_\lambda(x_1, ..., x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},$$

where the sum is over all SSYTs T whose entries belong to the set $\{1, ..., r\}$ (i.e. $t_i(T) = 0$ for $i > r$).
Take $\lambda = (2, 1) \vdash 3$. Then to calculate $s_\lambda(x_1, x_2, x_3)$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

So,

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

Theorem (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} = s_{\langle N^\beta \rangle}(1^{2\beta})$$

Theorem (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A\in U(N)}|P_N(A,\theta)|^{2\beta} = s_{\langle N^\beta \rangle}(1^{2\beta})$$

Corollary (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A\in U(N)}|P_N(A,\theta)|^{2\beta} = \prod_{j=0}^{N-1} \frac{j!(j+2\beta)!}{(j+\beta)!^2}$$

Theorem (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A\in U(N)}|P_N(A,\theta)|^{2\beta} = s_{\langle N^\beta \rangle}(1^{2\beta})$$

Corollary (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A\in U(N)}|P_N(A,\theta)|^{2\beta} = \prod_{j=0}^{N-1} \frac{j!(j+2\beta)!}{(j+\beta)!^2}$$

This also gives the interpretation that, for $\beta \in \mathbb{N}$, as $N \to \infty$

$$\mathbb{E}_{A\in U(N)}|P_N(A,\theta)|^{2\beta} \sim \frac{g_\beta}{\beta^2!} N^{\beta^2}$$

where g_β is the number of ways of filling a $\beta \times \beta$ array with the integers $1, 2, \ldots, \beta^2$ in such a way that the numbers increase along each row and down each column.

Proof of polynomial structure

Proof of polynomial structure

Recall

Theorem

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_N(k, \beta)$ is a polynomial in N.

Proof of polynomial structure

Recall

Theorem

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_N(k,\beta)$ is a polynomial in N.

Proposition (Bump and Gamburd)

$$\mathbb{E}_{A \in \mathrm{U}(N)} \left(\prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} \right) = \frac{s_{\langle N^{k\beta} \rangle}\left(e^{i\underline{\theta}}\right)}{\prod_{j=1}^{k} e^{iN\beta\theta_j}},$$

where $s_\nu(x_1,\ldots,x_n)$ is the Schur polynomial in n variables with respect to the partition ν. Here $\langle N^{k\beta} \rangle = (\overbrace{N,\ldots,N}^{k\beta})$, and

$$e^{i\underline{\theta}} = (\overbrace{e^{i\theta_1},\ldots,e^{i\theta_1}}^{\beta},\ldots,\overbrace{e^{i\theta_k},\ldots,e^{i\theta_k}}^{\beta},\overbrace{e^{i\theta_1},\ldots,e^{i\theta_1}}^{\beta},\ldots,\overbrace{e^{i\theta_k},\ldots,e^{i\theta_k}}^{\beta}).$$

Overview of proof of first theorem

Hence for $k, \beta \in \mathbb{N}$,

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{T} e^{i\theta_1(\tau_1 - N\beta)} \cdots e^{i\theta_k(\tau_k - N\beta)} \prod_{j=1}^{k} d\theta_j$$

$$= \sum_{\widetilde{T}} 1,$$

Overview of proof of first theorem

Hence for $k, \beta \in \mathbb{N}$,

$$\begin{aligned}\mathrm{MoM}_N(k,\beta) &= \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_T e^{i\theta_1(\tau_1 - N\beta)}\cdots e^{i\theta_k(\tau_k - N\beta)}\prod_{j=1}^k d\theta_j \\ &= \sum_{\widetilde{T}} 1,\end{aligned}$$

where the sum is now over $\widetilde{T}$, **restricted** SSYT

Overview of proof of first theorem

Hence for $k, \beta \in \mathbb{N}$,

$$\begin{aligned}\mathrm{MoM}_N(k,\beta) &= \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{T} e^{i\theta_1(\tau_1 - N\beta)}\cdots e^{i\theta_k(\tau_k - N\beta)}\prod_{j=1}^{k} d\theta_j \\ &= \sum_{\widetilde{T}} 1,\end{aligned}$$

where the sum is now over $\widetilde{T}$, **restricted** SSYT - require $N\beta$ entries from each of the sets $\{2\beta(j-1)+1,\dots,2j\beta\}$, for $j \in \{1,\dots,k\}$.

$$\mathrm{MoM}_N(k,\beta) = \sum_{\widetilde{T}} 1 < \sum_{T} 1 = \mathrm{Poly}_N(k^2\beta^2).$$

Overview of proof of first theorem

Overview of proof of first theorem

Conrey, Farmer, Rubinstein, Keating and Snaith give that

$$\mathbb{E}\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta} = \prod_{j=k\beta+1}^{2k\beta}\omega_j^{-N}\sum_{\sigma\in\Xi_{k\beta}}\frac{(\omega_{\sigma(k\beta+1)}\omega_{\sigma(k\beta+2)}\cdots\omega_{\sigma(2k\beta)})^N}{\prod_{l\leq k\beta<q}(1-\omega_{\sigma(l)}\omega_{\sigma(q)}^{-1})}$$

Overview of proof of first theorem

Conrey, Farmer, Rubinstein, Keating and Snaith give that

$$\mathbb{E}\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta} = \prod_{j=k\beta+1}^{2k\beta}\omega_j^{-N}\sum_{\sigma\in\Xi_{k\beta}}\frac{(\omega_{\sigma(k\beta+1)}\omega_{\sigma(k\beta+2)}\cdots\omega_{\sigma(2k\beta)})^N}{\prod_{l\leq k\beta<q}(1-\omega_{\sigma(l)}\omega_{\sigma(q)}^{-1})}$$

where $\Xi_{k\beta}$ is the set of $\binom{2k\beta}{k\beta}$ permutations $\sigma\in S_{2k\beta}$ such that $\sigma(1)<\sigma(2)<\cdots<\sigma(k\beta)$ and $\sigma(k\beta+1)<\cdots<\sigma(2k\beta)$, and

$$\underline{\omega} = (\underbrace{e^{i\theta_1},\ldots,e^{i\theta_1}}_{\beta},\ldots,\underbrace{e^{i\theta_k},\ldots,e^{i\theta_k}}_{\beta},\underbrace{e^{i\theta_1},\ldots,e^{i\theta_1}}_{\beta},\ldots,\underbrace{e^{i\theta_k},\ldots,e^{i\theta_k}}_{\beta}).$$

Examples

Examples

$$\text{MoM}_N(k,\beta) = \mathbb{E}_{A\in \text{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right)$$

Examples

$$\mathrm{MoM}_N(k,\beta) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right)$$

$\mathrm{MoM}_N(1,1) = N+1$

Examples

$$\mathrm{MoM}_N(k,\beta) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi} |P_N(A,\theta)|^{2\beta} d\theta\right)^k\right)$$

$$\mathrm{MoM}_N(1,1) = N+1$$

$$\mathrm{MoM}_N(2,1) = \frac{1}{6}(N+3)(N+2)(N+1)$$

Examples

$$\mathrm{MoM}_N(k,\beta) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right)$$

$$\mathrm{MoM}_N(1,1) = N+1$$

$$\mathrm{MoM}_N(2,1) = \frac{1}{6}(N+3)(N+2)(N+1)$$

$$\mathrm{MoM}_N(3,1) = \frac{1}{2520}(N+5)(N+4)(N+3)(N+2)(N+1)(N^2+6N+21)$$

Examples

$$\mathrm{MoM}_N(k,\beta) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right)$$

$$\mathrm{MoM}_N(1,1) = N+1$$

$$\mathrm{MoM}_N(2,1) = \frac{1}{6}(N+3)(N+2)(N+1)$$

$$\mathrm{MoM}_N(3,1) = \frac{1}{2520}(N+5)(N+4)(N+3)(N+2)(N+1)(N^2+6N+21)$$

$$\begin{aligned}\mathrm{MoM}_N(4,1) = \frac{1}{778377600}&(N+7)(N+6)(N+5)(N+4)(N+3)(N+2)\\ &\times (N+1)(7N^6+168N^5+1804N^4+10944N^3+\\ &+41893N^2+99624N+154440)\end{aligned}$$

Examples

$$\mathrm{MoM}_N(k,\beta) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k\right)$$

$$\mathrm{MoM}_N(1,1) = N+1$$

$$\mathrm{MoM}_N(2,1) = \frac{1}{6}(N+3)(N+2)(N+1)$$

$$\mathrm{MoM}_N(3,1) = \frac{1}{2520}(N+5)(N+4)(N+3)(N+2)(N+1)(N^2+6N+21)$$

$$\mathrm{MoM}_N(4,1) = \frac{1}{778377600}(N+7)(N+6)(N+5)(N+4)(N+3)(N+2)$$
$$\times (N+1)(7N^6+168N^5+1804N^4+10944N^3+$$
$$+41893N^2+99624N+154440)$$

$$\mathrm{MoM}_N(1,2) = \frac{1}{12}(N+1)(N+2)^2(N+3).$$

Examples

$$\mathrm{MoM}_N(2,2) = \frac{1}{163459296000}(N+7)(N+6)(N+5)(N+4)$$
$$\times (N+3)(N+2)(N+1)(298N^8 + 9536N^7 + 134071N^6$$
$$+ 1081640N^5 + 5494237N^4 + 18102224N^3 + 38466354N^2$$
$$+ 50225040N + 32432400).$$

Examples

$$
\begin{aligned}
\mathrm{MoM}_N(2,3) = {} & \frac{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)(N+9)(N+10)(N+11)}{172219132773102415494444188958720000000} \\
& \times \Big(12308743625763N^{24} + 1772459082109872N^{23} + 121902830804059138N^{22} + \\
& + 5328802119564663432N^{21} + 166214570195622478453N^{20} + 3937056259812505643352N^{19} \\
& + 73583663800226157619008N^{18} + 1113109355823972261429312N^{17} + 13869840005250869763713293N^{16} \\
& + 144126954435929329947378912N^{15} + 1259786144898207172443272698N^{14} \\
& + 9315726913410827893883025672N^{13} + 58475127984013141340467825323N^{12} \\
& + 311978271286536355427593012632N^{11} + 1413794106539529439589778645028N^{10} \\
& + 5427439874579682729570383266992N^{9} + 17564370687865211818995713096848N^{8} \\
& + 47561382824003032731805262975232N^{7} + 106610927256886475209611301000128N^{6} \\
& + 194861499503272627170466392014592N^{5} + 284303877221735683573377603640320N^{4} \\
& + 320989495108428049992898521600000N^{3} + 266974288159876385845370793984000N^{2} \\
& + 148918006780282798012340305920000N + 43144523802785397500411904000000 \Big)
\end{aligned}
$$

Overview of proof of second theorem

Overview of proof of second theorem

Recall,

Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then

$$\mathrm{MoM}_N(k, \beta) = c(k, \beta) N^{k^2\beta^2 - k + 1} + O(N^{k^2\beta^2 - k}),$$

where $c(k, \beta)$ is an explicit function of k, β.

Overview of proof of second theorem

Recall,

Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then

$$\mathrm{MoM}_N(k,\beta) = c(k,\beta)N^{k^2\beta^2-k+1} + O(N^{k^2\beta^2-k}),$$

where $c(k,\beta)$ is an explicit function of k, β.

Proof ingredients:

- Expand $\mathbb{E}\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta}$ as a multiple contour integral

Overview of proof of second theorem

Recall,

Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then

$$\mathrm{MoM}_N(k,\beta) = c(k,\beta) N^{k^2\beta^2 - k + 1} + O(N^{k^2\beta^2 - k}),$$

where $c(k,\beta)$ is an explicit function of k, β.

Proof ingredients:

- Expand $\mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta}$ as a multiple contour integral
- Deform and manipulate the integrals

Overview of proof of second theorem

Recall,

Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then

$$\mathrm{MoM}_N(k,\beta) = c(k,\beta) N^{k^2\beta^2 - k + 1} + O(N^{k^2\beta^2 - k}),$$

where $c(k,\beta)$ is an explicit function of k, β.

Proof ingredients:

- Expand $\mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta}$ as a multiple contour integral
- Deform and manipulate the integrals
- Analyse the result asymptotically as $N \to \infty$.

Overview of proof of second theorem

Overview of proof of second theorem

Define

$$I_{k,\beta}(\theta_1, \ldots, \theta_k) = \mathbb{E}_{A \in \mathrm{U}(N)} \left(\prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right),$$

Overview of proof of second theorem

Define

$$I_{k,\beta}(\theta_1,\ldots,\theta_k) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta}\right),$$

so

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi} I_{k,\beta}(\theta_1,\ldots,\theta_k)d\theta_1\cdots d\theta_k.$$

Overview of proof of second theorem

Define

$$I_{k,\beta}(\theta_1,\ldots,\theta_k) = \mathbb{E}_{A\in \mathrm{U}(N)}\left(\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta}\right),$$

so

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi} I_{k,\beta}(\theta_1,\ldots,\theta_k)d\theta_1\cdots d\theta_k.$$

Theorem [CFKRS]

For $k,\beta\in\mathbb{N}$,

$$I_{k,\beta}(\underline{\theta}) = \frac{(-1)^{k\beta}e^{-i\beta\sum_{j=1}^{k}\theta_j}}{(2\pi i)^{2k\beta}((k\beta)!)^2}\oint\cdots\oint \frac{e^{-N(z_{k\beta+1}+\cdots+z_{2k\beta})}\Delta(z_1,\ldots,z_{2k\beta})^2dz_1\cdots dz_{2k\beta}}{\prod_{m\le k\beta<n}(1-e^{z_n-z_m})\prod_{m=1}^{2k\beta}\prod_{n=1}^{k}(z_m+i\theta_n)^{2\beta}}.$$

Overview of proof of second theorem

Manipulation of MCI

$$I_{k,\beta}(\underline{\theta}) = \frac{(-1)^{k\beta} e^{-i\beta \sum_{j=1}^{k} \theta_j}}{(2\pi i)^{2k\beta}((k\beta)!)^2} \oint \cdots \oint \frac{e^{-N(z_{k\beta+1}+\cdots+z_{2k\beta})} \Delta(z_1, \ldots, z_{2k\beta})^2 dz_1 \cdots dz_{2k\beta}}{\prod_{m \leq k\beta < n}(1 - e^{z_n - z_m}) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k} (z_m + i\theta_n)^{2\beta}}.$$

Overview of proof of second theorem

Manipulation of MCI

$$I_{k,\beta}(\underline{\theta}) = \frac{(-1)^{k\beta} e^{-i\beta \sum_{j=1}^{k} \theta_j}}{(2\pi i)^{2k\beta}((k\beta)!)^2} \oint \cdots \oint \frac{e^{-N(z_{k\beta+1}+\cdots+z_{2k\beta})} \Delta(z_1, \ldots, z_{2k\beta})^2 dz_1 \cdots dz_{2k\beta}}{\prod_{m \le k\beta < n}(1 - e^{z_n - z_m}) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k} (z_m + i\theta_n)^{2\beta}}.$$

- Deform the contours
- Change of variables
- Carefully analyse remaining integrals

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2 - k + 1}$$

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$$

where

$$c(k,\beta) = \sum_{\substack{l_1,\dots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)^{f(k,\beta,\underline{l})} P_{k,\beta}(l_1,\dots,l_{k-1}),$$

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$$

where

$$c(k,\beta) = \sum_{\substack{l_1,\ldots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)^{f(k,\beta,\underline{l})} P_{k,\beta}(l_1,\ldots,l_{k-1}),$$

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$$

where

$$c(k,\beta) = \sum_{\substack{l_1,\ldots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)^{f(k,\beta,\underline{l})} P_{k,\beta}(l_1,\ldots,l_{k-1}),$$

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$$

where

$$c(k,\beta) = \sum_{\substack{l_1,\ldots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\left((k-1)\beta - \sum_{j=1}^{k-1} l_j\right)^{f(k,\beta,\underline{l})} P_{k,\beta}(l_1,\ldots,l_{k-1}),$$

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$$

where

$$c(k,\beta) = \sum_{\substack{l_1,\dots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)^{f(k,\beta,\underline{l})} P_{k,\beta}(l_1,\dots,l_{k-1}),$$

Overview of proof of second theorem

Leading order

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2-k+1}$$

where

$$c(k,\beta) = \sum_{\substack{l_1,\dots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)^{f(k,\beta,\underline{l})} P_{k,\beta}(l_1,\dots,l_{k-1}),$$

and

$$P_{k,\beta}(\underline{l}) = \frac{(-1)^{\sum_{\sigma<\tau}|S^-_{\sigma<\tau}|}}{(2\pi i)^{2k\beta}((k\beta)!)^2} \int_{\Gamma_0}\cdots\int_{\Gamma_0} \frac{e^{-\sum_{m=k\beta+1}^{2k\beta} v_m} \prod_{\substack{m<n \\ \alpha_m=\alpha_n}} (v_n - v_m)^2}{\prod_{\substack{m\le k\beta<n \\ \alpha_m=\alpha_n}} (v_n - v_m) \prod_{m=1}^{2k\beta} v_m^{2\beta}}$$

$$\times \Psi_{k,\beta,\underline{l}}\Big(\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)\underline{v}\Big) \prod_{m=1}^{2k\beta} dv_m.$$

Overview of proof of second theorem

Leading order

So for $k, \beta \in \mathbb{N}$ we have

$$\mathrm{MoM}_N(k,\beta) \sim c(k,\beta) N^{k^2\beta^2 - k + 1}.$$

Overview of proof of second theorem

Leading order

So for $k, \beta \in \mathbb{N}$ we have

$$\mathrm{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2\beta^2 - k + 1}.$$

The theorem follows if one can show that $c(k, \beta) \neq 0$. A lengthy computation shows that this is the case - in fact $c(k, \beta) > 0$.

Another alternative approach

Another alternative approach

One can recover the same asymptotic result using Gelfand-Tsetlin patterns.

Theorem [Assiotis-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then,

$$\mathrm{MoM}_N(k, \beta) = c(k, \beta) N^{k^2\beta^2 - k + 1} + O(N^{k^2\beta^2 - k}),$$

where $c(k, \beta)$ can be written explicitly as a volume of a certain region involving continuous Gelfand-Tsetlin patterns with constraints.

Translating to Number Theory

Translating to Number Theory

Analogue of $\mathrm{MoM}_N(k,\beta)$:

$$\mathrm{MoM}^{\zeta}_T(k,\beta) := \frac{1}{T}\int_T^{2T}\left(\frac{1}{2\pi}\int_0^{2\pi}|\zeta(1/2+i(t+\gamma))|^{2\beta}d\gamma\right)^k dt$$

$$= \frac{1}{T(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_T^{2T}\prod_{j=1}^{k}|\zeta(1/2+i(t+\gamma_j))|^{2\beta}dt\prod_{j=1}^{k}d\gamma_j.$$

Translating to Number Theory

Analogue of $\mathrm{MoM}_N(k,\beta)$:

$$\mathrm{MoM}_T^{\zeta}(k,\beta) := \frac{1}{T}\int_T^{2T}\left(\frac{1}{2\pi}\int_0^{2\pi}|\zeta(1/2+i(t+\gamma))|^{2\beta}d\gamma\right)^k dt$$

$$= \frac{1}{T(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_T^{2T}\prod_{j=1}^{k}|\zeta(1/2+i(t+\gamma_j))|^{2\beta}dt\prod_{j=1}^{k}d\gamma_j.$$

Conjecture Fyodorov & Keating

For $k\beta^2>1$, $\mathrm{MoM}_T^{\zeta}(k,\beta)\sim c'(k,\beta)(\log\frac{T}{2\pi})^{k^2\beta^2-k+1}$.

Translating to Number Theory

Conjectured expression for integrand (CFKRS):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\beta}dt = \frac{1}{T}\int_0^T \frac{(-1)^\beta}{(\beta!)^2}\frac{1}{(2\pi i)^{2\beta}} \times \oint\cdots\oint \frac{G_\zeta(z_1,\ldots,z_{2\beta})\Delta^2(z_1,\ldots,z_{2\beta})}{\prod_{j=1}^{2\beta} z_j^{2\beta}} \times e^{\frac{1}{2}\log\frac{t}{2\pi}\sum_{j=1}^{\beta} z_j - z_{\beta+j}} dz_1\cdots dz_{2\beta}\,dt + o(1).$$

Translating to Number Theory

Conjectured expression for integrand (CFKRS):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\beta} dt = \frac{1}{T}\int_0^T \frac{(-1)^\beta}{(\beta!)^2}\frac{1}{(2\pi i)^{2\beta}} \times \oint \cdots \oint \frac{G_\zeta(z_1,\ldots,z_{2\beta})\Delta^2(z_1,\ldots,z_{2\beta})}{\prod_{j=1}^{2\beta} z_j^{2\beta}} \times e^{\frac{1}{2}\log\frac{t}{2\pi}\sum_{j=1}^{\beta} z_j - z_{\beta+j}} dz_1 \cdots dz_{2\beta} dt + o(1).$$

where

$$G_\zeta(z_1,\ldots,z_{2\beta}) = A_\beta(z_1,\ldots,z_{2\beta}) \prod_{i,j=1}^{\beta} \zeta(1+z_i - z_{\beta+j}),$$

and $A_\beta(\underline{z})$ is an Euler product whose local factors are polynomials in p^{-1} and p^{-z_i}.

Symplectic, Orthogonal, and L-functions

Symplectic, Orthogonal, and L-functions

- Relationship between families of L-functions and other random matrix families

Symplectic, Orthogonal, and L-functions

- Relationship between families of L-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

Symplectic, Orthogonal, and L-functions

- Relationship between families of L-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

Symplectic, Orthogonal, and L-functions

- Relationship between families of L-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

$$\mathrm{USp}(2N) = \{M \in U(2N) : M^t \Omega M = \Omega\}, \quad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix},$$

$$\det(I - Ms) = \prod_{n=1}^{N} (1 - e^{i\theta_n} s)(1 - e^{-i\theta_n} s).$$

Symplectic, Orthogonal, and L-functions

- Relationship between families of L-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

$$\mathrm{USp}(2N) = \{M \in U(2N) : M^t \Omega M = \Omega\}, \quad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix},$$

$$\det(I - Ms) = \prod_{n=1}^{N} (1 - e^{i\theta_n} s)(1 - e^{-i\theta_n} s).$$

$$\mathrm{SO}(2N) = \{O \in O(2N) : \det(O) = 1\},$$

$$\det(I - Os) = \prod_{m=1}^{N} (1 - e^{i\theta_m} s)(1 - e^{-i\theta_m} s).$$

Symplectic, Orthogonal, and L-functions

Symplectic, Orthogonal, and L-functions

Theorem [Assiotis-B.-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_{\mathrm{USp}(2N)}(k,\beta)$ is a polynomial in N and further

$$\mathrm{MoM}_{\mathrm{USp}(2N)}(k,\beta) = c_{\mathrm{USp}}(k,\beta) N^{k\beta(2k\beta+1)-k} + O(N^{k\beta(2k\beta+1)-k-1}).$$

Symplectic, Orthogonal, and L-functions

Theorem [Assiotis-B.-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_{\mathrm{USp}(2N)}(k,\beta)$ is a polynomial in N and further

$$\mathrm{MoM}_{\mathrm{USp}(2N)}(k,\beta) = c_{\mathrm{USp}}(k,\beta) N^{k\beta(2k\beta+1)-k} + O(N^{k\beta(2k\beta+1)-k-1}).$$

Theorem [Assiotis-B.-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_{\mathrm{SO}(2N)}(k,\beta)$ is a polynomial in N. Further

$$\mathrm{MoM}_{\mathrm{SO}(2N)}(1,1) = 2(N+1),$$

otherwise

$$\mathrm{MoM}_{\mathrm{SO}(2N)}(k,\beta) = c_{\mathrm{SO}(2N)} N^{k\beta(2k\beta-1)-k} + O(N^{k\beta(2k\beta-1)-k-1}).$$

A word on the proof

A word on the proof

Again we have a number of forms for the matrix average:

$$\mathbb{E}_{\mathrm{USp}(2N)}\left(\prod_{j=1}^{k}|\det(I - Ae^{-i\theta})|^{2\beta}\right)$$
$$= \sum_{\mathcal{P}\in \mathrm{SP}_{\langle N^{k\beta}\rangle}} w(\mathcal{P}) \quad \text{(BG 2006)}$$

A word on the proof

Again we have a number of forms for the matrix average:

$$\mathbb{E}_{\mathrm{USp}(2N)}\left(\prod_{j=1}^{k}|\det(I - Ae^{-i\theta})|^{2\beta}\right)$$

$$= \sum_{\mathcal{P}\in \mathrm{SP}_{\langle N^k\beta\rangle}} w(\mathcal{P})$$

$$= \sum_{\varepsilon\in\{\pm 1\}^{2k\beta}} \frac{\prod_{j=1}^{2k\beta} x_j^{N(1-\varepsilon_j)}}{\prod_{i\leq j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})}$$ (CFKRS 2002)

A word on the proof

Again we have a number of forms for the matrix average:

$$\mathbb{E}_{\mathrm{USp}(2N)}\left(\prod_{j=1}^{k}|\det(I-Ae^{-i\theta})|^{2\beta}\right)$$

$$=\sum_{\mathcal{P}\in \mathrm{SP}_{\langle N^{k\beta}\rangle}} w(\mathcal{P})$$

$$=\sum_{\varepsilon\in\{\pm1\}^{2k\beta}} \frac{\prod_{j=1}^{2k\beta} x_j^{N(1-\varepsilon_j)}}{\prod_{i\leq j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}$$

$$=\frac{(-1)^{k\beta(2k\beta-1)}2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!}e^{-Ni\sum_{j=1}^{2k\beta}\theta_j} \quad \text{(CFRKS 2002)}$$

$$\times\oint\cdots\oint\frac{\Delta(z_1^2,\ldots,z_{2k\beta}^2)^2\prod_{j=1}^{2k\beta}z_je^{Nz_j}dz_j}{\prod_{l\leq m}(1-e^{-z_m-z_l})\prod_{m,n=1}^{2k\beta}(z_m-i\theta_n)(z_m+i\theta_n)}$$

Summary

Summary

- CUE: The integer k moments of the integer β moments are polynomials in N of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).

Summary

- CUE: The integer k moments of the integer β moments are polynomials in N of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.

Summary

- CUE: The integer k moments of the integer β moments are polynomials in N of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.
- Similar results are found for $SO(2N)$ and $USp(2N)$

Summary

- CUE: The integer k moments of the integer β moments are polynomials in N of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.
- Similar results are found for $SO(2N)$ and $USp(2N)$
- The polynomials can be explicitly computed.

Summary

- CUE: The integer k moments of the integer β moments are polynomials in N of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.
- Similar results are found for SO($2N$) and USp($2N$)
- The polynomials can be explicitly computed.
- Techniques carry over (under conjecture) to $\zeta(1/2 + it)$ and other L-functions.

Summary

- CUE: The integer k moments of the integer β moments are polynomials in N of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.
- Similar results are found for SO($2N$) and USp($2N$)
- The polynomials can be explicitly computed.
- Techniques carry over (under conjecture) to $\zeta(1/2 + it)$ and other L-functions.
- Proved the growth of certain representation theoretic sums.

THANK YOU