Moments of Moments

Emma Bailey
University of Bristol
Joint work with Jon Keating
Number Theoretic Motivation

Consider moments of the zeta function,

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} dt. \]

**Conjecture**

For \( \beta \in \mathbb{R}^+ \),

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} dt \sim f(\beta) c\zeta(\beta) \left(\frac{\log T}{2\pi}\right)^{\beta^2}, \]

as \( T \to \infty \).

\( f(\beta) \) is a known arithmetic function.
\( c\zeta(\beta) \) is another function depending on \( \beta \).
Consider moments of the zeta function,

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} dt. \]
Consider moments of the zeta function,

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} \, dt. \]

**Conjecture**

For \( \beta \in \mathbb{R}^+ \),

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} \, dt \sim f(\beta)c_\zeta(\beta) \left( \log \frac{T}{2\pi} \right)^{\beta^2}, \]

as \( T \to \infty \).
Consider moments of the zeta function,

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} \, dt. \]

**Conjecture**

For \( \beta \in \mathbb{R}^+ \),

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} \, dt \sim f(\beta)c_\zeta(\beta) (\log \frac{T}{2\pi})^{\beta^2}, \]

as \( T \to \infty \).

- \( f(\beta) \) is a known arithmetic function
Consider moments of the zeta function,

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} \, dt. \]

**Conjecture**

For \( \beta \in \mathbb{R}^+ \),

\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} \, dt \sim f(\beta) c_\zeta(\beta) \left( \log \frac{T}{2\pi} \right)^{\beta^2}, \]

as \( T \to \infty \).

- \( f(\beta) \) is a known arithmetic function
- \( c_\zeta(\beta) \) is another function depending on \( \beta \).
For $A \in \text{CUE}_N$ ($A \in \text{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$
For $A \in \text{CUE}_N$ ($A \in \text{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Then recall

$$\frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\beta} dt.$$
For $A \in \text{CUE}_N$ ($A \in U(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Instead

$$\int_{U(N)} |P_N(A, \theta)|^{2\beta} \, dA$$
For $A \in \text{CUE}_N$ ($A \in U(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,
For \( A \in \text{CUE}_N \) (\( A \in \text{U}(N) \) with Haar measure) set

\[
P_N(A, \theta) = \det(I - Ae^{-i\theta}).
\]

Keating and Snaith: for \( \beta > -1/2 \),

\[
\int_{\text{U}(N)} |P_N(A, \theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2\beta)}{(\Gamma(j + \beta))^2}
\]
For $A \in \text{CUE}_N$ ($A \in \text{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

$$\int_{\text{U}(N)} |P_N(A, \theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2\beta)}{(\Gamma(j + \beta))^2} \sim c_U(\beta)N^{\beta^2}$$
For $A \in \text{CUE}_N$ ($A \in \text{U}(N)$ with Haar measure) set

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

Keating and Snaith: for $\beta > -1/2$,

$$\int_{\text{U}(N)} |P_N(A, \theta)|^{2\beta} \, dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2\beta)}{(\Gamma(j + \beta))^2} \sim c_U(\beta)N^{\beta^2}$$

where

$$c_U(\beta) = \frac{G^2(\beta + 1)}{G(2\beta + 1)},$$

with $G(s)$ the Barnes $G$-function and if $\beta \in \mathbb{N}$,

$$c_U(\beta) = \prod_{j=0}^{\beta-1} \frac{j!}{(j + \beta)!}.$$
For \( A \in \text{CUE}_N \) (\( A \in \text{U}(N) \) with Haar measure) set

\[
P_N(A, \theta) = \det(I - Ae^{-i\theta}).
\]

Keating and Snaith: for \( \beta > -1/2 \),

\[
\int_{\text{U}(N)} |P_N(A, \theta)|^{2\beta} dA = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2\beta)}{(\Gamma(j + \beta))^2} \sim c_U(\beta)N^{\beta^2}
\]

where

\[
c_U(\beta) = \frac{G^2(\beta + 1)}{G(2\beta + 1)},
\]

with \( G(s) \) the Barnes \( G \)-function and if \( \beta \in \mathbb{N} \),

\[
c_U(\beta) = \prod_{j=0}^{\beta-1} \frac{j!}{(j + \beta)!}.
\]

Conjecture: \( c_U(\beta) = c_\zeta(\beta) \).
Consider fluctuations of moments of $\zeta(1/2 + it)$ over short ranges. For a fixed short range, model by a single matrix $A \in U(N)$ where $N \sim \log t / 2\pi$. Average fluctuations over many short intervals.
Instead

○ Consider fluctuations of moments of $\zeta(1/2 + it)$ over short ranges.
Instead

- Consider fluctuations of moments of \( \zeta(1/2 + it) \) over short ranges
- For a fixed short range, model by a single matrix \( A \in U(N) \) where \( N \sim \log t/2\pi \)
Instead

- Consider fluctuations of moments of $\zeta(1/2 + it)$ over short ranges.
- For a fixed short range, model by a single matrix $A \in U(N)$ where $N \sim \log t / 2\pi$.
- Average fluctuations over many short intervals.
Moments of Moments

Set \( \text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \int_0^{2\pi} |P_N(A, \theta)|^2 \beta d\theta \right)^k \).

Conjecture (Fyodorov & Keating)

As \( N \to \infty \),

\[
\text{MoM}_N(k, \beta) \sim \begin{cases} 
\left( G(1+\beta) \right)^2 G(1+2\beta) \Gamma(1-\beta^2) N^{k\beta^2} \left( k \Gamma(1-k\beta^2) \right) & \text{if } k\beta^2 < 1 \\
c(k, \beta) N^{k\beta^2-\frac{k}{2}+1} & \text{if } k\beta^2 > 1
\end{cases}
\]

where \( G(s) \) is the Barnes \( G \)-function and \( c(k, \beta) \) is some complicated function of \( k \) and \( \beta \).
Set

$$\text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).$$
Moments of Moments

\[ \textbf{MoM}_N(k, \beta) \]

Set

\[
\text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).
\]

**Conjecture** (Fyodorov & Keating)

As \( N \to \infty \),

\[
\text{MoM}_N(k, \beta) \sim \begin{cases} 
\left( \frac{(G(1+\beta))^2}{G(1+2\beta) \Gamma(1-\beta^2)} \right)^k \Gamma(1 - k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\
c(k, \beta) N^{k^2\beta^2 - k + 1} & k\beta^2 > 1,
\end{cases}
\]

where \( G(s) \) is the Barnes \( G \)-function and \( c(k, \beta) \) is some complicated function of \( k \) and \( \beta \).
\[ \text{MoM}_N(k, \beta) \sim \begin{cases} 
\left( \frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)} \right)^k 
\Gamma(1-k\beta^2)N^{k\beta^2} & k\beta^2 < 1 \\
\text{c}(k, \beta)N^{k^2\beta^2-k+1} & k\beta^2 > 1.
\end{cases} \]
\[ \text{MoM}_N(k, \beta) \sim \begin{cases} 
\left( \frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)} \right)^k \Gamma(1 - k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\
\frac{c(k, \beta) N^{k^2\beta^2-k+1}}{k} & k\beta^2 > 1.
\end{cases} \]

If \( k \in \mathbb{N} \),

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k. \]
\[ \text{MoM}_N(k, \beta) \sim \begin{cases} \left( \frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)} \right)^k \Gamma(1 - k\beta^2)N^{k\beta^2} & k\beta^2 < 1 \\ c(k, \beta)N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases} \]

If \( k \in \mathbb{N}, \)

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k. \]

Integrand can be expressed as a Toeplitz determinant
\[
\text{MoM}_N(k, \beta) \sim \begin{cases} 
\left( \frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)} \right)^k \Gamma(1 - k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\
 c(k, \beta) N^{k^2\beta^2 - k + 1} & k\beta^2 > 1.
\end{cases}
\]

If \( k \in \mathbb{N} \),

\[
\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k.
\]

- Integrand can be expressed as a Toeplitz determinant
- As \( N \to \infty \) and when \( \theta_1, \ldots, \theta_k \) are distinct and fixed, can use Fisher-Hartwig
\[ \text{MoM}_N(k, \beta) \sim \begin{cases} \left( \frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)} \right)^k \Gamma(1 - k\beta^2) N^{k\beta^2} & k\beta^2 < 1 \\ c(k, \beta) N^{k^2\beta^2 - k + 1} & k\beta^2 > 1. \end{cases} \]

If \( k \in \mathbb{N} \),

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} d\theta_1 \cdots d\theta_k. \]

- Integrand can be expressed as a Toeplitz determinant
- As \( N \to \infty \) and when \( \theta_1, \ldots, \theta_k \) are distinct and fixed, can use Fisher-Hartwig
- When \( k\beta^2 < 1 \), can then use Selberg to recover conjecture in this range
\[ \text{MoM}_N(k, \beta) \sim \begin{cases} \left( \frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)} \right)^k \Gamma(1-k\beta^2)N^{k\beta^2} & k\beta^2 < 1 \\ c(k, \beta)N^{k^2\beta^2-k+1} & k\beta^2 > 1. \end{cases} \]

If \( k \in \mathbb{N}, \)

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \, d\theta_1 \cdots d\theta_k. \]

- Integrand can be expressed as a Toeplitz determinant
- As \( N \to \infty \) and when \( \theta_1, \ldots, \theta_k \) are distinct and fixed, can use Fisher-Hartwig
- When \( k\beta^2 < 1 \), can then use Selberg to recover conjecture in this range
- However, if \( k\beta^2 \geq 1 \), then the expression diverges - coalescence of singularities becomes important
Previous results

MoM$^N(k, \beta) := \mathbb{E}_{A \in \mathbb{U}(N)} \left( \left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).$
Previous results

\[ \text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} \left| P_N(A, \theta) \right|^{2\beta} d\theta \right)^k \right). \]

- \( k = 1, \beta > -1/2 \): follows from Keating and Snaith, 2000 CMP
- \( k = 2 \) and \( \beta \in \mathbb{N} \): alternative proof from Bump and Gamburd, 2006
- \( k = 2 \) and all \( \beta \): can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018
- \( k = 2 \) small: Claeys and Krasovsky establish correct powers of \( N \), and relate \( c(2, \beta) \) to Painlevé, 2015
- \( k = 2 \) small: Webb, and Nikula, Saksman and Webb get consistent results
Previous results

\[
\text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).
\]

- \( k = 1, \beta > -1/2 \): follows from Keating and Snaith, 2000 \textit{CMP}
- \( k = 1, \beta \in \mathbb{N} \): alternative proof from Bump and Gamburd, 2006 \textit{CMP}

- \( k = 2 \) and \( \beta \in \mathbb{N} \): can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 Mathematische Zeitschrift
- \( k = 2 \) \small: Claeys and Krasovsky establish correct powers of \( N \), and relate \( c(2, \beta) \) to Painlevé, 2015 Duke

Emma Bailey
Moments of Moments
21st June
Previous results

\[ \text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right). \]

- **k = 1, \beta > -1/2**: follows from Keating and Snaith, 2000 CMP
- **k = 1, \beta \in \mathbb{N}**: alternative proof from Bump and Gamburd, 2006 CMP
- **k = 2 and \beta \in \mathbb{N}**: can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 *Mathematische Zeitschrift*
Previous results

\[
\text{MoM}_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).
\]

- \( k = 1, \beta > -1/2 \): follows from Keating and Snaith, 2000 *CMP*
- \( k = 1, \beta \in \mathbb{N} \): alternative proof from Bump and Gamburd, 2006 *CMP*
- \( k = 2 \) and \( \beta \in \mathbb{N} \): can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 *Mathematische Zeitschrift*
- \( k = 2 \) all \( \beta \): Claeys and Krasovsky establish correct powers of \( N \), and relate \( c(2, \beta) \) to Painlevé, 2015 *Duke*
Previous results

MoM_N(k, \beta) := \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).

- k = 1, \beta > -1/2: follows from Keating and Snaith, 2000 CMP
- k = 1, \beta \in \mathbb{N}: alternative proof from Bump and Gamburd, 2006 CMP
- k = 2 and \beta \in \mathbb{N}: can be deduced from Keating, Rodgers, Roditty-Gershon and Rudnick, 2018 Mathematische Zeitschrift
- k = 2 all \beta: Claeys and Krasovsky establish correct powers of N, and relate c(2, \beta) to Painlevé, 2015 Duke
- k\beta^2 small: Webb, and Nikula, Saksman and Webb get consistent results
Consider the case when $k, \beta \in \mathbb{N}$. 
Consider the case when \( k, \beta \in \mathbb{N} \). Then recall

\[
\text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} \, d\theta \right)^k \right) \\
= \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right) \, d\theta_1 \cdots d\theta_k.
\]
Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \left( \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} \right) d\theta_1 \cdots d\theta_k.$$
Consider the case when $k, \beta \in \mathbb{N}$. Then recall
\[
\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E}\left(\prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta}\right) d\theta_1 \cdots d\theta_k.
\]

Also $k\beta^2 > 1$ so we expect $\text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2\beta^2-k+1}$. 
Consider the case when \( k, \beta \in \mathbb{N} \). Then recall

\[
\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E}\left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right) d\theta_1 \cdots d\theta_k.
\]

Also \( k\beta^2 > 1 \) so we expect \( \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2\beta^2 - k + 1} \).

**Theorem** [B.-Keating (2018)]

Let \( k, \beta \in \mathbb{N} \). Then \( \text{MoM}_N(k, \beta) \) is a polynomial in \( N \).
Consider the case when $k, \beta \in \mathbb{N}$. Then recall

$$\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E}\left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right) d\theta_1 \cdots d\theta_k.$$ 

Also $k\beta^2 > 1$ so we expect $\text{MoM}_N(k, \beta) \sim c(k, \beta)N^{k^2\beta^2 - k + 1}$.

\textbf{Theorem} [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_N(k, \beta)$ is a polynomial in $N$.

\textbf{Theorem} [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then for $c(k, \beta)$, an explicit function of $k, \beta$,

$$\text{MoM}_N(k, \beta) = c(k, \beta)N^{k^2\beta^2 - k + 1} + O(N^{k^2\beta^2 - k}).$$
$$\text{MoM}_N(k, \beta)$$
Moments of Moments

\[ \text{Polyomial} \quad \text{MoM}_N(k, \beta) \quad \text{Power of N} \]
Combinatorial sum

Complex analysis

\( \text{Polynomial} \rightarrow \text{MoM}_N(k, \beta) \rightarrow \text{Power of } N \)
Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith

Complex analysis

Conrey, Farmer, Keating, Rubinstein and Snaith
Combinatorial sum
- Conrey, Farmer, Keating, Rubinstein and Snaith
- L’Hôpital

Complex analysis
Combinatorial sum

- Conrey, Farmer, Keating, Rubinstein and Snaith
- L’Hôpital
- Bump and Gamburd SSYT

Complex analysis
Combinatorial sum
- Conrey, Farmer, Keating, Rubinstein and Snaith
- L’Hôpital
- Bump and Gamburd SSYT

Complex analysis
- Exact representation of $\mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta}$
Combinatorial sum
○ Conrey, Farmer, Keating, Rubinstein and Snaith
○ L’Hôpital
○ Bump and Gamburd SSYT

Complex analysis
○ Exact representation of
\[ \mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \]
○ Multiple contour integrals
MoM_N(k, \beta)

Combinatorial sum
- Conrey, Farmer, Keating, Rubinstein and Snaith
- L’Hôpital
- Bump and Gamburd SSYT

Complex analysis
- Exact representation of \( \mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \)
- Multiple contour integrals
- Leading order analysis

Polynomial

Power of N
Aside

Representation-theoretic approach

A partition $\lambda$ is a sequence $$(\lambda_1, \ldots, \lambda_k)$$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Take the partition $\lambda = (6, 4, 2, 2)$. Then $\lambda$ corresponds to the Young diagram.
Partition

A partition $\lambda$ is a sequence $(\lambda_1, \ldots, \lambda_k)$ of positive integers satisfying
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k. \]
Aside

Representation-theoretic approach

**Partition**

A *partition* \( \lambda \) is a sequence \( (\lambda_1, \ldots, \lambda_k) \) of positive integers satisfying \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \).

Take the partition \( \lambda = (6, 4, 2, 2) \). Then \( \lambda \) corresponds to the Young diagram:

![Young diagram](image)
For $\lambda$ a partition, a \textit{semistandard Young tableau (SSYT)} of shape $\lambda$ is an array $T = (T_{ij})_{1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i}$ of positive integers such that $T_{ij} \leq T_{ij+1}$ and $T_{ij} < T_{i+1,j}$. It is common to write SSYTs in a Young diagram; e.g.

\[
\begin{array}{cccccc}
1 & 1 & 2 & 3 & 3 & 7 \\
2 & 3 & 3 & 4 \\
4 & 4 \\
6 & 7 \\
\end{array}
\]

is a SSYT of shape $(6, 4, 2, 2)$. 
SSYT

For $\lambda$ a partition, a *semistandard Young tableau (SSYT)* of shape $\lambda$ is an array $T = (T_{ij})_{1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i}$ of positive integers such that $T_{i,j} \leq T_{i,j+1}$ and $T_{ij} < T_{i+1,j}$. It is common to write SSYTs in a Young diagram; e.g.

\[
\begin{array}{cccccc}
1 & 1 & 2 & 3 & 3 & 7 \\
2 & 3 & 3 & 4 \\
4 & 4 \\
6 & 7
\end{array}
\]

is a SSYT of shape $(6, 4, 2, 2)$. $T$ has *type* $t = (t_1, t_2, \ldots)$ if $T$ has $t_i$ parts equal to $i$. The SSYT above has type $(2, 2, 4, 3, 0, 1, 2)$. 
SSYT

For $\lambda$ a partition, a *semistandard Young tableau (SSYT)* of shape $\lambda$ is an array $T = (T_{ij})_{1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i}$ of positive integers such that $T_{i,j} \leq T_{i,j+1}$ and $T_{ij} < T_{i+1,j}$. It is common to write SSYTs in a Young diagram; e.g.

\[
\begin{array}{cccccc}
1 & 1 & 2 & 3 & 3 & 7 \\
2 & 3 & 3 & 4 \\
4 & 4 \\
6 & 7 \\
\end{array}
\]

is a SSYT of shape $(6, 4, 2, 2)$. $T$ has type $t = (t_1, t_2, \ldots)$ if $T$ has $t_i$ parts equal to $i$. The SSYT above has type $(2, 2, 4, 3, 0, 1, 2)$.

It is common to use the multivariate notation

$$x^T = x_1^{t_1(T)} x_2^{t_2(T)} \ldots,$$

so for the example SSYT above,

$$x^T = x_1^2 x_2^2 x_3^4 x_4^3 x_6 x_7^2.$$
The combinatorial definition of *Schur functions* is as follows: For a partition \( \lambda \), the Schur function in the variables \( x_1, ..., x_r \) indexed by \( \lambda \) is a multivariable polynomial defined by

\[
s_\lambda(x_1, ..., x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},
\]

where the sum is over all SSYTs \( T \) whose entries belong to the set \( \{1, ..., r\} \) (i.e. \( t_i(T) = 0 \) for \( i > r \)).
Schur functions

The combinatorial definition of Schur functions is as follows: For a partition \( \lambda \), the Schur function in the variables \( x_1, \ldots, x_r \) indexed by \( \lambda \) is a multivariable polynomial defined by

\[
s_\lambda(x_1, \ldots, x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},
\]

where the sum is over all SSYTs \( T \) whose entries belong to the set \( \{1, \ldots, r\} \) (i.e. \( t_i(T) = 0 \) for \( i > r \)).

Take \( \lambda = (2, 1) \vdash 3 \).
The combinatorial definition of Schur functions is as follows: For a partition $\lambda$, the Schur function in the variables $x_1, \ldots, x_r$ indexed by $\lambda$ is a multivariable polynomial defined by

$$s_\lambda(x_1, \ldots, x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},$$

where the sum is over all SSYT$s$ $T$ whose entries belong to the set $\{1, \ldots, r\}$ (i.e. $t_i(T) = 0$ for $i > r$).

Take $\lambda = (2, 1) \vdash 3$. Then to calculate $s_\lambda(x_1, x_2, x_3)$:

$$
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 3 & 2 & 3 & 2 & 3 & 3 & 3 & 3 \\
\end{array}
$$
Schur functions

The combinatorial definition of *Schur functions* is as follows:

For a partition $\lambda$, the Schur function in the variables $x_1, \ldots, x_r$ indexed by $\lambda$ is a multivariable polynomial defined by

$$s_\lambda(x_1, \ldots, x_r) := \sum_T x^T = \sum_T x_1^{t_1(T)} \cdots x_r^{t_r(T)},$$

where the sum is over all SSYTs $T$ whose entries belong to the set $\{1, \ldots, r\}$ (i.e. $t_i(T) = 0$ for $i > r$).

Take $\lambda = (2, 1) \vdash 3$. Then to calculate $s_\lambda(x_1, x_2, x_3)$:

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 3 \\
2 & 3 & 2 & 3 & 2 & 3 & 3 & 3 & 3
\end{array}
\]

So,

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_3^2 + x_1 x_2^2 + 2 x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3 + x_2 x_3^2.$$
Theorem (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

\[
\mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} = s_{\langle N^\beta \rangle}(1^{2\beta})
\]
**Theorem** (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} = s_{\langle N^\beta \rangle}(1^{2\beta})$$

**Corollary** (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} = \prod_{j=0}^{N-1} \frac{j!(j + 2\beta)!}{(j + \beta)!^2}$$
**Theorem** (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A \in U(N)}|P_N(A, \theta)|^{2\beta} = s_{\langle N^\beta \rangle}(1^{2\beta})$$

**Corollary** (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$\mathbb{E}_{A \in U(N)}|P_N(A, \theta)|^{2\beta} = \prod_{j=0}^{N-1} \frac{j!(j + 2\beta)!}{(j + \beta)!^2}$$

This also gives the interpretation that, for $\beta \in \mathbb{N}$, as $N \to \infty$

$$\mathbb{E}_{A \in U(N)}|P_N(A, \theta)|^{2\beta} \sim \frac{g_{\beta}}{\beta!}N^{\beta^2}$$

where $g_{\beta}$ is the number of ways of filling a $\beta \times \beta$ array with the integers $1, 2, \ldots, \beta^2$ in such a way that the numbers increase along each row and down each column.
Proof of polynomial structure

Recall

Theorem

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_N(k, \beta)$ is a polynomial in $N$.

Proposition (Bump and Gamburd)

$E_A \in \mathbb{U}(N) \prod_{j=1}^{k} |P_N(A, \theta_j)|^2 = \langle N^k \beta \rangle (e^i \theta) \prod_{j=1}^{k} e^{i N \beta \theta_j}$,

where $s_\nu(x_1, \ldots, x_n)$ is the Schur polynomial in $n$ variables with respect to the partition $\nu$. Here $\langle N^k \beta \rangle = (N^k \beta, \ldots, N^k \beta)$, and $e^i \theta = (\beta e^i \theta_1, \ldots, \beta e^i \theta_k, \beta e^i \theta_1, \ldots, \beta e^i \theta_k)$. 

Proof of polynomial structure

Recall

**Theorem**

Let \( k, \beta \in \mathbb{N} \). Then \( \text{MoM}_N(k, \beta) \) is a polynomial in \( N \).
Proof of polynomial structure

Recall

**Theorem**

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_N(k, \beta)$ is a polynomial in $N$.

**Proposition (Bump and Gamburd)**

\[ \mathbb{E}_{A \in U(N)} \left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right) = \frac{s_{\langle N^{k\beta} \rangle} \left( e^{i\theta} \right)}{\prod_{j=1}^{k} e^{iN\beta \theta_j}}, \]

where $s_{\nu}(x_1, \ldots, x_n)$ is the Schur polynomial in $n$ variables with respect to the partition $\nu$. Here $\langle N^{k\beta} \rangle = (N, \ldots, N)$, and

\[ e^{i\theta} = \left( e^{i\theta_1}, \ldots, e^{i\theta_1}, \ldots, e^{i\theta_k}, \ldots, e^{i\theta_k}, e^{i\theta_1}, \ldots, e^{i\theta_1}, \ldots, e^{i\theta_k}, \ldots, e^{i\theta_k} \right). \]
Overview of proof of first theorem

Hence for $k, \beta \in \mathbb{N}$,

$$
\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_T e^{i\theta_1(\tau_1 - N\beta)} \cdots e^{i\theta_k(\tau_k - N\beta)} \prod_{j=1}^{k} d\theta_j
$$

$$
= \sum_{\tilde{T}} 1,
$$
Overview of proof of first theorem

Hence for \( k, \beta \in \mathbb{N} \),

\[
\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \ldots \int_0^{2\pi} \sum_{T} e^{i\theta_1(\tau_1 - N\beta)} \ldots e^{i\theta_k(\tau_k - N\beta)} \prod_{j=1}^{k} d\theta_j
\]

\[= \sum_{\tilde{T}} 1,\]

where the sum is now over \( \tilde{T} \), restricted SSYT
Overview of proof of first theorem

Hence for \( k, \beta \in \mathbb{N} \),

\[
\text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{\mathcal{T}} e^{i\theta_1(\tau_1 - N\beta)} \cdots e^{i\theta_k(\tau_k - N\beta)} \prod_{j=1}^{k} d\theta_j
\]

\[
= \sum_{\tilde{T}} 1,
\]

where the sum is now over \( \tilde{T} \), restricted SSYT - require \( N\beta \) entries from each of the sets \( \{2\beta(j - 1) + 1, \ldots, 2j\beta\} \), for \( j \in \{1, \ldots, k\} \).

\[
\text{MoM}_N(k, \beta) = \sum_{\tilde{T}} 1 < \sum_{\mathcal{T}} 1 = \text{Poly}_N(k^2\beta^2).
\]
Conrey, Farmer, Rubinstein, Keating and Snaith give that
\[ E_k \prod_{j=1}^{\infty} |P_N(A,\theta_j)|^2 \beta \]
\[ = 2^k \beta \prod_{l=k+1}^{2k} \omega - N_{j} \sum_{\sigma \in \Xi_k\beta} \omega_{\sigma}(k\beta+1) \omega_{\sigma}(k\beta+2) \cdots \omega_{\sigma}(2k\beta) \]
\[ \times N \prod_{l \leq k < q} (1 - \omega_{\sigma}(l) \omega_{\sigma}(q-1)) \]
where \( \Xi_k\beta \) is the set of \( (2^k \beta) \) permutations \( \sigma \in S_{2^k \beta} \) such that \( \sigma(1) < \sigma(2) < \cdots < \sigma(k\beta) \) and \( \sigma(k\beta+1) < \cdots < \sigma(2k\beta) \), and
\[ \omega = (e^{i\theta_1}, \ldots, e^{i\theta_1}, \ldots, e^{i\theta_k}, \ldots, e^{i\theta_k} \cdot \ldots \cdot e^{i\theta_k}) \]
Overview of proof of first theorem

Conrey, Farmer, Rubinstein, Keating and Snaith give that

\[
\mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} = \prod_{j=k\beta+1}^{2k\beta} \omega_j^{-N} \sum_{\sigma \in \Xi_{k\beta}} \left( \frac{\omega_{\sigma(k\beta+1)}\omega_{\sigma(k\beta+2)} \cdots \omega_{\sigma(2k\beta)}}{\prod_{l \leq k\beta < q}(1 - \omega_{\sigma(l)}\omega_{\sigma(q)}^{-1})} \right)^N
\]
Overview of proof of first theorem

Conrey, Farmer, Rubinstein, Keating and Snaith give that

\[ \mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} = \prod_{j=k\beta+1}^{2k\beta} \omega_j^{-N} \sum_{\sigma \in \Xi_{k\beta}} \frac{(\omega_{\sigma(k\beta+1)}\omega_{\sigma(k\beta+2)} \cdots \omega_{\sigma(2k\beta)})^N}{\prod_{l \leq k\beta < q}(1 - \omega_{\sigma(l)}\omega_{\sigma(q)}^{-1})} \]

where \( \Xi_{k\beta} \) is the set of \( \binom{2k\beta}{k\beta} \) permutations \( \sigma \in S_{2k\beta} \) such that \( \sigma(1) < \sigma(2) < \cdots < \sigma(k\beta) \) and \( \sigma(k\beta+1) < \cdots < \sigma(2k\beta) \), and

\[ \omega = (e^{i\theta_1}_\beta, \ldots, e^{i\theta_1}_\beta, \ldots, e^{i\theta_k}_\beta, \ldots, e^{i\theta_k}_\beta, e^{i\theta_1}_\beta, \ldots, e^{i\theta_1}_\beta, \ldots, e^{i\theta_k}_\beta, \ldots, e^{i\theta_k}_\beta). \]
Examples

$$\text{MoM}_N(k, \beta) = E_{A \in U(N)}(\left| P_N(A, \theta) \right|^2 \beta d\theta)^k$$

$$\text{MoM}_N(1, 1) = N + 1$$
$$\text{MoM}_N(2, 1) = \frac{1}{6}(N + 3)(N + 2)(N + 1)$$
$$\text{MoM}_N(3, 1) = \frac{1}{2520}(N + 5)(N + 4)(N + 3)(N + 2)(N + 1)(7N^6 + 168N^5 + 1804N^4 + 10944N^3 + 41893N^2 + 99624N + 154440)$$
$$\text{MoM}_N(4, 1) = \frac{1}{778377600}(N + 7)(N + 6)(N + 5)(N + 4)(N + 3)(N + 2) \times (N + 1)(7N^6 + 168N^5 + 1804N^4 + 10944N^3 + 41893N^2 + 99624N + 15440)$$
$$\text{MoM}_N(1, 2) = \frac{1}{12}(N + 1)(N + 2)^2(N + 3).$$
Examples

\[
\text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right)
\]
Examples

\[ \text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right) \]

\[ \text{MoM}_N(1, 1) = N + 1 \]
Examples

\[ \text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right) \]

\[ \text{MoM}_N(1, 1) = N + 1 \]

\[ \text{MoM}_N(2, 1) = \frac{1}{6} (N + 3)(N + 2)(N + 1) \]
Examples

\[ \text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} \, d\theta \right)^k \right) \]

MoM$_N(1, 1) = N + 1$

MoM$_N(2, 1) = \frac{1}{6} (N + 3)(N + 2)(N + 1)$

MoM$_N(3, 1) = \frac{1}{2520} (N + 5)(N + 4)(N + 3)(N + 2)(N + 1)(N^2 + 6N + 21)$
Examples

$$\text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} \, d\theta \right)^k \right)$$

$$\text{MoM}_N(1, 1) = N + 1$$

$$\text{MoM}_N(2, 1) = \frac{1}{6}(N + 3)(N + 2)(N + 1)$$

$$\text{MoM}_N(3, 1) = \frac{1}{2520}(N + 5)(N + 4)(N + 3)(N + 2)(N + 1)(N^2 + 6N + 21)$$

$$\text{MoM}_N(4, 1) = \frac{1}{778377600}(N + 7)(N + 6)(N + 5)(N + 4)(N + 3)(N + 2)$$
$$\times (N + 1)(7N^6 + 168N^5 + 1804N^4 + 10944N^3 +$$
$$+ 41893N^2 + 99624N + 154440)$$
Examples

\[ \text{MoM}_N(k, \beta) = \mathbb{E}_{A \in U(N)} \left( \left( \frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right) \]

\[ \text{MoM}_N(1, 1) = N + 1 \]
\[ \text{MoM}_N(2, 1) = \frac{1}{6}(N + 3)(N + 2)(N + 1) \]
\[ \text{MoM}_N(3, 1) = \frac{1}{2520}(N + 5)(N + 4)(N + 3)(N + 2)(N + 1)(N^2 + 6N + 21) \]
\[ \text{MoM}_N(4, 1) = \frac{1}{778377600}(N + 7)(N + 6)(N + 5)(N + 4)(N + 3)(N + 2) \]
\[ \times (N + 1)(7N^6 + 168N^5 + 1804N^4 + 10944N^3 + 41893N^2 + 99624N + 154440) \]
\[ \text{MoM}_N(1, 2) = \frac{1}{12}(N + 1)(N + 2)^2(N + 3). \]
Examples

$$\text{MoM}_N(2, 2) = \frac{1}{163459296000} (N + 7)(N + 6)(N + 5)(N + 4) \times (N + 3)(N + 2)(N + 1)(298N^8 + 9536N^7 + 134071N^6 + 1081640N^5 + 5494237N^4 + 18102224N^3 + 38466354N^2 + 50225040N + 32432400).$$
Examples

\[ \text{MoM}_N(2, 3) = \frac{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)(N+9)(N+10)(N+11)}{1722191327731024154944441889587200000000} \]
\[ \times \left( 12308743625763N^{24} + 1772459082109872N^{23} + 121902830804059138N^{22} + 
+ 5328802119564663432N^{21} + 166214570195622478453N^{20} + 3937056259812505643352N^{19} 
+ 73583663800226157619008N^{18} + 1113109355823972261429312N^{17} + 13869840005250869763713293N^{16} 
+ 144126954435929329947378912N^{15} + 1259786144898207172443272698N^{14} 
+ 9315726913410827893883025672N^{13} + 58475127984013141340467825323N^{12} 
+ 311978271286536355427593012632N^{11} + 1413794106539529439589778645028N^{10} 
+ 5427439874579682729570383266992N^9 + 17564370687865211818995713096848N^8 
+ 47561382824003032731805262975232N^7 + 10661092725688647520961130100128N^6 
+ 194861499503272627170466392014592N^5 + 28430387722173567353377603640320N^4 
+ 320989495108428049992898521600000N^3 + 266974288159876385845370793984000N^2 
+ 148918006780282798012340305920000N + 43144523802785397500411904000000 \right) \]
Overview of proof of second theorem

Recall, Theorem [B.-Keating (2018)]

Let \( k, \beta \in \mathbb{N} \). Then

\[
\text{Mom} \left( k, \beta \right) = c \left( k, \beta \right) N^{k^2 \beta^2 - k + 1} + O \left( N^{k^2 \beta^2 - k} \right),
\]

where \( c \left( k, \beta \right) \) is an explicit function of \( k, \beta \).

Proof ingredients:
1. Expand \( \prod_{j=1}^{k} |P_N (A_j, \theta_j)|^2 \) as a multiple contour integral.
2. Deform and manipulate the integrals.
3. Analyse the result asymptotically as \( N \to \infty \).
Overview of proof of second theorem

Recall,

**Theorem** [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then

$$\text{MoM}_N(k, \beta) = c(k, \beta)N^{k^2 \beta^2 - k + 1} + O(N^{k^2 \beta^2 - k}),$$

where $c(k, \beta)$ is an explicit function of $k, \beta$. 
Overview of proof of second theorem

Recall,

**Theorem [B.-Keating (2018)]**

Let \( k, \beta \in \mathbb{N} \). Then

\[
\text{MoM}_N(k, \beta) = c(k, \beta) N^{k^2 \beta^2 - k + 1} + O(N^{k^2 \beta^2 - k}),
\]

where \( c(k, \beta) \) is an explicit function of \( k, \beta \).

Proof ingredients:

- Expand \( \mathbb{E} \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} \) as a multiple contour integral
Overview of proof of second theorem

Recall,

**Theorem** [B.-Keating (2018)]

Let \( k, \beta \in \mathbb{N} \). Then

\[
\text{MoM}_N(k, \beta) = c(k, \beta) N^{k^2 \beta^2 - k + 1} + O(N^{k^2 \beta^2 - k}),
\]

where \( c(k, \beta) \) is an explicit function of \( k, \beta \).

Proof ingredients:
- Expand \( \mathbb{E} \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \) as a multiple contour integral
- Deform and manipulate the integrals
Overview of proof of second theorem

Recall,

**Theorem** [B.-Keating (2018)]

Let \( k, \beta \in \mathbb{N} \). Then

\[
\text{MoM}_N(k, \beta) = c(k, \beta) N^{k^2 \beta^2 - k + 1} + O(N^{k^2 \beta^2 - k}),
\]

where \( c(k, \beta) \) is an explicit function of \( k, \beta \).

Proof ingredients:

- Expand \( \mathbb{E} \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} \) as a multiple contour integral
- Deform and manipulate the integrals
- Analyse the result asymptotically as \( N \to \infty \).
Overview of proof of second theorem

Define

\[ I_{k,\beta}(\theta_1, \ldots, \theta_k) = E_{A \in U(N)} \left[ \prod_{j=1}^k |P_N(A, \theta_j)|^{2\beta} \right], \]

so

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(\theta_1, \ldots, \theta_k) \, d\theta_1 \cdots d\theta_k. \]

Theorem [CFKRS]

For \( k, \beta \in \mathbb{N} \),

\[ I_{k,\beta}(\theta) = (-1)^k \beta e^{-i\beta \sum_{j=1}^k \theta_j} (2\pi i)^{2k\beta} \left( \frac{N(z_{k\beta})}{N(z_{k\beta}^2 \cdots z_{2k\beta})} \right)^2 \prod_{m \leq k\beta < n} \left( 1 - e^{z_n - z_m} \right) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k\beta} (z_m + i\theta_n)^{2\beta}. \]
Overview of proof of second theorem

Define

\[ I_{k,\beta}(\theta_1, \ldots, \theta_k) = \mathbb{E}_{A \in U(N)} \left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right), \]
Overview of proof of second theorem

Define

\[ I_{k,\beta}(\theta_1, \ldots, \theta_k) = \mathbb{E}_{A \in U(N)} \left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right), \]

so

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(\theta_1, \ldots, \theta_k) d\theta_1 \cdots d\theta_k. \]
Overview of proof of second theorem

Define

\[ I_{k,\beta}(\theta_1, \ldots, \theta_k) = \mathbb{E}_{A \in \mathbb{U}(N)} \left( \prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right), \]

so

\[ \text{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(\theta_1, \ldots, \theta_k) d\theta_1 \cdots d\theta_k. \]

**Theorem** [CFKRS]

For \( k, \beta \in \mathbb{N}, \)

\[ I_{k,\beta}(\theta) = \frac{(-1)^k e^{-i\beta \sum_{j=1}^{k} \theta_j}}{(2\pi i)^{2k\beta}((k\beta)!)^2} \int \cdots \int \frac{e^{-N(z_{k\beta+1} + \cdots + z_{2k\beta})} \Delta(z_1, \ldots, z_{2k\beta})^2 dz_1 \cdots dz_{2k\beta}}{\prod_{m\leq k\beta<n}(1 - e^{z_n-z_m}) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k}(z_m + i\theta_n)^{2\beta}}. \]
Overview of proof of second theorem
Manipulation of MCI

\[ l_{k,\beta}(\theta) = \frac{(-1)^{k\beta} e^{-i\beta \sum_{j=1}^{k} \theta_j}}{(2\pi i)^{2k\beta} ((k\beta)!)^2} \oint \cdots \oint \frac{e^{-N(z_{k\beta+1} + \cdots + z_{2k\beta})} \Delta(z_1, \ldots, z_{2k\beta})^2 dz_1 \cdots dz_{2k\beta}}{\prod_{m \leq k\beta < n} (1 - e^{z_n - z_m}) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k} (z_m + i\theta_n)^{2\beta}}. \]
Overview of proof of second theorem
Manipulation of MCI

\[ I_{k,\beta}(\theta) = \frac{(-1)^k \beta e^{-i \beta \sum_{j=1}^{k} \theta_j}}{(2\pi i)^{2k\beta} ((k\beta)!)^2} \oint \cdots \oint \frac{e^{-N(z_{k\beta+1} + \cdots + z_{2k\beta})} \Delta(z_1, \ldots, z_{2k\beta})^2 dz_1 \cdots dz_{2k\beta}}{\prod_{m\leq k\beta < n} (1 - e^{z_n - z_m}) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k} (z_m + i\theta_n)^{2\beta}}. \]

- Deform the contours
- Change of variables
- Carefully analyse remaining integrals
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1} \]
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1} \]

where

\[
c(k, \beta) = \sum_{l_1, \ldots, l_{k-1}}^{2\beta} c_l(k, \beta)((k - 1)\beta - \sum_{j=1}^{k-1} l_j) f(k, \beta, l) P_{k, \beta}(l_1, \ldots, l_{k-1}),
\]
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2\beta^2 - k + 1} \]

where

\[ c(k, \beta) = \sum_{l_1, \ldots, l_{k-1} = 0}^{2\beta} c_l(k, \beta)((k - 1)\beta - \sum_{j=1}^{k-1} l_j) f(k, \beta, l) P_{k, \beta}(l_1, \ldots, l_{k-1}), \]
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1} \]

where

\[
c(k, \beta) = \sum_{l_1, \ldots, l_{k-1} = 0}^{2\beta} c_l(k, \beta) ((k - 1)\beta - \sum_{j=1}^{k-1} l_j) f^{(k, \beta, l)} P_{k, \beta}(l_1, \ldots, l_{k-1}),
\]
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1} \]

where

\[
c(k, \beta) = \sum_{\substack{l_1, \ldots, l_{k-1}=0 \\ (+)}} \left( \sum_{l_1, \ldots, l_{k-1}=0} \right) c_\perp(k, \beta) \left( (k - 1) \beta - \sum_{j=1}^{k-1} l_j \right)^{f(k, \beta, l)} P_{k, \beta}(l_1, \ldots, l_{k-1}),
\]
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1} \]

where

\[
c(k, \beta) = \sum_{l_1, \ldots, l_{k-1}=0}^{2\beta} c_1(k, \beta)((k - 1)\beta - \sum_{j=1}^{k-1} l_j)^{f(k, \beta, l)} P_{k, \beta}(l_1, \ldots, l_{k-1}),
\]
Overview of proof of second theorem

Leading order

\[ \text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1} \]

where

\[
c(k, \beta) = \sum_{l_1, \ldots, l_{k-1} = 0}^{2\beta} c_l(k, \beta)((k - 1)\beta - \sum_{j=1}^{k-1} l_j)f(k, \beta, l) P_{k, \beta}(l_1, \ldots, l_{k-1}),
\]

and

\[
P_{k, \beta}(l) = \frac{(-1)^{\sum_{\sigma < \tau} |S_{\sigma < \tau}^-|}}{(2\pi i)^{2k\beta} ((k\beta)!)^2} \int_{\Gamma_0} \cdots \int_{\Gamma_0} e^{-\sum_{m=k\beta+1}^{2k\beta} v_m} \prod_{m<n} (v_n - v_m)^2 \prod_{m \leq k\beta < n} (\alpha_m = \alpha_n) v_m^{2\beta} \prod_{m=1}^{2k\beta} v_m^{2\beta} \frac{\prod_{m \leq k\beta < n} (\alpha_m = \alpha_n) (v_n - v_m)^2}{\prod_{m=1}^{2k\beta} v_m^{2\beta}} \times \psi_{k, \beta, l}(((k - 1)\beta - \sum_{j=1}^{k-1} l_j)\nu) \prod_{m=1}^{2k\beta} dv_m.
\]
So for $k, \beta \in \mathbb{N}$ we have

$$\text{MoM}_{N}(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1}.$$
Overview of proof of second theorem
Leading order

So for $k, \beta \in \mathbb{N}$ we have

$$\text{MoM}_N(k, \beta) \sim c(k, \beta) N^{k^2 \beta^2 - k + 1}.$$ 

The theorem follows if one can show that $c(k, \beta) \neq 0$. A lengthy computation shows that this is the case - in fact $c(k, \beta) > 0$. 
Another alternative approach
Another alternative approach

One can recover the same asymptotic result using Gelfand-Tsetlin patterns.

**Theorem** [Assiotis-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then,

$$\text{MoM}_N(k, \beta) = c(k, \beta)N^{k^2\beta^2-k+1} + O(N^{k^2\beta^2-k}),$$

where $c(k, \beta)$ can be written explicitly as a volume of a certain region involving continuous Gelfand-Tsetlin patterns with constraints.
Translating to Number Theory

Analogue of MoM $\zeta_T(k, \beta) := \frac{1}{T} \int_0^T \left( \frac{1}{2 \pi} \int_0^{2\pi} |\zeta(1/2 + i(t + \gamma))|^{2\beta} d\gamma \right)^k dt = \frac{1}{T} (2\pi)^k \prod_{j=1}^{\infty} \left| \zeta(1/2 + i(t + \gamma_j)) \right|^{2\beta} d\gamma_j.$

Conjecture Fyodorov & Keating

For $k \beta^2 > 1$, $\zeta_T(k, \beta) \sim c'(k, \beta)(\log T)^{k^2 \beta^2 - k + 1}$. 

Emma Bailey

Moments of Moments

21st June 28 / 34
Translating to Number Theory

Analogue of $\text{MoM}_N(k, \beta)$:

$$
\text{MoM}^\zeta_T(k, \beta) := \frac{1}{T} \int_T^{2T} \left( \frac{1}{2\pi} \int_0^{2\pi} |\zeta(1/2 + i(t + \gamma))|^{2\beta} d\gamma \right)^k dt
$$

$$
= \frac{1}{T(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_T^{2T} \prod_{j=1}^k |\zeta(1/2 + i(t + \gamma_j))|^{2\beta} dt \prod_{j=1}^k d\gamma_j.
$$
Translating to Number Theory

Analogue of $\text{MoM}_N(k, \beta)$:

$$\text{MoM}_T^\zeta(k, \beta) := \frac{1}{T} \int_T^{2T} \left( \frac{1}{2\pi} \int_0^{2\pi} |\zeta(1/2 + i(t + \gamma))|^{2\beta} d\gamma \right)^k dt$$

$$= \frac{1}{T(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_T^{2T} \prod_{j=1}^k |\zeta(1/2 + i(t + \gamma_j))|^{2\beta} dt \prod_{j=1}^k d\gamma_j.$$

**Conjecture** Fyodorov & Keating

For $k\beta^2 > 1$, $\text{MoM}_T^\zeta(k, \beta) \sim c'(k, \beta)(\log \frac{T}{2\pi})^{k\beta^2 - k + 1}.$
Translating to Number Theory

Conjectured expression for integrand (CFKRS):

\[
\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\beta} dt = \frac{1}{T} \int_0^T \frac{(-1)^\beta}{(\beta!)^2} \frac{1}{(2\pi i)^{2\beta}} \\
\times \int \cdots \int \frac{G_\zeta(z_1, \ldots, z_{2\beta}) \Delta^2(z_1, \ldots, z_{2\beta})}{\prod_{j=1}^{2\beta} z_j^{2\beta}} \\
\times e^{\frac{1}{2} \log \frac{t}{2\pi}} \sum_{j=1}^{\beta} z_j^{-z_{\beta+j}} dz_1 \cdots dz_{2\beta} dt + o(1).
\]
Translating to Number Theory

Conjectured expression for integrand (CFKRS):

\[
\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\beta} \, dt = \frac{1}{T} \int_0^T \frac{(-1)^\beta}{(\beta!)^2} \frac{1}{(2\pi i)^{2\beta}} \\
\times \int \ldots \int \frac{G_{\zeta}(z_1, \ldots, z_{2\beta}) \Delta^2(z_1, \ldots, z_{2\beta})}{\prod_{j=1}^{2\beta} z_j^{2\beta}} \\
\times e^{\frac{1}{2} \log \frac{t}{2\pi} \sum_{j=1}^{\beta} z_j^{-z_{\beta}+j}} \, dz_1 \ldots \, dz_{2\beta} \, dt + o(1).
\]

where

\[G_{\zeta}(z_1, \ldots, z_{2\beta}) = A_{\beta}(z_1, \ldots, z_{2\beta}) \prod_{i,j=1}^{\beta} \zeta(1 + z_i - z_{\beta+j}),\]

and \(A_{\beta}(z)\) is an Euler product whose local factors are polynomials in \(p^{-1}\) and \(p^{-z_i}\).
Symplectic, Orthogonal, and $L$-functions

$\text{USp}(2^N) = \{ M \in \text{U}(2^N) : M^t \Omega M = \Omega \}$, $\Omega = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$,

$\det(I - Ms) = N \prod_{n=1}^{\infty} (1 - e^{i \theta_n s})(1 - e^{-i \theta_n s})$.

$\text{SO}(2^N) = \{ O \in \text{O}(2^N) : \det(O) = 1 \}$,

$\det(I - Os) = \prod_{m=1}^{\infty} (1 - e^{i \theta_m s})(1 - e^{-i \theta_m s})$. 
Symplectic, Orthogonal, and $L$-functions

- Relationship between families of $L$-functions and other random matrix families
Symplectic, Orthogonal, and $L$-functions

- Relationship between families of $L$-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc
Symplectic, Orthogonal, and $L$-functions

- Relationship between families of $L$-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

USp$(2^N) = \{ M \in U(2^N) : M^t \Omega M = \Omega \}$,
\[ \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \]
\[ \det(I - Ms) = N \prod_{n=1}^{\infty} (1 - e^{i \theta_n}s)(1 - e^{-i \theta_n}s). \]

SO$(2^N) = \{ O \in O(2^N) : \det(O) = 1 \}$,
\[ \det(I - Os) = N \prod_{m=1}^{\infty} (1 - e^{i \theta_m}s)(1 - e^{-i \theta_m}s). \]
Symplectic, Orthogonal, and \( L \)-functions

- Relationship between families of \( L \)-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

\[
\text{USp}(2N) = \{ M \in U(2N) : M^t \Omega M = \Omega \}, \quad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix},
\]

\[
\det(I - Ms) = \prod_{n=1}^{N} (1 - e^{i\theta_n} s)(1 - e^{-i\theta_n} s).
\]
Symplectic, Orthogonal, and $L$-functions

- Relationship between families of $L$-functions and other random matrix families
- Katz-Sarnak; Conrey, Farmer, Keating, Rubinstein, and Snaith etc

\[ \text{USp}(2N) = \{ M \in U(2N) : M^t \Omega M = \Omega \}, \quad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \]
\[ \det(I - Ms) = \prod_{n=1}^{N} (1 - e^{i \theta_n s})(1 - e^{-i \theta_n s}). \]

\[ \text{SO}(2N) = \{ O \in O(2N) : \det(O) = 1 \}, \]
\[ \det(I - Os) = \prod_{m=1}^{N} (1 - e^{i \theta_m s})(1 - e^{-i \theta_m s}). \]
Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_{\text{USp}(2^N)}(k, \beta)$ is a polynomial in $N$ and further
$$\text{MoM}_{\text{USp}(2^N)}(k, \beta) = c_{\text{USp}}(k, \beta) N^k \beta (2^k \beta + 1)^{-k} + O(N^k \beta (2^k \beta + 1)^{-k-1}).$$

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_{\text{SO}(2^N)}(k, \beta)$ is a polynomial in $N$. Further
$$\text{MoM}_{\text{SO}(2^N)}(1, 1) = 2(N + 1), \quad \text{otherwise} \quad \text{MoM}_{\text{SO}(2^N)}(k, \beta) = c_{\text{SO}(2^N)} N^k \beta (2^k \beta - 1)^{-k} + O(N^k \beta (2^k \beta - 1)^{-k-1}).$$
**Theorem** [Assiotis-B.-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_{\text{USp}(2N)}(k, \beta)$ is a polynomial in $N$ and further

$$\text{MoM}_{\text{USp}(2N)}(k, \beta) = c_{\text{USp}}(k, \beta) N^{k\beta(2k\beta+1)-k} + O(N^{k\beta(2k\beta+1)-k-1}).$$
Symplectic, Orthogonal, and $L$-functions

**Theorem** [Assiotis-B.-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_{\text{USp}(2N)}(k, \beta)$ is a polynomial in $N$ and further

$$\text{MoM}_{\text{USp}(2N)}(k, \beta) = c_{\text{USp}}(k, \beta) N^{k\beta(2k\beta+1)-k} + O(N^{k\beta(2k\beta+1)-k-1}).$$

**Theorem** [Assiotis-B.-Keating (2019)]

Let $k, \beta \in \mathbb{N}$. Then $\text{MoM}_{\text{SO}(2N)}(k, \beta)$ is a polynomial in $N$. Further

$$\text{MoM}_{\text{SO}(2N)}(1, 1) = 2(N + 1),$$

otherwise

$$\text{MoM}_{\text{SO}(2N)}(k, \beta) = c_{\text{SO}(2N)} N^{k\beta(2k\beta-1)-k} + O(N^{k\beta(2k\beta-1)-k-1}).$$
A word on the proof
A word on the proof

Again we have a number of forms for the matrix average:

\[
\mathbb{E}_{USp(2N)} \left( \prod_{j=1}^{k} \left| \det(I - Ae^{-i\theta}) \right|^{2\beta} \right) = \sum_{P \in SP} w(P) \quad (BG 2006)
\]
A word on the proof

Again we have a number of forms for the matrix average:

$$\mathbb{E}_{\text{USp}(2N)} \left( \prod_{j=1}^{k} |\det(I - Ae^{-i\theta})|^{2\beta} \right)$$

$$= \sum_{\mathcal{P} \in \text{SP}_{\langle Nk\beta \rangle}} w(\mathcal{P})$$

$$= \sum_{\varepsilon \in \{\pm 1\}^{2k\beta}} \frac{\prod_{j=1}^{2k\beta} x_j^{N(1-\varepsilon_j)}}{\prod_{i \leq j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})} \quad (\text{CFKRS 2002})$$
A word on the proof

Again we have a number of forms for the matrix average:

\[
\mathbb{E}_{\text{USp}(2N)} \left( \prod_{j=1}^{k} \left| \det(l - Ae^{-i\theta}) \right|^{2\beta} \right)
\]

\[= \sum_{\mathcal{P} \in \text{SP}_{\langle Nk\beta \rangle}} w(\mathcal{P}) \]

\[= \sum_{\varepsilon \in \{\pm 1\}^{2k\beta}} \frac{\prod_{j=1}^{2k\beta} x_j^{N(1-\varepsilon_j)}}{\prod_{i \leq j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})} \]

\[= \frac{(-1)^{k\beta(2k\beta-1)} 2^{2k\beta}}{(2\pi i)^{2k\beta}(2k\beta)!} e^{-Ni \sum_{j=1}^{2k\beta} \theta_j} \quad \text{(CFRKS 2002)} \]

\[\times \int \cdots \int \frac{\Delta(z_1^2, \ldots, z_{2k\beta}^2) \prod_{j=1}^{2k\beta} z_j e^{Nz_j} dz_j}{\prod_{l \leq m}(1 - e^{-z_m-z_l}) \prod_{m,n=1}^{2k\beta}(z_m - i\theta_n)(z_m + i\theta_n)}\]
The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2 \beta^2 - k + 1$ (in line with FK conjecture). We recover a formula for the leading coefficient in this case. Similar results are found for $\text{SO}(2^N)$ and $\text{USp}(2^N)$. The polynomials can be explicitly computed. Techniques carry over (under conjecture) to $\zeta(1/2 + it)$ and other $L$-functions. Proved the growth of certain representation theoretic sums.
CUE: The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
CUE: The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).

We recover a formula for the leading coefficient in this case.
Summary

- CUE: The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.
- Similar results are found for SO(2$N$) and USp(2$N$).
CUE: The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).

We recover a formula for the leading coefficient in this case.

Similar results are found for SO($2N$) and USp($2N$).

The polynomials can be explicitly computed.
Summary

- CUE: The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).
- We recover a formula for the leading coefficient in this case.
- Similar results are found for $\text{SO}(2N)$ and $\text{USp}(2N)$.
- The polynomials can be explicitly computed.
- Techniques carry over (under conjecture) to $\zeta(1/2 + it)$ and other $L$-functions.
CUE: The integer $k$ moments of the integer $\beta$ moments are polynomials in $N$ of degree $k^2\beta^2 - k + 1$ (in line with FK conjecture).

- We recover a formula for the leading coefficient in this case.
- Similar results are found for SO($2N$) and USp($2N$)
- The polynomials can be explicitly computed.
- Techniques carry over (under conjecture) to $\zeta(1/2 + it)$ and other $L$-functions.
- Proved the growth of certain representation theoretic sums.
Thank you